THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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OBLIQUE DEVIATION AND REFRACTION PRODUCED BY PRISMS.

By H. S. UHLER, Yale University.

The occurrence in the MONTHLY of a few problems[1] in geometrical optics indicates that some of its readers are interested in this subject. As the field is a fairly fertile one for applied mathematics, work in it deserves to be encouraged. Since, in general, there are many discouraging difficulties and pitfalls that the mathematical reader will encounter if he attempts to gain information first hand from the monographs and advanced treatises on geometrical optics, it occurred to the present writer that a relatively short essay dealing with oblique refraction by prisms might be of some interest and perhaps value in the pages of the MONTHLY.

It is but fair to state, at the outset, that none of the results obtained—save the last one—is new. On the other hand, most of the proofs are original and they have been developed with special reference to brevity and rigor.

The only physical law that will be needed is called Snell's law. The complete statement of this law involves two facts which are embodied respectively in the sentence: "The angles of incidence and refraction are coplanar," and in the formula:

$$n_1 \sin a_1 = n_2 \sin a_2. \tag{1}$$

If the ray of light passes from medium 1 into medium 2, then a_1 and a_2 denote respectively the angles of incidence and refraction. n_1 and n_2 symbolize the absolute indices of refraction for the first and second media. The "absolute" index means the ratio of the velocity of light in empty space to the velocity of light in the material dispersive medium in question. The ratio $n_2/n_1 \equiv n$ is called the "relative" index of refraction of medium 2 with respect to medium 1.

In Fig. 1 let the plane IZ suggest the first face of the prism, and let FI and IS indicate respectively the incident ray and the refracted ray. Air and glass may be imagined at the left and right of this plane respectively. Stated broadly, our problem will be to investigate briefly the behavior of the emergent ray beyond the second face of the prism (intentionally omitted from Fig. 1) as the incident ray FI is moved around the point of incidence, I, into all possible positions.

Experience has shown that it is not advantageous to determine the angular positions of the rays by employing a rectangular coördinate frame, direction

[1] THE AMER. MATH. MONTHLY, vol. 26, 80, 1919; vol. 27, 35, 1920.

cosines, component vectors, etc. Instead, the analysis and the resulting theorems are very appreciably simplified by introducing a "principal plane," NJ, and by projecting the rays orthogonally on this plane.

Definitions: The line in which the two refracting planes or faces intersect is called the "refracting edge" of the prism. The angle between the refracting planes is named the "refracting angle." Any plane passing through the prism perpendicular to the refracting edge is termed a "principal plane." Hence, a principal plane is parallel to the bases of a right prism of solid geometry. The orthogonal projection, GIG_1, on a principal plane of the oblique ray, FIS, is called the "projected ray." Although both the oblique ray and the projected ray are mathematical fictions,[1] it is conducive to clearness of thought to treat the former as a physical reality (that is, the path along which the energy is transmitted in a non-crystalline medium), and the latter as a purely geometric auxiliary line. The angles which the oblique ray and the projected ray beyond the second or emergence face of the prism make with the corresponding rays before incidence at the first face are called respectively the "deviation of the (oblique) ray" and the "deviation of the projected ray," or the "oblique deviation" and the "projected deviation." The angles, v_1, v_2, $\cdots$,[2] which the segments of the complete oblique ray make with the orthogonal projections on a principal plane will be called the "altitudes" of the rays. Similarly the angles, h_1, h_2, $\cdots$,[2] which the segments of the complete projected ray form with the normals, $NIN_1, \cdots$, to the faces of the prism, at the two points where refraction occurs, will be called the "azimuths" of the oblique ray.

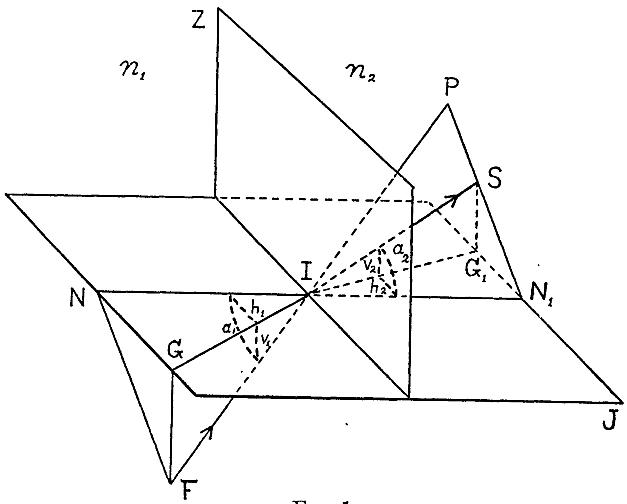


Fig. 1.

In Fig. 1, $NIF \equiv a_1$ and $N_1IS \equiv a_2$ denote respectively the angle of incidence and the angle of refraction of the oblique ray (at the first face of the prism). The outline $FNIN_1P$ indicates the plane of incidence at the first face. The mathematical reader should now be prepared to follow the rest of the paper without undue distraction arising from justifiable unfamiliarity with the highly specialized subject.

For the sake of both consistency and brevity we shall not prove the first formulas of oblique refraction by adding certain highly artificial construction

[1] The wave-length of light is quite finite, diffraction phenomena exist, etc.

[2] h and v to suggest horizontal and vertical respectively.

lines to (the definitional) Fig. 1 and by employing certain theorems of solid geometry (as is done in the older[1] treatises); instead we shall make use of a far more powerful, elegant and general method. This method consists in moving the rays (which are unlocalized vectors), without changing their relative angular directions, until they radiate from the center of a sphere of unit radius. The points where the rays intersect the spherical surface may then be used to define the directions of the rays, and the problems will be reduced primarily to the realm of spherical trigonometry.

Let us now imagine a spherical diagram drawn with the equatorial circle representing the principal plane passing through the point of incidence. Let the points N_1, P, and S indicate respectively the normal to the first face of the prism at the point of incidence, the incident ray, and the refracted ray. Let H mark the pole of the equatorial circle on the same side of the plane of the circle as the points P and S. Imagine great circle arcs connecting H to the points P and S. Let the points of intersection of these two arcs with the equatorial circle be denoted by p and s respectively.

Then the arcs N_1p and N_1s measure the azimuths h_1 and h_2 of the incident and refracted rays. Similarly the arcs pP and sS measure the corresponding altitudes v_1 and v_2.

If a great circle arc be imagined connecting the points N_1 and P it will also pass through the point S because the angle of refraction must lie in the same plane as the angle of incidence (Snell's law in part). Hence the arcs N_1P and N_1S measure respectively the angle of incidence, a_1, and the angle of refraction, a_2.

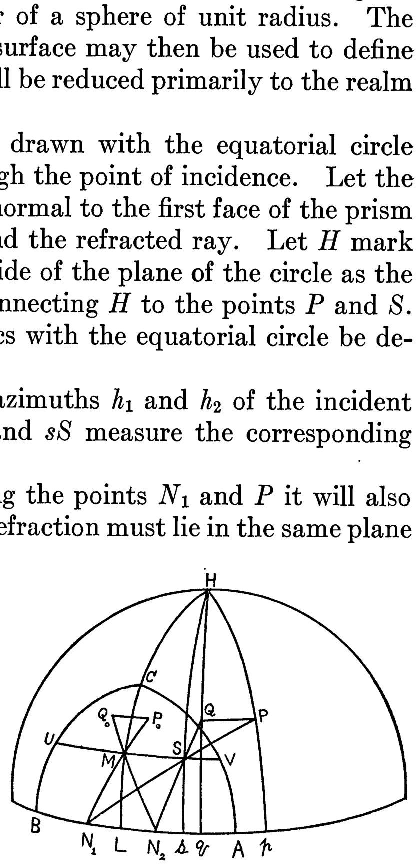


FIG. 2.

We shall now prove that

$$n_1 \sin v_1 = n_2 \sin v_2, \tag{2}$$

$$n_1 \cos v_1 \sin h_1 = n_2 \cos v_2 \sin h_2. \tag{3}$$

In the right spherical triangles N_1pP and N_1sS each member of the following equation is equal to the sine of the common angle pN_1P

$$\frac{\sin v_1}{\sin a_1} = \frac{\sin v_2}{\sin a_2}.$$

Combining this result with equation (1) we obtain formula (2), at a glance. Thus we have the theorem: *The altitudes of the incident and refracted rays obey the trigonometric part of the law of refraction.* Obviously, the angles v_1 and v_2 are not coplanar.

In the right triangles used above, the tangent of the angle pN_1P gives the following equation

$$\frac{\tan v_1}{\sin h_1} = \frac{\tan v_2}{\sin h_2}$$

[1] *Cf.* R. S. Heath, *Geometrical Optics*, p. 21, 1887.

or

$$\sin v_2 \cos v_1 \sin h_1 = \sin v_1 \cos v_2 \sin h_2.$$

Combining this equation with formula (2) we obtain immediately formula (3). If $n_1 \cos v_1$ and $n_2 \cos v_2$ be replaced by n_1' and n_2' respectively, equation (3) assumes the form

$$n_1' \sin h_1 = n_2' \sin h_2,$$

which is algebraically of the same type as formulæ (1) and (2). By analogy, it is customary to speak of $n_1 \cos v_1$ and $n_2 \cos v_2$ as "effective" absolute indices of refraction. This is appropriate since there is no justifiable danger of confusing effective indices with true ones, and n_1' and n_2' play an important rôle in many problems of advanced geometrical optics. This being understood, we may now state the theorem: *The azimuths of the incident and refracted rays obey the law of refraction with effective indices whose ratio is proportional to the ratio of the cosines of the corresponding altitudes.* In this case the planes of the angles h_1 and h_2 coincide.[1]

We are now prepared to follow the path of the oblique ray through and beyond the complete prism.

The points N_2 and Q represent respectively the normal to the second face of the prism at the point of emergence, and the emergent ray. By Snell's law N_2, S, and Q lie on the same great circle. $N_2S \equiv a_3$ is the angle of incidence (within the prism) at the second face. $N_2Q \equiv a_4$ is the angle of refraction of the emergent ray. $qQ \equiv v_4$ is the altitude of the emergent ray, while $N_2q \equiv h_4$ is the azimuth of the same. $N_2s \equiv h_3$ is the azimuth of the incident ray at the second face. $N_1N_2 \equiv \beta$ is the refracting angle of the prism. $QP \equiv D$ is the total deviation of the oblique ray. $qp \equiv D'$ is the total deviation of the projected ray. Angles in the plane N_1sp will be defined as positive when their initial sides have to be rotated in a counterclockwise direction—as viewed from H—in order to bring them into coincidence with the terminal sides. None of the angles measured from normals can exceed $\pi/2$.

Assuming the same medium (usually air) to be in contact with both faces of the prism, formula (2), when applied to the second or emergence face, becomes[2]

$$n_1 \sin v_4 = n_2 \sin v_2. \tag{4}$$

Inspection of equations (2) and (4) shows that $v_4 = v_1$, therefore: *When a prism is surrounded by a single medium, the incident and emergent rays make equal angles with their orthogonal projections on a principal plane.*[3]

It should be noted that this result is independent of the values of the indices

[1] Since only one refracting plane was involved in the preceding proofs it should be clear that the theorems are not restricted to prisms or even to plane interfaces. In Fig. 1, say, IZ may be the tangent plane to any smooth curved surface at the point of incidence. Also the projected ray is a possible path for light of wave-length different from that along the oblique ray.

[2] Obviously $v_3 \equiv v_2$.

[3] This would not hold for the element of a compound prism having air in contact with one face and a different kind of glass with the other face.

of refraction n_1 and n_2. Also it is just what we might have expected, since the elements of the surfaces of the prism which are perpendicular to the principal plane act as a plane-parallel slab of refracting material for the vertical (parallel to the refracting edge) components of the rays, and a slab of this shape (in a single medium) produces no deviation but only a parallel displacement of the rays.

Consequently, in Fig. 2, $qQ = pP = v_1$, and the triangle PHQ is isosceles. Since $\angle\, qHp = D'$, $QP = D$, and $PH = \pi/2 - v_1$ we may imagine a great circle drawn through H so as to bisect QP (and qp). Either of the resulting halves of the triangle PHQ gives

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}D' \cos v_1. \tag{5}$$

Now

$$qp = N_1p - (N_2q + N_1N_2)$$

or

$$D' = h_1 - h_4 - \beta. \tag{6}$$

But, as stated above, $|h_1| \not> \pi/2$, $|h_4| \not> \pi/2$, hence $|h_1 - h_4| \not> \pi$, and relation (6) shows that $D'/2$ must be acute ($\beta > 0$).

For $v_1 \neq 0$, $\cos v_1 < 1$; hence formula (5) shows that $D < D'$ or: *The total deviation of an oblique ray by a triangular prism surrounded by a single medium is less than that of the orthogonal projection of the ray on a principal plane.*

This accounts for the fact that the monochromatic images of a rectilinear slit (spectral lines) formed by any prism spectroscope or spectrograph are appreciably curved. This phenomenon may be observed by looking through any[1] prism toward the vertical boundary of a window, the refracting edge of the prism being vertical also, of course.

Before proceeding farther with the properties of the deviations, attention should be called to a condition which must be fulfilled in order that light may be transmitted directly by a triangular prism. The greatest value that the angle of incidence a_1 can attain is $\pi/2$ ("grazing" incidence) and, when this is the case, equation (1) shows that the greatest value of the angle of refraction a_2 is given by

$$a_2 = \sin^{-1}\frac{1}{n} \equiv c,$$

where $n \equiv n_2/n_1 > 1$. This limiting value of a_2 is called the "critical" angle, c. Similarly, in order that the ray may emerge from the second face of the prism, a_3 must not exceed c. Accordingly, if arcs of small circles be described with N_1 and N_2 as centers and with spherical radii of length c, a lune-like area will be enclosed having the property that the point S,—representative of the internal ray,—must lie within, or on the boundary of, this region in order that transmission without total reflection may occur.[2] In Fig. 2, the upper half of this area is bounded by the arcs AC and BC. If the lower intersection of the limiting small circles be designated as C', then the arcs $C'AC$ and $C'BC$ correspond respectively to grazing incidence and grazing emergence. The points C and C' each signify simultaneous grazing incidence and emergence.

[1] The refracting angle must not be so large as to preclude transmission.

[2] For $n = 1.5$, $c = 41° 48' 37''$, and the greatest value of $\beta = 2c = 83° 37' 14''$.

The most complete and satisfactory method of exploring the domain of transmission $CBC'AC$, for minima of D and D', etc., is to proceed in two steps: (a) to keep the altitude of the ray incident upon the first face of the prism constant while varying the azimuth, and (b) to follow out the behavior of the deviations corresponding to the salient features found under (a) while suitably varying the altitude of the incident ray. More precisely: (a) to keep v_2 [and hence, by (2), v_1 also] constant and explore along a small circle of altitude v_2, such as UV in Fig. 2, and (b) to pick out the interesting points on UV and investigate how the deviations at these points vary as v_2 is changed in such a manner as to sweep over the entire domain from C to BA to C'. The associated analysis will be appreciably simplified by drawing an arc through H and C, cutting the principal plane in L ($N_1L = LN_2 = \beta/2$), and by introducing a new azimuthal angle x, for S, which is reckoned from L as origin. Then equation (3) gives

$$\sin h_1 = n' \sin (x + \tfrac{1}{2}\beta), \tag{7}$$

$$\sin h_4 = n' \sin (x - \tfrac{1}{2}\beta), \tag{8}$$

where

$$n' \equiv n \cos v_2/\cos v_1 = [n^2 + (n^2 - 1) \tan^2 v_1]^{1/2} > 1 \text{ (for } n > 1).$$

Now let us see what happens along UV. Keeping n' constant while differentiating equations (6), (7), and (8) with respect to x, we find

$$\frac{\partial D'}{\partial x} = \frac{\partial h_1}{\partial x} - \frac{\partial h_4}{\partial x},$$

$$\cos h_1 \frac{\partial h_1}{\partial x} = n' \cos (x + \tfrac{1}{2}\beta),$$

$$\cos h_4 \frac{\partial h_4}{\partial x} = n' \cos (x - \tfrac{1}{2}\beta),$$

whence

$$\frac{\partial D'}{\partial x} = n'[\cos (x + \tfrac{1}{2}\beta)/\cos h_1 - \cos (x - \tfrac{1}{2}\beta)/\cos h_4].$$

Omitting the limiting cases $h_1 = \pi/2$ and $h_4 = \pi/2$ we may multiply and divide by the sum of the two fractions within the bracket, thus obtaining

$$\frac{\partial D'}{\partial x} = \frac{n'[\cos^2 (x + \tfrac{1}{2}\beta)/\cos^2 h_1 - \cos^2 (x - \tfrac{1}{2}\beta)/\cos^2 h_4]}{\cos (x + \tfrac{1}{2}\beta)/\cos h_1 + \cos (x - \tfrac{1}{2}\beta)/\cos h_4}.$$

By elementary trigonometric transformations, the last equation may be readily reduced to

$$\frac{\partial D'}{\partial x} = \frac{2n'(n'^2 - 1) \sin \beta \sin x \cos x}{\cos h_1 \cos h_4(\cos h_1 \cos h_3 + \cos h_2 \cos h_4)}. \tag{9}$$

Since $\beta \ngtr \pi$, $n' > 1$, and none of the remaining angles can exceed $\pi/2$, we see that the sign of the derivative is the same as that of $\sin x$. For any two

permissible values of x having the same magnitude but opposite signs, the derivative also has equal numerical values with opposite signs. Moreover it vanishes when $x = 0$. Consequently, the values of the deviation of the projected ray, D', are symmetrical with respect to the great circle $x = 0$, and they have a minimum both in the algebraic and in the arithmetic sense when, and only when, $x = 0$. Inspection of equations (6), (7), and (8) shows that no exception to the last statement arises when either h_1 or h_4 equals $\pi/2$. When $x = 0$, $h_2 = - h_3 = \beta/2$, and

$$h_1 = - h_4 = \sin^{-1} (n' \sin \tfrac{1}{2}\beta).$$

Since v_1 is being kept constant, equation (5) shows that D has precisely the same minimum properties as D'.

These results may be interpreted in terms of the actual prism in the following manner. Imagine a plane diaphragm constructed in the interior of the prism and fulfilling three conditions; (a) of passing through the point of incidence at the first face, (b) of being parallel to the refracting edge, and (c) of making equal angles $(\pi - \beta)/2$ with the refracting faces of the prism, that is, crossing the prism symmetrically. Then, as the altitude of the ray incident upon the first face is kept constant while its azimuth is varied, the oblique deviation and the projected deviation will both have minimum values at the instant when the internal ray lies in the plane of symmetry just defined. The winglike planes containing the minimum positions of the incident and emergent rays, which may be imagined outside of the prism, will also be symmetrically situated, but the common value of the equal angles, $\pi/2 - \sin^{-1} [n' \sin (\beta/2)]$, that these planes make with the refracting faces of the prism will not be independent of the altitude of the incident ray, hence their positions will be altered if a new constant value for v_1 be taken, since n' is a function of v_1.

With this interpretation of the term "symmetrical," we may now state the theorem: *When the altitude of the internal ray, or of the external rays, is kept constant, both the oblique deviation and the projected deviation have least values when, and only when, the rays are situated symmetrically with respect to the refracting faces of the prism.*

For the sake of completeness it may be remarked that, when the altitude of the incident ray is kept constant while the azimuth is varied, the incident ray and the internal ray each describe portions of the lateral surfaces of right circular cones having as common axis a line (in the incidence face) passing through the point of incidence and parallel to the refracting edge, the semi-apical angles of the cones being respectively the complements of the altitudes v_1 and v_2.

Let D_0 and D_0' denote respectively the minimum values of D and D' which occur when x equals zero and v_1 is constant. The next step will be to keep x zero and to investigate the behavior of the (partial) minima D_0 and D_0' as the point S moves along the great circle arc from C to L to C'. The independent variable is now v_2. With direct reference to the prism, our immediate problem is to cause the altitudes of the three parts of the complete ray to vary in such a

manner as to maintain the above mentioned symmetry and to investigate the way in which the minima of deviation depend upon the values of the altitudes. Under these circumstances the internal ray will be constrained to move up or down (fanwise) in the transverse diaphragm introduced earlier. It will be convenient to take up D_0 before D_0'.

Consider the halves of the isosceles triangles N_1MN_2 and P_0MQ_0, and let each of the four equal acute angles having the common vertex M be denoted by e. Since $Q_0P_0/2 = D_0/2$, $N_1L = \beta/2$, $MP_0 = a_1 - a_2$, and $N_1M = a_2$, we have

$$\sin e = \sin \tfrac{1}{2}D_0/\sin (a_1 - a_2) = \sin \tfrac{1}{2}\beta/\sin a_2$$

or

$$\sin \tfrac{1}{2}D_0 = (\sin a_1 \cos a_2 - \cos a_1 \sin a_2) \sin \tfrac{1}{2}\beta/\sin a_2.$$

As equation (1) is

$$\sin a_1 = n \sin a_2$$

it is clear that

$$\cos a_1 = + [n^2 \cos^2 a_2 - (n^2 - 1)]^{1/2}$$

and

$$\sin \tfrac{1}{2}D_0 = \{n \cos a_2 - [n^2 \cos^2 a_2 - (n^2 - 1)]^{1/2}\} \sin \tfrac{1}{2}\beta. \tag{10}$$

As $LM = v_2$, the right triangle N_1LM gives

$$\cos a_2 = \cos \tfrac{1}{2}\beta \cos v_2, \tag{11}$$

which, combined with the process of rationalizing the braced expression in equation (10), leads to

$$\sin \tfrac{1}{2}D_0 = \frac{(n^2 - 1) \sin \tfrac{1}{2}\beta}{n \cos \tfrac{1}{2}\beta \cos v_2 + [(n \cos \tfrac{1}{2}\beta \cos v_2)^2 - (n^2 - 1)]^{1/2}}. \tag{12}$$

Since $\cos v_2$ increases as v_2 decreases numerically, we see from equation (12) that D_0 attains its least value when $v_2 = 0$, that is, when the entire ray lies in the principal plane. This does not show explicitly that D_0 has a minimum in the algebraic sense when $v_2 = 0$, for, a cusp of the first species might be present at the lowest point of a surface constructed on a rectangular coördinate frame and exhibiting D as a function of x and v_2.

In order to show that D_0 has a true algebraic minimum when $v_2 = 0$ we must form the derivative of D_0 with respect to v_2. It will be found, without difficulty, by differentiating equations (10) and (11), and then eliminating da_2/dv_2, that

$$\frac{dD_0}{dv_2} = \frac{n \sin \beta \sin v_2}{\cos \tfrac{1}{2}D_0} \left\{ \frac{n \cos \tfrac{1}{2}\beta \cos v_2}{[(n \cos \tfrac{1}{2}\beta \cos v_2)^2 - (n^2 - 1)]^{1/2}} - 1 \right\}.$$

The expression enclosed by braces in the last equation, $\sin \beta$, and $\cos \frac{1}{2}D_0$ are always positive, hence the derivative has the same sign as $\sin v_2$. The derivative also approaches zero simultaneously with v_2. It should be evident, therefore, that D_0 has an algebraic minimum when, and only when, $v_2 = 0$.

When $x = 0$, equation (5) may be written

$$\sin \tfrac{1}{2}D_0' = \sin \tfrac{1}{2}D_0 \sec v_1,$$

which shows, at a glance, that D_0' has an algebraic minimum simultaneously with D_0, for both $\sin (D_0/2)$ and $\sec v_1$ decrease as v_2 becomes smaller.

The whole region bounded by $CBC'AC$ has now been explored and we have proved the theorem: *The deviation produced by direct transmission through a triangular prism has the smallest value, which is also an absolute algebraic minimum, when the ray lies in a principal plane and makes equal angles with the refracting faces.* This is subject to the qualifications that the prism is surrounded by one medium and that its relative index of refraction is greater than unity.

As far as the writer is aware, the following generalization of formula (5) is entirely new. Suppose we have k prisms with their refracting edges parallel, and surrounded by one and the same medium. The prisms must be so situated, of course, as to permit the passage of light through the system.

The hypothesis of a single surrounding medium causes v_1 to be constant throughout the entire prism train, quite independently of the various indices of refraction of the materials composing the different prisms. [See equation (4) and the associated context.] Also, the refracting angles need not bear any particular relation to one another. Since the emergent ray from any prism (save the last) becomes the incident ray of the next succeeding prism, the representative points P and Q of Fig. 2 will be distributed along a small circle arc of altitude v_1. Accordingly, the separate deviations produced by the individual prisms will form collectively a fluting of great circle arcs having salient points on the small circle just mentioned. With these flutings, however, we are not directly concerned, for the single great circle arc connecting the terminus of the last or kth emergent ray to that of the first incident ray will represent the total deviation D produced by the entire train of prisms. Incidentally, $D < \sum_1^k D_j$.

As in the case of a single prism, so also here, the triangle PHQ,—where P and Q now mark the extreme ends of the fluting,—will be isosceles, so that

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}D' \cos v_1. \tag{5'}$$

Since the segments of the oblique ray are all projected orthogonally on the common principal plane, the following sub-relation between the total projected deviation and the separate projected deviations obviously holds:

$$D' = \sum_1^k D_j'.$$

Bibliographical and Historical Notes: The most reliable and elegant presentation of geometrical optics in English is the second edition of J. P. C. Southall's *Principles and Methods of Geometrical Optics*, New York (1913). The most complete general discussion of prisms was written by H. Konen and published in H. Kayser's *Handbuch der Spectroscopie*, vol. I, pp. 253–394, (1900). On page 258 may be found a very complete list of references to proofs of the theorem of minimum deviation *in a principal plane.*

This theorem dates from the work of Newton. The reference given by Konen

is "Lect. opt. London 1729. P. I., Sect. II, Art. 31." For oblique deviation, the fact that $v_4 = v_1$ was published by A. Bravais in 1845, *Journal de l'Ecole Polytechnique*, vol. 18, 30^e Cahier, p. 79. The priority for formula (5) seems to belong to Mascart. This equation is given correctly in his *Traité d'Optique*, vol. 1, p. 84 (1889). By drawing an incorrect diagram, R. S. Heath (*l.c.* p. 32) derived the formula

$$\cos \tfrac{1}{2}D = \cos \tfrac{1}{2}D' \cos v_1.$$

This was copied by almost all later writers regardless of the fact that J. Larmor called attention to the error in the *Proceedings of the Cambridge Philosophical Society*, vol. 9, p. 108 (1896). An unbiased discussion of this matter (including the part which the present writer has taken) is given in Southall's treatise, p. 127. Finally, Konen in attempting to generalize formula (5) extended the error even farther by stating that the cosine equation still held (*l.c.* p. 267).

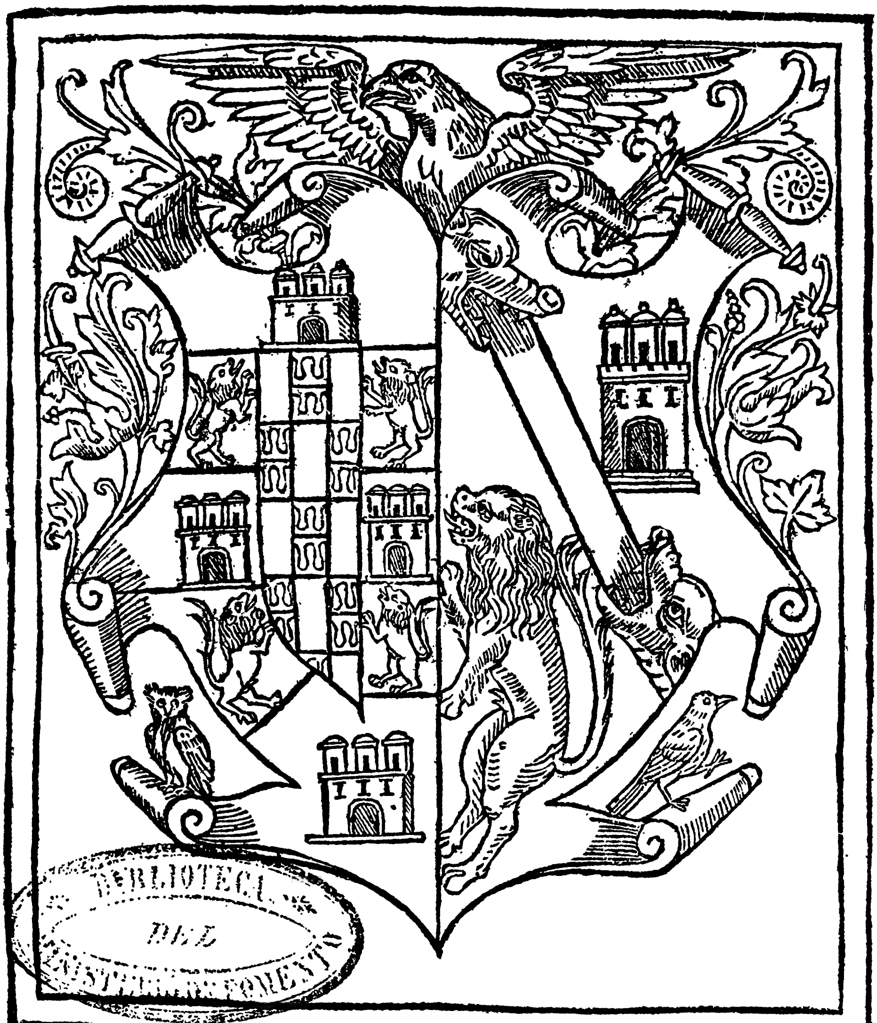
THE FIRST WORK ON MATHEMATICS PRINTED IN THE NEW WORLD.[1]

By DAVID EUGENE SMITH, Columbia University.

I. General Description. If the student of the history of education were asked to name the earliest work on mathematics published by an American press, he might, after a little investigation, mention the anonymous arithmetic that was printed in Boston in the year 1729. It is now known that this was the work of Isaac Greenwood who held for some years the chair of mathematics in what was then Harvard College. If he should search the records still further back, he might come upon the American reprint of Hodder's well-known English arithmetic, the first textbook on the subject, so far as known, to appear in our language on this side of the Atlantic. If he should look to the early Puritans in New England for books of a mathematical nature, or to the Dutch settlers in New Amsterdam, he would look in vain; for, so far as known, all the colonists in what is now the United States were content to depend upon European textbooks to supply the needs of the relatively few schools that they maintained in the seventeenth century.

The earliest mathematical work to appear in the New World, however, antedated Hodder and Greenwood by more than a century and a half. It was published long before the Puritans had any idea of migrating to another continent, and fifty years before Henry Hudson discovered the river that bears his name. Of this work, known as the *Sumario Compendioso*, there remain perhaps only four copies, and it is desirable, not alone because of its rarity but because of its im-

[1] Address delivered before The Mathematical Association of America, and the section on the History of Science of the American Association for the Advancement of Science, at the University of Chicago, December 28, 1920. The extracts are from a fac-simile reprint of the original work soon to published by Ginn & Company, Boston, with translation and notes by Professor Smith.



Sumario cõpẽdioso delas quẽtas de plata y oro q̃ en los reynos del Piru son necessarias a los mercaderes: y todo genero de tratantes. Cõ algunas reglas tocantes al Arithmetica.

Fecho por Juan Diez freyle.

Title page of the first work on mathematics printed in the New World, Mexico City, 1556.

portance in the history of education on the American continent, that some record of its contents should be made known to scholars.

In order to understand the *Sumario Compendioso* it is necessary to consider briefly the political and social situation in Mexico in the middle of the sixteenth century. Cortés entered the ancient city of Tenochtitlan, later known as Mexico, in the year 1519, but its capture and destruction occurred two years later, in 1521. Thus, in the very year that Luther was attacking certain ancient customs and privileges in the Old World, the representatives of other ancient customs and privileges were attacking and destroying a worthy civilization in the newly discovered continent.

The first viceroy of New Spain, which included the present Mexico, was a man of remarkable genius and of prophetic vision,—Don Antonio de Mendoza. He assumed his office in 1535, and for fifteen years administered the affairs of the colony with such success as to win for himself the name of "the good viceroy." He founded schools, established a mint, ameliorated the condition of the natives, and encouraged the development of the arts. In his efforts at improving the condition of the people he was ably assisted by Juan de Zumárraga, the first Bishop of Mexico. Among the various activities of these leaders was the arrangement made with the printing establishment of Juan Cromberger of Seville whereby a branch should be set up in the capital of New Spain.

As a result of this arrangement there was sent over as Cromberger's representative one Juan Pablos, a Lombard printer, and so the "casa de Juan Cromberger" was established, prepared to spread the doctrines of the Church to the salvation of the souls of the unbelievers. Cromberger himself never went to Mexico, but his name appears either on the *portadas* or in the colophons of all the early books. From and after 1545, however, the name is no longer seen, Cromberger having already died in 1540.

The author of the *Sumario* was one Juan Diez, a native of the Spanish province of Galicia, a companion of Cortés in the conquest of New Spain, and the editor of the works of Juan de Avila, "the apostle of Andalusia," and of the *Itinerario* of the Spanish fleet to Yucatan in 1518. He is sometimes confused with another Juan Diaz (the name being spelled both ways), a contemporary theologian and author. In a letter written to Charles V in 1533 he is mentioned as a "clérigo anciano y honrado," so that he must have been advanced in years when the *Sumario* appeared. That this was the case is also apparent from a record of the expedition of 1518 in which it is stated that "triximus vn clerigo que dezia joan diaz," doubtless a young and adventurous apostle, full of zeal and desire to make known the gospel in the New World.

Juan Diez undertook the work primarily for the purpose of assisting those who were engaged in the buying of the gold and silver which was already being taken from the mines of Peru and Mexico for the further enriching of the moneyed class and the rulers of Spain. He felt that he could best serve this purpose by preparing such a set of tables as should relieve these merchants as far as possible from any necessity for computation. Apparently, however, he was prompted by

the further demand for a brief treatment of arithmetic which should be suited to the needs of apprentices in the counting houses of the New World, and so he devotes eighteen pages to the subject of computation and presents it in a manner not unworthy of the European writers of the period.

The most interesting feature of the work, however, is neither the tables nor the arithmetic; it consists of six pages devoted to algebra, chiefly relating to the quadratic equation.

The book consists of one hundred and three folios, generally numbered. After the dedication (folios i, *v*, and ij, *r*) there is an elaborate set of tables, including those relating to the purchase price of various grades of silver (folio iij, *v*), to per cents (folio xlix, *r*), to the purchase price of gold (folio lvij, *v*), to assays (folio lxxxj, *r*), and to monetary affairs of various kinds. The mathematical text (folio xcj, *v*) consists of twenty-four pages besides the colophon (folio ciij, *v*). As already stated, eighteen of these pages relate chiefly to arithmetic, and six to algebra.

The book was printed in the City of Mexico in the year 1556, being the first work on mathematics to be printed outside the boundaries of Europe, except for the ancient block books of China.

In order to give some idea of the general nature of the work, a few of the problems will be set forth, chiefly those which illustrate the application of algebra as we understand the term today.

II. Typical Problems not listed under Algebra. 1. I bought 10 varas of velvet at 20 pesos less than cost, for 34 pesos plus a vara of velvet. How much did it cost a vara?

Rule: Add 20 pesos to 34 pesos, making 54 pesos, which will be your dividend. Subtract one from 10 varas, leaving 9. Divide 54 by 9, giving 6, the price per vara.

Proof: 10 varas at 6 pesos is 60 pesos. This minus 20 pesos is 40. You paid 34 pesos plus a vara costing 6 pesos, and this gives the result, 40 pesos.

2. I bought 9 varas of velvet for as much more than 40 pesos as 13 varas at the same price is less than 70 pesos. How much did a vara cost?

Rule: Add the pesos, 40 and 70, making 110. Add the varas, 9 and 13, making 22. Dividing 110 by 22 the quotient is 5, the price of each vara.

Proof: 9 varas at 5 pesos is 45 pesos, which is 5 more than 40 pesos; and 13 varas at 5 pesos is 65, which is 5 pesos less than 70, as you see.

3. Required a number which if 8 is added to it will be a square, and if 8 is subtracted from it will also be a square. Take half of eight, which is 4; square it, making 16; add 1, making 17, and this is the number to which if you add 8 you have 25, the root of which is 5; and if 8 is taken from it there is left 9, the root of which is 3; for 3 times 3 is 9, as you see.

4. Find 2 numbers the sum of the squares of which will make a square number which has an integral root. The first numbers are 3 and 4, for their squares are 9 and 16, and these added together make 25, the root of which is 5. Observe that

you have 5 numbers; the first are 2 and 3; the next are 3 and 4, the proposed numbers; and there is also 5, which is their root. Place these numbers as you see in the figure below. Then use cross multiplication, saying "3 times 3 is 9, and 2 times 4 is 8." Place these numbers at the right-hand side, one under the other. Then multiply again at the top, 2 times 3 is 6; and underneath, 3 times 4 is 12. Now subtract the less from the greater, that is, 6 from 12, and there remains 6. Divide this by 5, the root of the assumed numbers, and the quotient is $1\frac{1}{5}$, one of the numbers required. Now add 8 and 9, the products of the first multiplication, and the sum is 17. Divide this by 5 and the quotient is $3\frac{2}{5}$, and this is the second required number.

Proof: The square of $1\frac{1}{5}$ is $1\frac{11}{25}$; the square of $3\frac{2}{5}$ is $11\frac{14}{25}$; and these added together, as you see, make 13.

$$
\begin{array}{c|cccc|c}
 & & 6 & & & \\
5 & 2 & 6 & 3 & 9 & 17 \\
\hline
 & 3 & & 4 & 8 & \\
 & & 12 & & &
\end{array}
\qquad
\begin{array}{l}
02 \\
17 \mid 3\frac{2}{5} \\
\;\,5
\end{array}
\qquad
\begin{array}{l}
1 \\
6 \mid 1\frac{1}{5} \\
5
\end{array}
$$

III. Typical Problems listed under Algebra. Although the above problems are solved by arithmetical rules, they are essentially algebraic. Under the title *Arte Mayor* the author gives a number of examples generally involving quadratic equations, of which the following are types:

1. Find a square from which if $15\frac{3}{4}$ is subtracted the result is its own root.

Rule: Let the number be *cosa* (x). The square of half a *cosa* is equal to $\frac{1}{4}$ of a *zenso* (x^2). Adding 15 and $\frac{3}{4}$ to $\frac{1}{4}$ makes 16, of which the root is 4, and this plus $\frac{1}{2}$ is the root of the required number.

Proof: Square the square root of 16, plus half a *cosa*, which is four and a half, giving 20 and $\frac{1}{4}$, which is the square number required. From $20\frac{1}{4}$ subtract 15 and $\frac{3}{4}$ and you have 4 and $\frac{1}{2}$, which is the root of the number itself.

2. A man takes passage in a ship and asks the master what he has to pay. The master says that it will not be any more than for the others. The passenger on again asking how much it would be, the master replies: "It will be the number of pesos which, multiplied by itself and added to the number, will give 1260." Required to know how much the master asked.

Rule: Let the cost be a *cosa* of pesos. Then half of a *cosa* squared makes $\frac{1}{4}$ of a *zenso*, and this added to 1260 makes 1260 and a quarter, the root of which less $\frac{1}{2}$ of a *cosa* is the number required. Reduce 1260 and $\frac{1}{4}$ to fourths; this is equal to $\frac{5041}{4}$, the root of which is 71 halves; subtract from it half of a *cosa* and there remains 70 halves, which is equal to 35 pesos, and this is what was asked for the passage.

Proof: Multiply 35 by itself and you have 1225; adding to it 35, you have 1260, the required number.

3. A man is selling goats. The number is unknown except that it is stated that a merchant asked how many there were and the seller replied: "There are

so many that, the number being squared and the product quadrupled, the result will be 90,000." Required to know how many goats he had.[1]

A DETERMINATION OF THE CURVE MINIMIZING THE AREA ENCLOSED BY IT AND ITS EVOLUTE.

By OTTO DUNKEL, Washington University, St. Louis, Mo.

One of the problems treated in the calculus of variations as an example in which the second derivative appears in the integrand is that of determining the curve in a plane passing through two fixed points which with its evolute and its two normals of given directions at these two points enclose a minimum area.[2] The special form of the integrand permits this problem to be solved without resorting to the general theory of the calculus of variations, and the conditions appear as necessary and sufficient simultaneously. The solution given here appears to be new so far as the writer can learn from the references consulted. It is adapted only to this special form of the integrand, but it is quite possible that there may be other problems of this form to which it applies. For example, the problem of determining the curve which with its caustic produced by parallel rays of light, and the reflected rays at two of its given points, enclose a minimum area may also be solved in the same way.[3] It will be observed that the method admits of a slight generalization.

The Necessary and Sufficient Conditions for a Minimum. Let one of the fixed points be the origin and the other (x_2, y_2) and let the inclination of the curve to the y-axis at these two points be θ_1 and θ_2, respectively, $\theta_2 > \theta_1$. Let s be the length of the arc of the curve measured from the origin and, R, the radius of curvature. Then

$$R = \frac{ds}{d\theta} \tag{1}$$

[1] In the above brief extracts the archaic forms of expression have been retained so far as the circumstances of translation permit. No effort has been made to explain, in this presentation, the method of attacking the problems, or to consider the sources from which the author drew his materials.

[2] This problem is discussed in many well-known works, for example: I. Todhunter, *Researches in the Calculus of Variations*, London, 1871, chapter 13; H. Hancock, *Lectures on the Calculus of Variations*, Cincinnati, 1904, pp. 75–76; A. Kneser, *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, pp. 203, 219; and O. Bolza, *Vorlesungen über Variationsrechnung*, Leipzig, 1909, p. 152. It originated with Euler (*Methodus Inveniendi Lineas Curvas . . .* Lausannæ & Genevæ, 1744, pp. 64–66): "Invenire curvam *Am*, quæ cum sua evoluta *AR* & radio osculi *mR* in quavis loco applicato, minimum spatium *ARm* includat"; the solution which Euler gives employs the calculus of variations. In *Annals of Mathematics*, new series, vol. 14, pp. 14–26, 1912, E. J. Miles discusses the "Determination of the constants in Euler's problem concerning the minimum area between a curve and its evolute," and in this MONTHLY, *1917*, 420–422, P. R. Rider discusses "An intrinsic equation solution of a problem of Euler."

[3] A paper on this problem was read by Professor Dunkel before the American Mathematical Society, November 27, 1920.—EDITOR..

determines R as a positive quantity, as we shall suppose that the curve is always concave downward. We shall also assume that R is a continuous function of θ. The integral for the area, S, between the curve, its evolute, and the two normals at the end points

$$S = \frac{1}{2}\int_{\theta_1}^{\theta_2} R^2 d\theta \tag{2}$$

is to be made a minimum subject to the conditions

$$x_2 = \int_{\theta_1}^{\theta_2} R \sin\theta\, d\theta, \qquad y_2 = \int_{\theta_1}^{\theta_2} R \cos\theta\, d\theta. \tag{3}$$

The problem is handled more easily by replacing these by equivalent conditions involving similar integrals equated to zero. This is easily accomplished by setting

$$R = A\sin\theta + B\cos\theta + \eta, \tag{4}$$

where A and B are constants to be determined so that

$$\begin{aligned} x_2 &= A\int_{\theta_1}^{\theta_2} \sin^2\theta\, d\theta + B\int_{\theta_1}^{\theta_2} \sin\theta\cos\theta\, d\theta, \\ y_2 &= A\int_{\theta_1}^{\theta_2} \sin\theta\cos\theta\, d\theta + B\int_{\theta_1}^{\theta_2} \cos^2\theta\, d\theta, \\ 0 &= \int_{\theta_1}^{\theta_2} \eta\sin\theta\, d\theta, \qquad 0 = \int_{\theta_1}^{\theta_2} \eta\cos\theta\, d\theta. \end{aligned} \tag{4'}$$

The last two of these four equations follow from the first two and (3) and (4). The first pair of equations can always be solved for A and B, since their determinant is never zero. This may be seen as follows: the integral

$$\int_{\theta_1}^{\theta_2} (\lambda\sin\theta + \cos\theta)^2\, d\theta$$

is a quadratic form in λ which is always positive; also the coefficient of λ^2 is always positive. Hence its discriminant is positive; but this discriminant is the determinant of the pair of equations in (4′).[1] The equation (2) is now replaced by

$$S = \frac{1}{2}\int_{\theta_1}^{\theta_2} (A\sin\theta + B\cos\theta)^2\, d\theta + \frac{1}{2}\int_{\theta_1}^{\theta_2} \eta^2\, d\theta, \tag{2'}$$

where two integrals are omitted since they are each zero by virtue of the new conditions on η in the last line of (4′). Since R is continuous, η is continuous, and it is immediately obvious that $\eta = 0$ is a necessary and a sufficient condition

[1] It may be shown by computation that the value of the determinant is $[(\theta_2 - \theta_1)^2 - \sin^2(\theta_2 - \theta_1)]/4 > 0$, but the above method is given in preference on account of its generality. It should be noticed that the method does not depend upon the special functions $\sin\theta$ and $\cos\theta$.

for a minimum. This value of η satisfies the two equations in (4′). Hence the curve for which

$$R = A \sin \theta + B \cos \theta, \tag{5}$$

where A and B are given by (4′), gives a minimum area.

The Minimizing Curve and its Evolute. The equations for x and y differ from those in the first two lines of (4′) by having θ, x, y in place of θ_2, x_2, y_2. Since not both x_2 and y_2 are zero, A and B cannot both be zero and hence we may set $A = 4a \cos \alpha$, $B = 4a \sin \alpha$, $\psi = \alpha + \theta$, where $4a = \sqrt{A^2 + B^2}$. Inserting these in (5) and rotating the axes through the angle α, we find $R = 4a \sin \psi$, where ψ is the angle that the tangent to the curve makes with the new y'-axis. The integration of the new differential equations,

$$dx' = 4a \sin^2 \psi d\psi, \qquad dy' = 4a \sin \psi \cos \psi d\psi,$$

gives

$$\begin{aligned} x' + a[2(\theta_1 + \alpha) - \sin 2(\theta_1 + \alpha)] &= a(2\psi - \sin 2\psi), \\ y' + a[1 - \cos 2(\theta_1 + \alpha)] &= a(1 - \cos 2\psi). \end{aligned} \tag{6}$$

Hence the curve is a cycloid generated by a circle of radius a rolling on a straight line parallel to the x'-axis and at a distance $a[1 - \cos 2(\theta_1 + \alpha)]$ below it with a cusp at $x' = - a[2(\theta_1 + \alpha) - \sin 2(\theta_1 + \alpha)]$ and at intervals of $2\pi a$. The angle through which the circle has rolled is given by 2ψ.

If σ is the length of arc of the evolute, the radius of curvature of the evolute is

$$\rho = \frac{d\sigma}{d\psi} = 4a \cos \psi = 4a \sin \left(\frac{\pi}{2} + \psi \right).$$

The values of ψ which make R vanish determine the cusps of the cycloid and hence these cusps must be points of the evolute. Moreover at such points of the evolute ρ has its maximum absolute value $4a$. This shows that the evolute is an equal cycloid having the same direction and with its vertices at the cusps of the former.[1]

Fixed End Points and a Variable Slope at One End. We now consider the case in which the end points are fixed, the slope at the initial point is given but that at the final point is not assigned. If there is a curve C which gives a minimum area, a necessary condition may be obtained as follows. Suppose that the inclination of C at the terminal point is θ_2 and let us consider only those curves $[C]$ which have this terminal inclination and satisfy the remaining conditions. Since the area given by C does not exceed that resulting from any one of the curves $[C]$, the previous work shows that the curve C must be a cycloid. It remains now to determine the terminal inclination θ_2 of a cycloid which satisfies all the other end conditions and which gives a smaller area than any cycloid

[1] This result was given by Huygens in *Christiani Hvgenii . . . Horologivm Oscillatorivm . . .*, Paris, 1673, page 67, prop. VI: "Semicycloidis evolutione, à vertice cœpta, alia semicyclois describitur evolutæ æqualis & similis, cujus basis est in ea recta quæ cycloidem evolutam in vertice contingit."—EDITOR.

having a different terminal inclination. This is merely a problem of minimizing a definite function S of θ_2. Returning to the equations (2), (4′), (5) the expression for the area becomes

$$(7)\qquad \begin{aligned} 2S &= A^2\int_{\theta_1}^{\theta_2}\sin^2\theta\,d\theta + 2AB\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta\,d\theta + B^2\int_{\theta_1}^{\theta_2}\cos^2\theta\,d\theta,\\ &= Ax_2 + By_2, \end{aligned}$$

where A and B are functions of θ_2. Hence

$$(8)\qquad 2\frac{dS}{d\theta_2} = x_2\frac{dA}{d\theta_2} + y_2\frac{dB}{d\theta_2}.$$

From (4′) we have

$$(9)\qquad \begin{aligned} 0 &= \frac{dA}{d\theta_2}\int_{\theta_1}^{\theta_2}\sin^2\theta\,d\theta + \frac{dB}{d\theta_2}\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta\,d\theta + A\sin^2\theta_2 + B\sin\theta_2\cos\theta_2,\\ 0 &= \frac{dA}{d\theta_2}\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta\,d\theta + \frac{dB}{d\theta_2}\int_{\theta_1}^{\theta_2}\cos^2\theta\,d\theta + A\sin\theta_2\cos\theta_2 + B\cos^2\theta_2. \end{aligned}$$

By multiplying the first of these equations by A, the second by B, and adding the results, we have, after using (4′) and (5)

$$x_2\frac{dA}{d\theta_2} + y_2\frac{dB}{d\theta_2} + R_2^2 = 0,$$

where R_2 is the radius of curvature of the cycloid at (x_2, y_2). Hence

$$(10)\qquad 2\frac{dS}{d\theta_2} = -R_2^2,$$

and therefore the area decreases as θ_2 increases. There is therefore no finite value of θ_2 rendering S a minimum. We may say, however, that for a portion of a single arch the cycloid having a cusp at the terminal point gives a smaller area than any cycloid having a smaller terminal inclination. From this it follows that the portion of a single arch of a cycloid having a terminal cusp gives a smaller area than any other curve having the same or a smaller inclination at the terminal point.

If we suppose now that the inclination at the terminal point is given, but that at the initial point is variable, it will be found that

$$(10')\qquad 2\frac{dS}{d\theta_1} = R_1^2,$$

where R_1 is the radius of curvature of the cycloid at the initial point. Hence the area decreases as θ_1 decreases.

Consider now the case in which the inclination at each end point is variable. The area S decreases as θ_1 decreases and θ_2 increases. Suppose for simplicity

that $y_2 = 0$; then a cycloid with the respective inclinations 0 and π at the end points, *i.e.*, a single arch with cusps at the end points, gives a smaller area than any portion of an arch with intermediate inclinations at these same end points. It thus follows that the area given by a single complete arch of a cycloid is smaller than that given by any other curve whose initial inclination is not smaller than 0 and whose terminal inclination is not greater than π.

The method employed above applies to integrals of the form (2) with any number of conditions such as (3) in which other functions of θ as well as the trigonometric may appear in the integrand. Certain light restrictions are to be placed upon such functions.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. REMARKS BY J. P. BALLANTINE, Pennsylvania State College.

In remarks by the editor on this question [*1920*, 302], attention was called to the related equation

$$\int_\alpha^\beta f(x)dx = \frac{\beta - \alpha}{6}\left[f(\alpha) + 4f\left(\frac{\alpha + \beta}{2}\right) + f(\beta)\right], \tag{1}$$

where both α and β are allowed to vary; and it was pointed out that if $f(x)$ is a solution of this equation possessing a fifth derivative, then $f(x)$ must be a polynomial of degree $\leqq 3$. It will be proved here that the same conclusion follows from the assumption that $f(x)$ is merely continuous.

I. *Let x_0, x_2, x_4, x_6 be four equally spaced points, and x_1, x_3, x_5 the points of bisection of the three equal intervals formed. Then any integrable function which satisfies* (1) *and vanishes at x_0, x_2, x_4, x_6 vanishes also at x_1, x_3, x_5.*

Let $x_2 - x_0 = 2a$. We find from (1)

$$\int_{x_0}^{x_4} f(x)dx = 0, \qquad \int_{x_2}^{x_6} f(x)dx = 0.$$

Hence, with the notation

$$\int_{x_2}^{x_4} f(x)dx = G, \tag{2}$$

we find

$$\int_{x_0}^{x_2} f(x)dx = -G, \tag{3}$$

$$\int_{x_4}^{x_6} f(x)dx = -G; \tag{4}$$

and therefore, by addition,

$$\int_{x_0}^{x_6} f(x)dx = -G + G - G = -G. \tag{5}$$

By (1) and (2),

$$G = \tfrac{4}{3}af(x_3);$$

by (1) and (5),

$$-G = 4af(x_3);$$

hence

$$G = 0, \qquad f(x_3) = 0.$$

Finally, by (1) and (3),

$$\tfrac{4}{3}af(x_1) = -G = 0,$$

and by (1) and (4),

$$\tfrac{4}{3}af(x_5) = -G = 0;$$

so that

$$f(x_1) = f(x_3) = f(x_5) = 0,$$

as stated.

II. *If an integrable function $f(x)$ satisfying* (1) *vanishes at four equally spaced points, it vanishes also at all points obtainable by repeated bisection of the intervals between the points.*

This is proved by repeated application of I.

III. *If a continuous function $f(x)$ satisfying* (1) *vanishes at four equally spaced points, it vanishes at every point of the interval bounded by the extreme points.*

Since the set of vanishing points obtained in II is everywhere dense, the identical vanishing of $f(x)$ follows at once from the added hypothesis of continuity.

IV. *Any continuous function $f(x)$ satisfying* (1) *is a polynomial of degree* $\leqq 3$.

Choose at random four equally spaced points x_0, x_2, x_4, x_6. It is possible to find $\varphi(x) = A + Bx + Cx^2 + Dx^3$, so that $\varphi(x_0) = f(x_0)$, $\varphi(x_2) = f(x_2)$, $\varphi(x_4) = f(x_4)$, $\varphi(x_6) = f(x_6)$; for on writing out these conditions, we have for the determination of A, B, C, D four linear algebraic equations whose determinant does not vanish. Then the function $f_1(x) = f(x) - \varphi(x)$ satisfies (1) and vanishes at x_0, x_2, x_4, x_6. Therefore, by III, $f_1(x)$ vanishes, $x_0 \leqq x \leqq x_6$. That is, $f(x)$ is a polynomial of degree $\leqq 3$ in the interval from x_0 to x_6. By choosing x_0, x_2, x_4, x_6 properly, any real value of x may be included. It is also easily seen that widening the interval cannot alter the values of A, B, C, D; for, if such alteration were possible, we should have two different polynomials identically equal in the smaller interval, which is impossible.

II. Remarks by Louis Weisner, New York City.

The solution of this problem, published in the Monthly, *1920*, 301–2, is based on the assumption that $f(x)$ is analytic; this assumption leading to the conclusion that "$f(x)$ can be at worst a cubic polynomial." The following solution, based upon a different assumption, leads to a more general result. Assume that

$$f(x) = c_0 + c_1x^{r_1} + c_2x^{r_2} + \cdots + c_nx^{r_n} + \cdots,$$

in which the r's are to be found, if they exist, the only restriction placed upon them being that they may not be negative. Then $f(0) = c_0$.

Substituting for $f(x)$ in the proposed equation, transposing and collecting terms, we find that c_0 disappears and that

$$\sum_{n=1}^{n=\infty}\left[c_n\left(\frac{1}{6}\cdot\frac{4}{2^{r_n}}+\frac{1}{6}-\frac{1}{r_n+1}\right)h^{r_n+1}\right] = 0.$$

This equation will be satisfied if we choose the r's so that they are the roots of the auxiliary equation

$$\frac{1}{6}\cdot\frac{4}{2^r}+\frac{1}{6}-\frac{1}{r+1} = 0,$$

or of the equation

$$(r+1)(2^{-r+2}+1) = 6.$$

It is seen by inspection that there are no negative or fractional roots of the auxiliary equation; the only positive integral roots are $r = 1$, $r = 2$, $r = 3$. However, the auxiliary equation has in fact an infinite number of roots, real or complex.

Consequently, we have

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \sum_{n=4}^{n=\infty} c_nx^{r_n},$$

in which the r's are the incommensurable or complex roots of the auxiliary equation. The c's are independent arbitrary constants, since no assumption was made regarding them.

III. Remarks by the Editor.

Mr. Ballantine's result, dealing not with the given equation but with the similar one in which both limits of integration are allowed to vary, is of considerable interest, as it not only shows that the hypothesis of continuity alone suffices to restrict solutions to cubic polynomials, but also effects the proof in a very simple and elementary way. It will be remembered that in virtue of the previous remarks of Professor Bennett and the editor [*1920*, 462], a similar result is not obtainable for the equation as stated in the question, even with the hypothesis of a continuous first derivative, and that it remains undecided what number of derivatives between one and six will suffice.

Mr. Weisner, seeking solutions other than cubic polynomials, is led to a result practically equivalent to that stated by the editor [*1920*, 463]. This result was passing through the press at the time of receipt of Mr. Weisner's manuscript. It may be observed that the editor's statement shows that in Mr. Weisner's notation, when $n > 3$, r_n is necessarily of the form $2 + \beta i$, where β is a solution of a certain transcendental equation.

DISCUSSIONS.

The two discussions in this number deal with questions of analytic geomet y. Professor Borger shows how curves may frequently be plotted advantageously by geometric construction of the points rather than by computation. He illustrates the method by a number of examples, both in rectangular and in polar coördinates, and also in parametric form. It is fairly clear that this method may at times be more useful and at other times much less useful than the usual plan. Surely the student should begin learning as early as possible that the mere laborious plotting of points, either arithmetically or geometrically, is generally to be used only as an incidental aid in determining the form of the curve, the important information being obtained from a functional study of the equation, first by the machinery of algebra, later also by that of the calculus.

Professor Bradshaw gives an interesting exposition of the inaccuracy of the figures usually given in the text-books on solid analytic geometry. The reader who has never before given attention to this matter will scarcely credit, until after direct examination, the uniformity of this error in our usual texts.

I. On Some Geometric Methods for Curve Tracing.[1]

By R. L. Borger, Ohio University.

In beginning courses in analytic geometry, curve tracing is confined almost exclusively to the process of making a table and plotting the points of the curve from the computed coördinates. This destroys in a great measure the geometric aspect of the problem, and at the same time develops in the student a quality of dependence upon the table even in cases where a geometric treatment would be simpler. For many curves in polar form, plotting from a table becomes excessively onerous. If the radius vector is a trigonometric function of the vectorial angle, the interposition of two tables becomes necessary. The student finds too that he has no means of detecting the character of a curve as he has in rectangular coördinates. Such simple curves as

[1] Read before the Mathematical Association of America, Ohio Section, Columbus, April 2, 1920.

$$\rho = a\cdot\sec\theta, \qquad \rho = 2a\cdot\cos\theta, \qquad \rho = a\cdot\sec^2\frac{\theta}{2},$$

do not reveal themselves to him unless perhaps he transforms their equations to the rectangular form, and, in this degree, he regards the polar treatment as inconvenient if not redundant. There are some cases in which this defect may be remedied by a geometric process. It is the purpose of this paper to show that if $\rho = P(f_1, f_2, f_3, f_4, f_5, f_6) \equiv P(f), f_i$ being any one of the six trigonometric functions, and $P(f)$ any polynomial in these functions, then a geometric representation of ρ may be effected for any θ.

By means of the Peano construction for curves in rectangular coördinates, and by means of well-known artifices most of the curves ordinarily occurring may be plotted without the use of a table. If the rectangular equation is of the form $f(x, y) = 0$, a parametric representation enables us to employ the method indicated above for polar coördinates. This unifies the treatment in the various representations of a curve and also gives the student greater power in the study of curve tracing. The paper contains:

1. Some methods for the study of curves in polar coördinates, and their application to curves in parameter form.
2. The Peano construction.
3. The construction of some curves.

To exhibit a construction of $\rho = P(f)$, P being a polynomial in the six trigonometric functions, we show a construction for $\rho = a\cdot f_i^n(\theta)$, n being a positive integer and f_i any one of the trigonometric functions.

1. The product $\rho = a\cdot\sin^n\theta$ (n being any integer).

Let OX be the initial line, $OA = a$ a perpendicular to it, and OP a ray such that the angle $XOP = \theta$. Draw $AP_1 \perp OP$, $P_1P_2 \perp OA$, $P_2P_3 \perp OP$, $P_3P_4 \perp OA$, and so on. Then

$$OP_1 = a\cdot\sin\theta, \qquad OP_2 = a\cdot\sin^2\theta,$$

and in general

$$OP_n = a\cdot\sin^n\theta.$$

With OX, OA, OP as before, draw $AP_1 \perp OA$, $P_1P_2 \perp OP$, $P_2P_3 \perp OA$, and so on. Then

$$OP_1 = a\cdot\csc\theta, \qquad OP_2 = a\cdot\csc^2\theta, \qquad \text{and} \qquad OP_n = a\cdot\csc^n\theta.$$

Combining these results we have

$$OP_n = a\cdot\sin^n\theta \text{ (n being any integer).}$$

The construction of the product $\rho = a\cdot\cos^n\theta$ is immediate. We measure $OA = a$ on the x-axis and project alternately on OP and OA; while for $\rho = a\sec^n\theta$ we erect perpendiculars alternately to OA and OP.

To construct $\rho = a\tan^n\theta$, lay off $OA = a$ on OX. Draw $AQ \perp OA$ cutting OP in Q and $QP_1 \perp OY$ cutting OY in P_1; draw $P_1P_2 \perp OP$ cutting OX in P_2.

Repeating this set of operations, we have $OP_n = a \tan^n \theta$; a slight change shows that we may obtain $a \cot^n \theta$ also. Thus the equation $\rho = a \cdot \tan^n \theta$, may be constructed when n is any integer. The construction of any cross product is obvious, and it follows that if $\rho = P(f)$ is a polynomial in the six trigonometric functions ρ may be constructed geometrically.

2. We turn now to the consideration of rectangular coördinates and the Peano[1] construction.

(*a*) To construct the product $y = F_1(x) \cdot F_2(x)$, $F_1(x)$, $F_2(x)$ being represented for any value of x by the points F_1, F_2, select the x-unit OU, and the y-unit OV, and draw the diagonal $y = x$. Let M be the foot of the ordinate through F_1. Translate either point (say F_1) parallel to the x-axis to the unit line in F_1'; and the other (F_2) to the diagonal in F_2'. Draw through F_2' a perpendicular to the x-axis cutting it in M' and cutting OF_1' in Q. Draw $QP \perp F_1M$ cutting it in P. From the similar triangles $OM'Q$ and OUF_1' it may be seen that $M'Q = MP$ is the product F_1F_2. Changing the order of translation a check on the product may be found.

(*b*) To construct the reciprocal $y = 1/F_2$.

As before, let M be the foot of the ordinate through F_2, and let U, V be unit points on the axes. Translate F_2 parallel to the x-axis to the diagonal line at F_2', draw the ordinate through F_2', meeting the horizontal unit-line at S and the x-axis at M', and draw OS meeting the vertical unit-line at P'; translate P' horizontally to P on F_2M.

From the similar triangles OUP' and $OM'S$

$$UP'/OU = M'S/OM' \qquad \text{or} \qquad y = 1/F_2 = MP.$$

By means of (*a*) and (*b*) the quotient $y = F_1/F_2$ may be at once found.

3. The Construction of Curves. (*a*) The class $y = a \cdot x^n$. Multiplication of the abscissa of each point on $y = x$ by x, by means of (*a*) above, gives the curve $y = x^2$; and the higher powers may be found by a repetition of the process. By (*b*) we construct the reciprocal of x, and the curve $y = 1/x \doteq x^{-1}$. From this we get the nth power, and thus the curve $y = x^n$, n being any integer. A mere change in the y-scale is sufficient to give the curves $y = a \cdot x^n$, desired.

(*b*) The curves $y = a \cdot x^2 + b \cdot x + c$. The method in (*a*) enables us to get the curve $y = a \cdot x$ and a shift of the x-axis in a direction indicated by the sign of b, gives the graph of $y = a \cdot x + b$. Multiplying this by x and repeating the shift gives $y = a \cdot x^2 + b \cdot x + c$. These may be condensed into a single step.

The construction of $y = P(x)$, and of $y = P_1(x)/P_2(x)$ may now be effected.[2]

(*c*) For the curve $y = a^x$, we may construct a scale for all integral values of x and plot such points.

(*d*) The inverse functions may be graphically obtained by revolving the graph of the given function about the line $y = x$. By this means then we secure the

[1] Cf. Lunn, "Outline of a coherent course in college algebra," AMERICAN MATHEMATICAL MONTHLY, *1905*, 123–129.

[2] Cf. Runge, *Graphical Methods*, pp. 6–7.

graphs of the inverse trigonometric functions; from (*c*) the logarithmic curve, and from (*a*) the curves $y = x^{1/n}$ and further $y = x^{m/n}$.

By a transformation to polar coördinates, or by a parameter representation for curves of the form $F(x, y) = 0$, we may frequently find a convenient construction. We shall now consider some such cases.

1. $\rho = a \cdot \sec \theta$, $\rho = a \cdot \sec (\theta - \alpha)$. Any point P on the locus 1_1 is located by erecting a perpendicular to the initial line at A. Its intersection with the radius vector corresponding to the vectorial angle θ gives a point on the locus. Since these perpendiculars all coincide the points for all θ's must be on the straight line AP. In 1_2 the initial line makes an angle α with OA, and we have the general equation of the straight line.

2. $\rho = 2a \cdot \cos \theta$. Take $OA = 2a$; the projection of OA upon the various radii vectors makes it evident that the curve is a circle of diameter $2a$.

3. $\rho = a \sec^2 \theta/2$. Take $OA = a$ on the initial line; let $AOP = \theta$, $AOQ = \frac{1}{2}\theta$. Draw $AQ \perp OA$, meeting OQ in Q, and $QP \perp OQ$, meeting OP in P. Then the point P is a point of the locus, which may be seen, as follows, to be a parabola.[1]

At D on OA, making $OA = AD$, erect the perpendicular DR. Prolong OQ to meet DR in R. The triangle $OPQ = RPQ$, PR is perpendicular to DR, and $OP = PR$. Since P is any point of the locus it is a parabola. The line QP may be shown to be a tangent line, and the construction of a tangent to the curve is obvious; while the following theorems may be readily proved.

(*a*) The angle between two tangents is half the angle between the focal radii drawn to the point of contact.

(*b*) The perpendicular upon a tangent from the focus meets the tangent at the point of intersection of the tangent at the vertex.

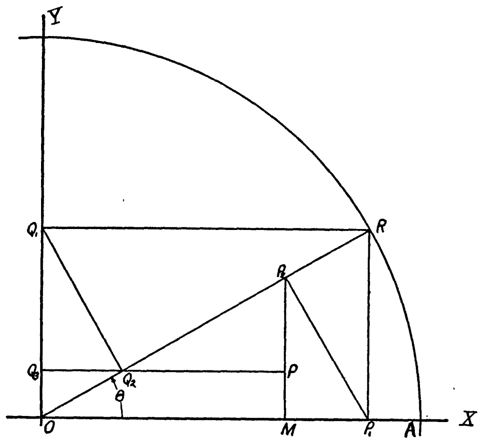


Fig. 1.

(*c*) Tangents at the extremities of a focal chord meet at right angles.

(*d*) Tangents to a parabola from a point on the directrix are perpendicular.

(*e*) Two perpendicular tangents meet on the directrix.

4. $\rho = 2a \cdot \cos 2\theta \cdot \sec \theta$. Let $OA = 2a$, the angle $AOP = \theta$, $AOR = 2\theta$. The perpendicular AR upon OR meets the terminal line of θ in P a point of the curve. If we increase θ by $\pi/2$, OR remains invariant and the prolongation of AP intersects the terminal line of the new θ, $\theta + (\pi/2)$, in another point P' of the curve. A is easily seen to be a point of the locus. Denoting by Q the intersection of PP' and OY, the angle at P' is seen to be equal to the angle QOP', and since POP' is a right angle Q is equidistant from P and P'. Thus the curve is the strophoid.

[1] Cf. Smith and Gale, *New Analytic Geometry*, p. 123.

Curves such as those given by the equations below are readily plotted by this method.

(a) $\rho = a \cdot \sin^n \theta$.

(b) $\rho = a \cdot \sin n\theta$.

(c) $\rho = a \cdot \tan^n \theta$, etc.

The method of geometric multiplication employed in the case of polar coördinates frequently becomes serviceable for curves in implicit form given in rectangular coördinates after representing the curve in parameter form.

$x^{2/3} + y^{2/3} = a^{2/3}$; $x = a \cdot \cos^3 \theta$, $y = a \cdot \sin^3 \theta$. Corresponding to the parameter θ, the projection of OR (Fig. 1) upon OX gives $OP_1 = a \cdot \cos \theta$, $OP_2 = a \cdot \cos^2 \theta$ and $OM = a \cdot \cos^3 \theta$. Thus $x = OM$; and similarly for $y = MP$. The equations show that at the same time points may be located in all four quadrants.

The curves $x^2 y = 4a^2(2a - y)$ with the parameter form

$$x = 2a \cdot \tan \theta,$$

$$y = 2a \cdot \cos^2 \theta,$$

and $y^2 = x^3/(2a - x)$ with the parameter form

$$x = 2a \cdot \cos^2 \theta,$$

$$y = 2a \cdot \cos^2 \theta \cdot \cot \theta,$$

furnish good examples.

As a final illustration we consider the curve given in parameter form:[1]

$$x = 2r \cdot \cos \theta + r \cdot \cos 2\theta - r,$$

$$y = 2r \cdot \sin \theta - r \cdot \sin 2\theta.$$

These equations may be written in the form:

$$x = 4r \cdot \cos \theta \cdot \cos^2 \theta/2 - 2r,$$

$$y = 4r \cdot \sin \theta \cdot \sin^2 \theta/2.$$

Then the following construction may be effected (Fig. 2):

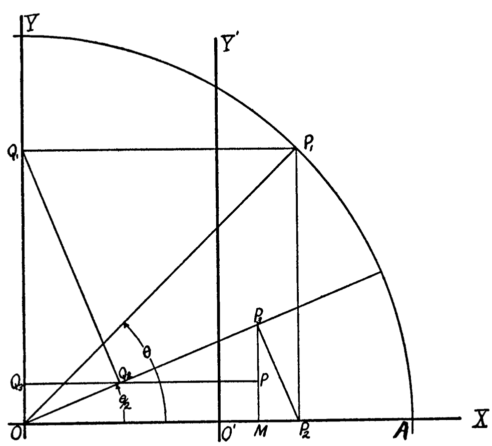


Fig. 2.

Projecting OP_1 upon the x-axis, $OP_2 = 4r \cdot \cos \theta$ $(OA = 4r)$, $OP_3 = 4r \cos \theta \cdot \cos \theta/2$ and $OM_4 = 4r \cdot \cos \theta \cdot \cos^2 \theta/2$. The projection upon the y-axis gives the ordinate of the point, which is represented at P. A shift of the y-axis $2r$ units to the right $(r > 0)$ completes the location of the curve.

This is a brief sketch of an effort to make the problem of curve tracing more a geometric than an arithmetic one. An acquaintance with the plans here indicated leads to a great variety of applications that serve to interest the student.

[1] Cf. Smith and Gale, *New Analytic Geometry*, p. 206, where this curve is drawn from table.

II. A Figure of Solid Analytic Geometry.

By John W. Bradshaw, University of Michigan.

There may well be a difference of opinion as to the particular system of axonometry that should be used in the figures of solid analytic geometry. Some would prefer drawings in perspective, as giving the best likeness and therefore furnishing the greatest assistance to the student's geometrical imagination; others would choose the simplest form of oblique parallel projection, that in which the picture-plane is assumed parallel to one of the coördinate planes, as lending itself most readily to reproduction by the student. The former type of figure is represented chiefly in the resort to photography of models,—Is there here a confession of inadequacy on the part of the draftsman?—the latter type is that commonly found in the texts. Perhaps a judicious use of both types with a comparison of the two would render the greatest service to the student.

Whatever attitude be taken on this question, however, it seems hardly necessary to contend that when a decision is once reached each figure should be consistently and accurately drawn according to the system chosen. Nor is much argument needed to show that definiteness is desirable, such definiteness as is attained by marking on the pictures of the coördinate axes three segments to represent three equal segments, or units, on the axes in space. Unless some such indication is given, one has no means of determining the point, or the direction, from which the figure should be viewed, a matter of the utmost importance if it is to produce the best possible impression. The drawings in most English and American texts with which I am acquainted are little short of disgraceful in their careless disregard of truth.

Let me call attention to the defects of just one simple figure as it appears in many books, the figure that accompanies the derivation of the equation of a plane in the normal form. Figure 1, but for the dotted lines and the lettering, is copied from a well-known text. To facilitate the description and distinguish relations in space from those of the plane figure, I shall designate points in space by Roman capitals and their pictures by the same letters starred.

In Figure 1, then, O^*X^*, O^*Y^*, and O^*Z^* represent three mutually perpendicular axes in space, OX, OY, and OZ. A plane η cuts these axes in U, V, W, represented by U^*, V^*, W^*. A perpendicular is let fall upon this plane from the origin, N is the foot of this perpendicular, O^*N^* is its picture. P is any point of the plane, reached from the origin by the broken line $OMLP$, whose parts OM, ML, LP give the coördinates of P. Since O^*X^* and O^*Z^* are perpendicular to each other it seems a reasonable assumption that the author had in mind a picture-plane parallel to the plane XOZ, though this cannot be positively asserted since he does not indicate the units. The first criticism of the figure does not depend on this assumption, the second does.

The point P, as end of the broken line $OMLP$, is not a point of the plane η. The dotted lines O^*K^* and K^*W^* are drawn to show this. The parallels OZ and LP determine a plane which cuts the plane XOY in OK and the plane η

in KW. If P lies in η it must lie in KW; so that Q^*, and not P^*, should be taken to represent a point of the plane. To be sure P^* is the picture of some point of the plane but not the point whose coördinates are indicated.

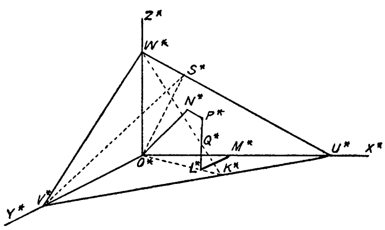


FIG. 1.

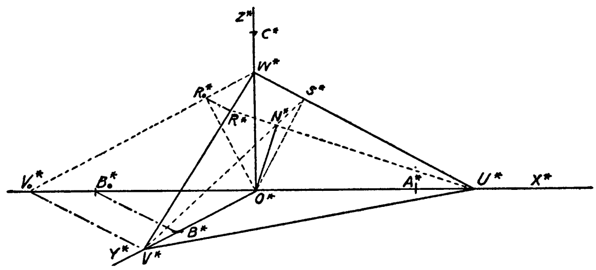


FIG. 2.

Again, N is an impossible position for the foot of the normal. For the plane ν of ON and OY is perpendicular both to XOZ and to η, and hence to their intersection UW. It cuts XOZ in a line OS perpendicular to UW. This must be represented by O^*S^* perpendicular to U^*W^*, since in this plane perpendiculars are represented by perpendiculars. Further, ν cuts η in VS and N must lie on this line, N^* on V^*S^*. The exact position of N^* cannot be given unless the unit on O^*Y^* be marked.

In Figure 2, O^*A^*, O^*B^*, O^*C^* represent equal segments on the axes in space, and the position of N is determined by rabatting the plane YOZ about the Z-axis till it coincides with XOZ. B^* thereby falls at B_0^*, such that $O^*B_0^*=O^*A^*$. The construction proceeds by drawing $V^*V_0^* \| B^*B_0^*$, $O^*R_0^* \perp V_0^*W^*$, $R_0^*R^* \| B_0^*B^*$. Then N^* is given as the intersection of U^*R^* and V^*S^*. We might instead have rabatted XOY about OX into coincidence with XOZ. The two constructions combined yield a useful check on the accuracy of the drawing.

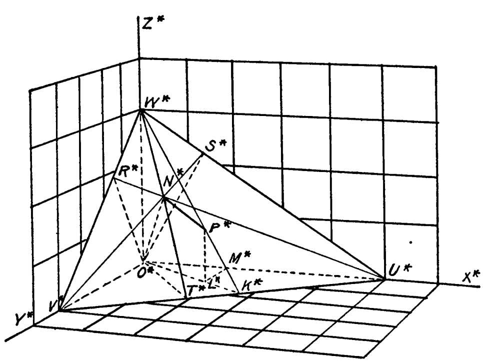


FIG. 3

As an illustration of what may be done with perspective Fig: 3 is offered. Would not many a student for whom Fig. 1 presents difficulties find Fig. 3 easily readable?

Has perhaps the author of the Solid Analytics been tricked into a careless and unwarranted sense of security by the generality of the theorem of Pohlke, that fundamental theorem of axonometry which asserts that any three segments of a plane, O^*X^*, O^*Y^*, O^*Z^* emanating from a point, so long as not more than three of the four points lie in a line, may be considered as the parallel projection of three equal, mutually perpendicular segments of space? Their lengths may

be taken at pleasure and the angles that they form with each other are entirely arbitrary. A superficial view of this theorem might lead one to think that the figures of axonometry may be drawn haphazard and that they will then represent the space figures in mind from some point of vision. On closer acquaintance, however, the theorem is seen to be associated with law, rather than license. It relates to setting up in the plane figure a standard of measurement which makes for definiteness.

RECENT PUBLICATIONS.

REVIEWS.

Girolamo Saccheri's Euclides Vindicatus. Edited and translated by GEORGE BRUCE HALSTED. Chicago and London, Open Court, 1920. 8vo. 30 + 246 pp. Price $2.00.

The Jesuit Girolamo Saccheri was born in 1667 and first taught in a college of his order in Milan where Tommaso Ceva, a brother of the Giovanni Ceva whose triangle theorem is well known, was teacher of mathematics. The influence of these brothers is apparent in the first two mathematical works which Saccheri published: (1) *Quœsita geometrica a Comite Rugerio de Vigintimilliis omnibus proposita, ab Hieronymo Saccherio Genuensi Societatis Jesu soluta.* Mediolani, 1693 (37 pages); another edition Parma, 1694 (dealing mainly with the solution of six problems in conic sections). (2) *Neostatica auctore Hieronymo Saccherio e Societate Jesu Excellentissimo senatui Mediolanensi dicata.* Mediolani, 1708 (168 pages) (discussing questions of statics and dynamics).

His third mathematical work *Euclides ab omni nœvo vindicatus* was published at Milan in 1733 (16 + 142 pages + 6 plates). Saccheri died in October, 1733, and there is doubt as to whether he lived to see his completed master work issue from the press. While the work is frequently referred to during the next one hundred and fifty years it was not till the publication of an article by Beltrami[1] in 1889 that Saccheri became generally recognized as a forerunner of Legendre, Lobatchevsky, and Bolyai. Indeed Saccheri gave many of their propositions. For example, in his discussions of the parallel postulate, 1794–1833, Legendre proved, by using only the first twenty-eight propositions of Euclid's Elements, that: The sum of the angles of a triangle cannot be greater than two right angles; and that the sum must be equal to two right angles if this is true for a single triangle. Both of these propositions are proved in more general form by Saccheri.

In Saccheri's time the conception of parallels as equidistant straight lines was a favorite one, but Saccheri, like some of his predecessors, as Sommerville remarks, "saw that it would not do to assume this in the definition. He starts with two equal perpendiculars AC and BD to a line AB. When the ends C, D

[1] "Un precursore italiano di Legendre et di Lobatschewsky," *Atti della Reale Accademia dei Lincei*, Anno 1889, series 4, Vol. 5, pp. 441–448.

are joined, it is easily proved that the angles at C and D are equal; but are they right angles? Saccheri keeps an open mind, and proposes three hypotheses: (1) The hypothesis of the right angle; (2) The hypothesis of the obtuse angle; and (3) The hypothesis of the acute angle. The object of his work is to demolish the last two hypotheses and leave the first, the Euclidean hypothesis, supreme. . . . If Saccheri had had a little more imagination and been less bound down by tradition and by a firmly planted belief that Euclid's hypothesis was the only true one, he would have anticipated by a century the discovery of the two non-euclidean geometries which follow from his hypotheses of the obtuse and the acute angles."

Saccheri's work is divided into two books: the first, pages 1–101, containing the discussion indicated above; the second, a defense of the treatment of proportion found in book V of Euclid's Elements. The first complete translation of the first book was into German by Engel and Stäckel in their *Die Theorie der Parallellinien von Euklid bis auf Gauss*, Leipzig, 1895. This contains very full references to the literature of the work. An Italian translation, by G. Boccardini of both books (the first very slightly, the latter much abbreviated), appeared in the Manuali Hoepli series in 1904.

As to Dr. Halsted's translation there is no indication in the work before us that any part of the work has appeared in print elsewhere before. Nevertheless this is the case. The last paragraph of a circular advertising the book under review, and signed by Dr. Halsted, is as follows: "The English translation is a revision of the first ever made into any language, published in 1894, but long unprocurable." Hardly a single statement in this sentence is accurate. The translation in question (of propositions I–XXXVI in the first book) appeared in this MONTHLY, volumes 1–5, June, 1894–December, 1898. In "1894" the English translation of thirteen propositions only had appeared. The complete German translation of the first book was published before half of the English translation had appeared. The early numbers of the MONTHLY are not "unprocurable."

The differences between the part of Saccheri's work which appeared in this MONTHLY, and its reprint, are numerous—nearly three hundred were noticed in the first twenty-five propositions—but these have rarely introduced radical changes in the sense of the original passages. The figures are all new and are black on white instead of white on a black background. They are not repeated nearly as often as in the translation of Engel and Stäckel and the reading is consequently much less easy. In Saccheri's work the figures were on folding plates.

The additions to the original English translation are considerable: propositions 37 to 39 (the last of book 1), Saccheri's preface to the reader and synopsis of contents, and the original Latin of the whole, each page of text having the translation on the opposite page.

Dr. Halsted's "Introduction" emphasizes the importance of Saccheri's *Logica demonstrativa* first published in 1697 (the second edition appearing in 1701

and the third in 1735) in studying the *Euclides Vindicatus*. He has also appended two pages of notes, a subject index, and an index of proper names. The whole has been issued by the Open Court Publishing Company in a volume of attractive appearance.

This enterprising company and Dr. Halsted have placed mathematicians much in their debt by making the masterpieces of Saccheri,[1] Bolyai, and Lobatchevsky so readily accessible to American and English readers.

R. C. ARCHIBALD.

The Elementary Differential Geometry of Plane Curves. By R. H. FOWLER. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 20.) Cambridge, at the University Press, 1920. 7 + 105 pages. Price 6 shillings.

Preface: "This tract is intended to present a precise account of the elementary differential properties of plane curves. The matter contained is in no sense new, but a suitable connected treatment in the English language has not been available.

"As a result, a number of interesting misconceptions are current in English text books. It is sufficient to mention two somewhat striking examples. (*a*) According to the ordinary definition of an envelope, as the locus of the limits of points of intersection of neighbouring curves, a curve is not the envelope of its circles of curvature, for neighbouring circles of curvature do not intersect. (*b*) The definitions of an asymptote—(1) a straight line, the distance from which of a point on the curve tends to zero as the point tends to infinity; (2) the limit of a tangent to the curve, whose point of contact tends to infinity—are not equivalent. The curve may have an asymptote according to the former definition and the tangent may exist at every point, but have no limit as its point of contact tends to infinity.

"The subjects dealt with, and the general method of treatment, are similar to those of the usual chapters on geometry in any *Cours d'Analyse*, except that in general plane curves alone are considered. At the same time extensions to three dimensions are made in a somewhat arbitrary selection of places, where the extension is immediate, and forms a natural commentary on the two dimensional work, or presents special points of interest (Frenet's formulæ). To make such extensions systematically would make the tract too long. The subject matter being wholly classical, no attempt has been made to give full references to sources of information; the reader however is referred at most stages to the analogous treatment of the subject in the *Cours* or *Traité d'analyse* of de la Vallée Poussin, Goursat, Jordan or Picard, works to which the author is much indebted.

"In general the functions, which define the curves under consideration, are (as usual) assumed to have as many continuous differential coefficients as may be mentioned. In places, however, more particularly at the beginning, this rule is deliberately departed from, and the greatest generality is sought for in the enunciation of any theorem. The determination of the *necessary and sufficient* conditions for the truth of any theorem is then the primary consideration. In the proofs of the elementary theorems, where this procedure is adopted, it is believed that this treatment will be found little more laborious than any rigorous treatment, and that it provides a connecting link between Analysis and more complicated geometrical theorems, in which insistence on the precise necessary conditions becomes tedious and out of place, and suitable sufficient conditions can always be tacitly assumed. At an earlier stage the more precise formulation of conditions may be regarded as (1) an important grounding for the student of Geometry, and (2) useful practice for the student of Analysis.

"The introductory chapter is a collection of somewhat disconnected theorems which are

[1] For other discussions of Saccheri's *Euclides Vindicatus* the curious reader may turn to Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. 3, 2te Aufl., 1901; to P. Mansion, "Analyse des recherches du P. Saccheri, S. J., sur le postulatum d'Euclide," *Annales de la Société scientifique de Bruxelles*, 1889–1890, Vol. 14, 2d part; reprinted in a supplement to *Mathesis*, January, 1891; to G. Veronese, *Grundzüge der Geometrie*, Leipzig, 1894, pp. 636–639; and to H. S. Carslaw's English translation of Bonola's *Non-Euclidean Geometry* (Open Court, 1912).

It seems curious that there is no reference to Saccheri in *The Catholic Encyclopedia*.

required for reference. The reader can omit it, and refer to it as it becomes necessary for the understanding of later chapters."

Contents—Chapter I: Introduction, 1–8; II: The elementary properties of tangents and normals, 8–23; III: The curvature of plane curves, 24–44; IV: The theory of contact, 45–58; V: The theory of envelopes, 58–79; VI: Singular points of plane curves, 80–89; VII: Asymptotes of plane curves, 89–103.

NOTES.

The second edition of W. A. Robertson and F. A. Ross's *Actuarial Theory* (Edinburgh, Oliver and Boyd, 23 + 431 pages) appeared in the early summer of 1920. The changes in the first edition are mainly in the correction of errors.

The Oxford University Press announces the following books: *Roger Bacon and the State of Science in the Fourteenth Century* by R. Steele; *Archimedes' Principle of the Balance and some criticisms upon it* by J. M. Child; *A History of Greek Mathematics*, 2 volumes, by Sir Thomas Heath—The Cambridge University Press has published a third edition of E. T. Whittaker and G. N. Watson's *A Course of Modern Analysis* (8 + 608 pages; price 40 shillings). It contains about 230 pages more than the first edition by Whittaker alone.

The second part of the sixth volume of the *Encyklopädie der Mathematischen Wissenschaften* is devoted to Astronomy. Between 1905 and 1915 six Hefte were published; the last of these included Professor E. W. BROWN'S "Theorie des Erdmondes," translated by A. v. Brunn. The first thirty-five pages of Heft 7, published in 1920, contain "Die Satelliten" by Professor KURT LAVES, of the University of Chicago. This section was "abgeschlossen im Sommer 1916." The rest of the Heft (pages 843–895) is devoted to "Bestimmung und Zusammenhang der astronomischen Konstanten" by J. Bauschinger.

In *Proceedings of the Benares Mathematical Society*, volume 2, 1920, D. K. Sen, research scholar in mathematics at Benares Hindu University, has an eleven page article entitled: "On the application of Burgess's method for determining the uniform motion of an ellipsoid of revolution through a viscous liquid along its axis of revolution." The reference here is to Professor R. W. Burgess's doctor's thesis at Cornell, on "The uniform motion of a sphere through a viscous liquid," published in the *American Journal of Mathematics*, 1916.

America's influence is being wielded in mathematics of the elementary and secondary schools of Cuba and South America through the works in Spanish by "Jorge Wentworth y David Eugenio Smith" published by the enterprising firm of Ginn and Company. These works are: *Geometría Plana y del de Espacio* (1915, 8 + 469 pp.; price $1.72; translation of the authors' English work), *Elementos de Algebra* (1917, 6 + 458 pp.; price $1.72; translation, with slight changes, of book 1 and part of book 2 of the authors' *School Algebra*), and *Aritmética Moderna*, 2 books (1916, 6 + 265 + 6 + 317 pp.; price $.64 + $.80; not an exact translation of any American editions).

From an editorial in *The New York Times,* October 3, 1920, on James Russell Lowell as a teacher at Harvard, with special reference to an article on this topic by W. R. Thayer: "Mr. Thayer describes and seems to wonder at Mr. Lowell's illimitable mustaches—frowning over a square beard. They were a work of art, nor can we understand why Mr. Thayer found them 'neither becoming nor beautiful.' There were barbate giants in those days. One remembers Professor BENJAMIN PEIRCE and his son, Professor JAMES MILLS PEIRCE, bearded both like the pard or the ball players of the House of David. Mr. James T. Fields, one of Mr. Lowell's publishers and his successor as editor of *The Atlantic,* had as copious and melodious a beard as ever inflamed a barber's eyes; at one time there were strange sausages or curls each side of the cheek."

The Göttingen *Nachrichten,* Mathematisch-physikalische Klasse for June, 1920, contains the first of a series of memoirs by G. H. Hardy and J. E. Littlewood in which they propose to develop in detail the new analytic method which they found for the discussion of Waring's problem and a number of allied problems in 'additiver Zahlentheorie.' The general lines of this method, in so far as it concerns Waring's problem in particular were explained in a recent paper in the *Quarterly Journal of Mathematics* [see this MONTHLY, *1920,* 272] where full references to the literature of the problem were given. The object of the authors now is "to give full details of the proofs, up to the point at which Hilbert's famous theorem, first proved in this journal in 1909, emerges as a corollary from our analysis." The Geschäftliche Mitteilungen of the *Nachrichten,* for July, 1920, contains sketches of the life and work of Woldemar Voigt [*1920,* 280], pages 45–52, by C. Runge, and of Adolf Hurwitz [*1920,* 191], pages 75–83, by D. Hilbert.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 42, no. 3, July (published in October), 1920: "The failure of the Clifford chain" by W. B. Carver, 137–167; "On the representations of numbers as sums of 3, 5, 7, 9, 11 and 13 squares" by E. T. Bell, 168–188; "On a certain class of rational ruled surfaces" by A. Emch, 189–210.

BULLETIN DES SCIENCES MATHEMATIQUES, volume 55, June, 1920: Review by R. Garnier of O. Veblen and J. W. Young's *Projective Geometry,* volume 2 (Boston, 1918), 105–112 [Last paragraph: "Par ce rapide apercu on aura pu pressentir la variété des problèmes abordés, la rigueur des développements et l'importance des résultats établis. A ces qualités essentielles, le Traité joint tous les avantages d'une forme attrayante; rédigé dans un style clair et précis, complèté par un choix très étendu d' 'exercises'—dont quelques-uns, d'un niveau élevé, sont empruntés à de recents Mémoires—, l'Ouvrage de MM. Veblen et Young sera pour l'étudiant francais le plus sûr des initiateurs."]

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 1, October, 1920: "The Seattle meeting of the San Francisco section" by E. T. Bell, 1–4; "Note on a generalization of a theorem of Baire" by E. W. Chittenden, 5–6; "Certain iterative characteristics of bilinear operations" by N. Wiener, 6–10; "Necessary and sufficient conditions that a linear transformation be completely continuous" by C. A. Fischer, 10–17; "On the relation of the roots and poles of a rational function to the roots of its derivative" by B. Z. Linfield, 17–21; "Moritz Cantor, the historian of mathematics" by F. Cajori, 21–28; "Shorter notices," 28–38 [Reviews by D. E. Smith of A. Eymieu's *La Part des Croyants dans les Progrès de la Science au XIXe Siècle,* 3e éd. (Paris, 1920) and of J. M. Child's *Early Mathematical Manuscripts of Leibnig* (Chicago, 1920); reviews by R. D. Carmichael of A. N. Whitehead's *An Enquiry Concerning the Principles of Natural Knowledge* (Cambridge, 1919) and of H. Bateman's *Differential Equations* (London, 1918)]; "Notes," 39–46; "New publications," 46–48.

Mathematical Gazette, volume 10, July, 1920: "The Leeds meeting" by R. C. Fawdry, 81–82; "The training of the mathematical teacher" by W. P. Milne, 83–85; "The mathematics of thread and cloth construction: an historical survey" by A. F. Barker, 86–91; "The mathematical theory of the sateen arrangement" by S. A. Shorter, 92–97; "The teaching of mathematics to textile students" by J. H. Whitwam, 98–102; "On the teaching and applications of dynamics" by A. Gray, 103–114; "Mathematics and commerce" by A. N. Shimmin, 115–118; "Life of James Stirling, the Venetian" by C. Tweedie, 119–128.

Messenger of Mathematics, volume 49, nos. 7, 8, November, December, 1919: "The tetrahedron and pentaspherical coördinates" by T. C. Lewis, 97–106; "Two trigonometrical determinants" by E. H. Neville, 107–111; "The dissection of rectilineal figures" (concluded) by W. H. Macaulay, 111–121; "On Laplace's integrals for a Legendre polynomial" by S. Pollard, 121–125; "A form for $(d/dn)P_n(\mu)$, where $P_n(\mu)$ is the Legendre polynomial of degree n" by A. E. Jolliffe, 125–127; "On a property of algebraic numbers" by W. Burnside, 127–128—January, February, March, 1920: "On plane curves of degree n with tangents of n-point contact" by H. Hilton, 129–134; "An integral equation occurring in a mathematical theory of retail trade" by H. Bateman, 134–137 [First two paragraphs: "A tradesman, who buys and sells various articles, will be supposed to have worked up his business to such an extent that he can be sure of selling his goods at a constant rate so that a new supply of any article will be completely exhausted at the end of an interval of time T after the date of purchase.[1] Our problem is to find the law according to which goods must be purchased in order that the total value of the stock may remain constant.

"To simplify matters the process of buying and selling will be treated as continuous instead of discontinuous. This is approximately true if business is brisk all the time and if the working hours and days are pieced together so that 'business time' can be treated as a continuous variable."]; "On apolar and co-apolar triangles for a cubic, and on apolarly conjugate triangles" by E. B. Elliott, 137–149; "Notes on some points in the integral calculus" (liii) by G. H. Hardy, 149–155; "Four-vector algebra and analysis" by C. E. Weatherburn, 155–176—April: "On the congruence $(p-1)! \equiv -1 \pmod{p^2}$" by N. G. W. H. Beeger, 177–178; "On Pascalian collinearities and concurrencies" by E. B. Elliott, 178–180; "Transitive constituents of the group of isomorphisms of any abelian group of order p^m" by G. A. Miller, 180–186; "Is there an analogue in solid geometry to Feuerbach's theorem ?" by T. C. Lewis, 187–192 [A discussion of a problem proposed in J. L. Coolidge, *Treatise on the Circle and the Sphere*, p. 247; Mr. Lewis finds: "It is easy to prove that there is no analogue, in space of three dimensions, to Feuerbach's theorem, and it follows that there is nothing corresponding to the Hart systems"].

The Monist, volume 30, no. 3, July, 1920: "Space and time" by R. W. Sellars, 321–364; "Lord Rayleigh," 474–475.

Nature, volume 105, July 29, 1920: "Tycho Brahe" by J. K. Fotheringham, 672–673 [Review of *Tychonis Brahe Dani Opera Omnia*, volume 6, Copenhagen, 1919].—August 5: "Relativity and reality" by R. A. Sampson, 708.—August 12: "Complex elements in geometry" by G. B. Mathews, 736–737 ["P.S.—Since the above was written, I have had time to reflect further upon Prof. Hatton's book, and I have read Prof. G. H. Hardy's review of it in a recent number of the *Mathematical Gazette*. I do not wholly agree with Prof. Hardy's attitude, because I still think that there are geometrical notions not reducible to arithmetic—still less to formal logic. But I do agree with him that Prof. Hatton's book has no theoretical value, and disagreeable as it is, I think it is my duty to say so, especially as I have been informed that another reviewer has praised the book in absurdly exaggerated terms. G.B.M." Cf. *1920*, 268–269]; "Obituary, Professor John Perry, F.R.S.," by H.E.A. and W.E.D., 751–753; [Note], 762 ["News has just reached us that Prof. A. T. De Lury was appointed some months ago to be the head of the department of mathematics in the University of Toronto by the Board of Governors on the recommendation of the president of the University, Sir R. A. Falconer. The staff, council, and senate have nothing to do with appointments, and the only check upon the action of the president and the Board of Governors is public opinion. Prof. De Lury has been a member of the teaching staff of the University for many years, and is the author of a number of mathematical text-books which have done service in the schools of the province of Ontario. He possesses high teaching ability, but has not been associated with the research activities which it should be the essential

[1] "Strictly this should be the date at which the new supply is made available for sale. We shall suppose for simplicity that in the case of any special article the date at which the old supply is exhausted coincides approximately with the date when a new package is opened."

function of a university to create and foster. Without men engaged in the production of new knowledge the work of a university differs little from that of a secondary school preparing students for examinations. Toronto has won much distinction by the scientific investigations of such men as Profs. Maccallum, McLennan, and Brodie, and it was hoped that the chair of mathematics would have been filled by someone who possesses the highest research qualifications in mathematics that Canada could produce. If Prof. De Lury can and will build up a strong research staff under him, he will be doing the best service to his University and extend the stimulating atmosphere which some of his scientific colleagues have given to the institution by their work"].—August 19: "The mathematician as anatomist" by A. Keith, 767–770 [Review of two parts of K. Pearson's "A Study of the long bones of the English skeleton" Cambridge, 1919]; "Obituary, Sir Norman Lockyer," 781–784.—August 26: "Professor Alexander's Gifford Lectures" by Viscount Haldane, 798–801 [Review of S. Alexander's *Space, Time, and Deity*, 2 volumes, London, 1920]; "Use of Sumner lines in navigation" by J. Ball, 806–808; "Relativity and hyperbolic space" by A. McAulay, 808; "Sir Norman Lockyer's contributions to astrophysics," by A. Fowler, 831–833.—Volume 106, September 2: "Internal constitution of the stars" by A. S. Eddington, 14–20; "Memorial tributes to Sir Norman Lockyer," 20–25 [by various authors]—September 16: "Ewing's 'Thermodynamics'" by H.L.C., 72–73 [Review of J. A. Ewing's *Thermodynamics for Engineers*, Cambridge, 1920]; "Associated squares and derived squares of order 5" by J. C. Burnett, 79 [shows that there are nearly 700,000 such magic squares of the first 25 integers].

La Nature, volume 482, July 10, 1920: "La 'multi,' nouvelle machine à multiplier" by J. Boyer, 30–32—July 17: "L'heure exacte sans instruments et sans calculs" by Viguier, 43–45.

Nouvelles Annales de Mathématiques, volume 79, May, 1920: "Sur l'application de la loi de Gauss à la position probable d'un point dans le plan ou dans l'espace" (continued) by J. Haag, 161–178; "Sur une propriété caractéristique des cylindres et du cylindroïde" by R. Harmegnies, 178–180; "Sur la surface dont tous les points sont des ombilics" by R. Harmegnies, 180–181; "Courbes gauches liées par échange des directions des tangentes et des binormales. Les formules de Frenet sont intuitives" by G. Fontené, 181–188; "Correspondance," "Chronique," "Solutions de questions proposées," "Questions," 188–200—June: "Sur l'application de la loi de Gauss à la position probable d'un point dans le plan ou dans l'espace" (concluded) by J. Haag, 201–208; "Sur un système remarquable de cinq droites" by R. Bricard, 209–214; "Rayon de courbure de la courbe qui est le lieu des centres des sphères osculatrices à une courbe gauche" by G. Fontené, 214–219; "Chronique," "Certificats d'analyse supérieure," "Solutions de questions proposées," "Questions," 220–240—July: "Sur un défaut de la méthode d'interpolation par les polynomes de Lagrange" by M. Fréchet, 241–249; "Transformation polaire interaxiale" by M. D'Ocagne, 248–260; "Agrégation des sciences mathématiques (juillet, 1919): Problème de calcul différentiel et intégral, et composition de calcul différentiel et intégral" by R. Garnier, 260–275; "Chronique" and "Questions," 275–280—August: "Surfaces de translation applicables l'une sur l'autre" by B. Gambier, 281–295; "Notes sur les congruences de normales" by M. Bayard, 295–297; "Concours spécial d'agrégation de 1919. Solution du problème de mécanique" by E. Delassus, 297–307; "Certificat de géométrie supérieure, Paris, 1919," 307–311; "Certificat de mécanique appliquée, Lille, 1919," 314–316; "Concours d'admission a l'Ecole Polytechnique en 1920," 316–319; "Questions," 319–320.

Proceedings of the Royal Society of Edinburgh, volume 40, part 1, January, 1920: "Notices of fellows . . . recently deceased" [E. C. Fisher; Lord Rayleigh, J. W. Strutt; J. M. Bernard], 1–2.

Rendiconti del Circolo Matematico di Palermo, volume 44, no. 1, January–April, 1920: "The foundations of the elliptic functions" by H. Hancock, 87–102.

Revue de l'Enseignement des Sciences, volume 14, March–April, 1920: "Sur le sens de la variation d'une fonction" by G. Fontené, 49–53; "Sur l'ensemble de deux équations et la fraction rationnelle du second degré" by J. Juhel-Rénoy, 53–60; "Perpendiculaire menée d'un point sur une droite" by J. Lemaire, 60; "La fonction exponentielle et les fonctions circulaires" by C. Michel, 61–84; "Examens et concours de 1919," 84–91; "Problèmes de mathématiques donnés au baccalauréat en juillet 1919," 91–96—May–June, 1920: "Première leçon sur les séries entières" by P. Flamant, 97–107; "Note de géométrie" by R. Malloizel, 107; "Sur les orbiformes" by C. Bioche, 108–110; "Sur les ovales de Descartes" by R. Bérard, 110–119; "Sur le développement d'un cône et les points d'inflexion de la transformée d'une courbe tracée sur le cône" by R. Bérard, 119–121.

Revue de Mathématiques Spéciales, volume 13, no. 9, June, 1920: "Sur la folium double" by G. Loria, 201–202; Solutions of questions in analytic geometry, algebra and analysis, mechanics, 203–207, 213–220; "Sujets de concours, agrégation des sciences mathématiques, session spéciale d'octobre, 1919," 210–213.—No. 10, July: Solutions of questions in analytic geometry, mechanics, calculus, analysis, 225–233, 238–243; "Ecole Normale Supérieure et Bourses de Licence, concours de 1919," 234–237.—No. 11, August: "Ecole Normale Supérieure et Bourses de Licence, concours de 1914," 249–257 [Solutions of problems by L. Simon]; "Sujets de concours, Ecole Polytechnique, 1920," 258–260; Sujets de concours, Ecole Navale, 1916 et 1917, 268–272.

Revue Générale des Sciences, volume 31, June 30, 1920: "Charles Ange Laisant (1841–1920)" by J. Boyer, 397–398—August 15–30, 1920: "Les relations entre la science et l'industrie et les sociétés de perfectionnements industriels" by R. d'Adhémar, 513–519.

Science, new series, volume 52, July 23, 1920: "The structure of the universe" by W. D. MacMillan, 67–74—August 6: "A priori use of the Gaussian law" by E. G. Boring, 129–130—August 13: "Transverse vibrating rods" by A. G. Webster, 154; "Mathematische Zeitschrift" by G. A. Miller, 155—August 20: "Efficiency in thermal calculations" by A. W. Forbes and A. G. Webster, 175–176; Review by F. H. Garrison of C. Singer's *Greek Science and Modern Science* (Oxford Univ. Press, 1920), 178–179—September 3: Review by F. H. Garrison of D. W. Singer's *Hand-List of Scientific Manuscripts in the British Isles dating from before the sixteenth century* (London, 1919) and *Survey of Medical Manuscripts in the British Isles dating from before the sixteenth century* (London, 1920), 216–227—September 10: "The internal constitution of the stars" by A. S. Eddington, 233–240 [Address before the mathematical and physical section of the British Association for the Advancement of Science]—September 17: "Galileo's experiment from the leaning tower" by E. A. Partridge, 272–273—September 24: "The national committee on mathematical requirements," 289—October 1: "Electricity and gravitation" by H. Bateman, 314–315.

Scientific Monthly, volume 11, September, 1920: "Giant suns" by H. H. Turner, 228–234; "A simplified musical notation" by E. V. Huntington with an introduction by A. T. Davison, 276–283 [First sentences of the article: "The purpose of this paper is to present a new musical notation which, while retaining all the excellent features of the present notation, would, it is believed, greatly simplify the processes of reading, studying and composing musical scores.

"The plan is not one of the artificial mnemonic devices of which the literature is full, but is based directly on the fundamental principle of music, namely, the equi-tempered scale. This scale, introduced by J. S. Bach about two centuries ago, and now dominating all musical composition, makes use of only twelve notes in each octave, these twelve notes dividing the octave into twelve mathematically equal intervals, called semi-tones. Now, if we draw an ordinary five-line staff with one ledger line, we see that exactly twelve notes can be accommodated on the lines and spaces of such a staff. What is then more natural than to assign one place on the staff to each of the twelve notes of the scale, thus doing away with all 'sharps' and 'flats,' and representing every musical interval correctly to the eye as well as to the ear? This, in brief, is precisely the plan here proposed."]

Transactions of the American Mathematical Society, volume 21, no. 3, July (published in October), 1920: "On the representation of a number as the sum of any number of squares, and in particular of five" by G. H. Hardy, 255–284; "A memoir upon formal invariancy with regard to binary modular transformations. Invariants of relativity" by O. E. Glenn, 285–312; "Properties of the subgroups of an abelian prime power group which are conjugate under its group of isomorphisms" by G. A. Miller, 313–320; "On the order of magnitude of the coefficients in trigonometric interpolation" by D. Jackson, 321–332; "Concerning simple continuous curves" by R. L. Moore, 333–347; "On the iteration of rational functions" by J. F. Ritt, 348–356.—October: "Minima of functions of lines" by Elizabeth Le Stourgeon, 357–383; "Invariants of infinite groups in the plane" by E. F. Simonds, 384–390; "On triply orthogonal congruences" by J. B. Shaw, 391–408; "A set of properties characteristic of a class of congruences connected with the theory of functions" by E. J. Wilczynski, 409–445; "On the equilibrium of a fluid mass at rest" by J. W. Alexander, 446–450; "Concerning approachability of simple closed and open curves" by J. R. Kline, 451–458.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 51, no. 6, published June 15, 1920: "Zur Erfindung des Zeichens $\times$" by H. Wieleitner, 145–148; "Optische Geometrie" by R. Böger, 148–164; "Kleine Mitteilungen," 164–167; "Bücherbesprechungen," 167–169.

The sound of a gun fired at B reaches M_1 at the time T, and M_2 at the time $T + \tau_1$ sec., and it reaches M_3 at the time $T + \tau_2$ sec. How far is B from M_1?

2874. Proposed by J. L. RILEY, Stephenville, Texas.

Show that the equation,

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + k = 0,$$

has some imaginary roots if $a^2 - 2b < n\sqrt[n]{k^2}$; $a, b, \cdots k$ are supposed real.

Note. This result is a particular case of a theorem contained in a paper published a few years ago.—EDITORS.

2875.

Show how to draw through a given point a straight line dividing a given triangle into two parts of equal area.

Three cases of this well-known problem (proposed by an Association member) were discussed by Euclid over two thousand years ago. A new solution is sought. —EDITORS.

PROBLEMS—NOTES

1. Mr. R. M. Winger, of the University of Washington, suggested the problem: "Six of the points of the Feuerbach circle of a triangle, namely, the middle points of the sides of the triangle and the middle points of the junctions of the opposite vertices with the orthocenter, lie in pairs at the extremities of diameters." This result is the basis of several proofs that Feuerbach's circle goes through the six points noted above as well as through the feet of the altitudes. See, for example, J. Casey, *Sequel to . . . the Elements of Euclid*, third edition, 1884, p. 58.

H. P. M.

2. The following problem was proposed and solved in the *Journal of the Indian Mathematical Club*, 1910: "All the significant figures except four in a sum of division were replaced by dots, and the result was:

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It is required to recover the lost figures."

Similar problems with "four fours" and "seven sevens" are due to Mr. W. E. H. Berwick, lecturer in University College, Bangor, England:

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The first of these was proposed in *The Mathematical Gazette*, March, 1920, page 43, with the statement: "There are just four ways of filling up the missing figures so as to leave a correctly-worked long division sum (in scale ten)." The problem was reproduced in *The Observatory*, July, 1920, page 274, and the solutions were given on page 372 of the issue for October. The "seven sevens" problem was proposed in *The School World*, London, July 1906, volume 8, page 280; its solution appeared in the following August number, page 320. The proposer remarked that "there is one, and only one solution possible." ARC.

3. A member of the Association has proposed as a problem the derivation of the familiar formula:

$$\cos nx = 2^{n-1}\cos^n x - 2^{n-3}\, n\cos^{n-2} x + 2^{n-5}\frac{n(n-3)}{2\,!}\cos^{n-4} x - \cdots$$

$$+ (-1)^r \frac{2^{n-2r-1}n(n-r-1)(n-r-2)\cdots(n-2r+1)}{r\,!}\cos^{n-2r}x + \cdots$$

where n is a positive integer. Such a derivation may be found in many well known works, for example: R. Levett and C. Davison, *Elements of Plane Trigonometry*, London, 1892, pp. 299–300; S. L. Loney, *Plane Trigonometry*, Cambridge, 1893, pp. 323–354; J. A. Serret, *Traité de Trigonométrie*, 6e édition, Paris, 1880, pp. 204–236; T. J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series*, London, 1908, p. 177; and H. Weber and J. Wellstein, *Encyklopädie der Elementar-Mathematik*, Band 2, 2te Auflage, Leipzig, 1907, pp. 320–321. In obtaining the result, DeMoivre's theorem, the expansion of log $(1-2p\cos x+p^2)$,[1] and mathematical induction are variously employed (cf. problem 2828, *1920*, 186).

References may be given also to discussions in Lagrange, *Lecons sur le Calcul des Fonctions*, seconde édition, Paris, 1806; in *Cambridge Mathematical Journal*, vol. 4, November, 1844, pp. 250–252, by R. Moon; and in *Nouvelles Annales de Mathématiques*, 2e série, vol. 12, 1873, pp. 408, 425, by Mourgue and LeBesgue; vol. 14, 1875, p. 385, by Desboves.

In a letter to Leibnitz dated June 16, 1676, Newton gave a formula which, in modern notation, may be written:

$$\sin nx = n\sin x + \frac{(1-n^2)n}{3\,!}\sin^3 x + \frac{(1-n^2)(9-n^2)n}{5\,!}\sin^5 x + \cdots.$$

About twenty-five years later Johann Bernoulli derived the formula which, in modern notation, may be written:

$$\cos nx = \cos^n x - \binom{n}{2}\sin^2 x\cos^{n-2} x + \binom{n}{4}\sin^4 x\cos^{n-4} x - \cdots.$$

The formula with which this note opens seems to have been first given in 1748 by Euler, who arrived at the result by mathematical induction. (See *Introductio in analysin infinitorum*. Tomus primus. Lausannae, 1748; French edition,

[1] This method was used by E. Heine, *Mathematische Annalen*, vol. 2, 1870, pp. 187–188.

Paris, 1796, pp. 194–195. Also "Dilucidationes super formulis, quibus sinus et cosinus angulorum multiplorum exprimi solent, ubi simul ingentes difficultates diluuntur," *Nova acta acad. sc. Petrop.*, volume 9 (1791), 1795, pp. 54–55; "Conventui exhib. die 6 Mart. 1777.") In a posthumous paper, "Enodatio insignis cujusdam paradoxi circa multiplicationem angulorum observati," (*Opera postuma*, vol. 1, 1862, pp. 299–314) Euler discussed cos $n\phi$ as a power series in cos ϕ.

The developments of sin nx and cos nx into series in sin x and cos x, when n is not restricted to integral values, seem to have been first thoroughly discussed by Poinsot and forms the subject of his *Recherches sur l'Analyse des Sections Angulaires*, Paris, 1825, 80 quarto pages. ARC.

SOLUTIONS OF PROBLEMS.

417 (Algebra) [1914, 156]. Proposed by A. J. RICHARDSON, Marquette, Mich.

Required to reduce the quartic

$$x^4 + px^2 + qx + r = 0$$

to the form

$$(x^2 + k)^2 = [(2k - p)x + (2k - q)]^2,$$

wherein k is the solution of a certain cubic.[1] Hence, express the solution of the given quartic in terms of p, q, r, and k.

SOLUTION BY H. S. UHLER, Yale University.

We shall show that the hypothesis that the given quartic can be reduced to the required form destroys the mutual independence of the coefficients p, q, r, and that k is not "the solution of a certain cubic."

Transposing the right-hand member of

$$(x^2 + k)^2 = [(2k - p)x + (2k - q)]^2,$$

expanding, and collecting terms, we find

$$x^4 + [2k - (2k - p)^2]x^2 - 2(2k - p)(2k - q)x + [k^2 - (2k - q)^2] = 0.$$

Since, by hypothesis, the last equation is a modified form of the given quartic the coefficients of like powers of x in the two equations must be equal, that is, the three following conditions must hold

$$2k - (2k - p)^2 = p, \tag{1}$$

$$- 2(2k - p)(2k - q) = q, \tag{2}$$

$$k^2 - (2k - q)^2 = r. \tag{3}$$

Condition (1) gives

$$2k - p = 0, \tag{4}$$

or

$$2k - p = 1. \tag{5}$$

In either case k is a linear function of p alone and hence k is not dependent upon the solution of any cubic.

Combining relations (2) and (4) we see that q must have the special value zero. Then the given quartic reduces to a quadratic in x^2 and the solution is immediate.

---

[1] There must have been some mistake in writing this problem. The following may have been intended:

$$(x^2 + k)^2 = [x\sqrt{2k - p} - \sqrt{k^2 - r}]^2.$$

See Todhunter, *An Elementary Treatise on the Theory of Equations*, London, 1880, p. 117.—EDITORS.

Combining relations (2) and (5) we obtain

$$k = q/4. \tag{6}$$

Equating the values of k given by relations (5) and (6) we find that p and q are connected by the linear equation

$$2p - q + 2 = 0. \tag{7}$$

Substituting k from relation (6) in relation (3) and then employing equation (7) we find

$$r = -3q^2/16 = -3(p+1)^2/4. \tag{8}$$

Incidentally, the last result shows that r must be negative in order that p and q may be real.

Relations (7) and (8) enable the given quartic to be expressed in terms of (say) p alone, namely

$$x^4 + px^2 + 2(p+1)x - 3(p+1)^2/4 = 0,$$

or

$$[x^2 + (p+1)/2]^2 = [x - (p+1)]^2.$$

Consequently, the roots of the quartic are

$$x = (1 \pm \sqrt{-6p-5})/2,$$

$$x = (-1 \pm \sqrt{2p+3})/2.$$

For $-3/2 \leqq p \leqq -5/6$ all four roots will be real. For values of p outside of this range two roots will be real and two complex.

The reduction of the general quartic to the difference of two squares is given in Burnside and Panton's *Theory of Equations*, vol. I, p. 129, Art. 63 (4th or 7th editions.)

2783 [1919, 312]. Proposed by C. C. BRAMBLE, U. S. Naval Academy.

Two players, A and B, take turns throwing a single dice, A leading. The one first making a score of three aces is to be the winner. Find the probability that A will win.

SOLUTION BY H. P. MANNING, Brown University.

The probability of throwing three aces in the first three throws is $1/6^3$; in general, the probability of throwing exactly two aces in the first n throws, and an ace in the $(n+1)$th throw, is $\dfrac{n(n-1)}{2} \cdot \dfrac{5^{n-2}}{6^{n+1}}$. That is, the probabilities of the different ways of throwing the first three aces are for A or B

$$\frac{1}{6^3},\quad 3\frac{5}{6^4},\quad 6\frac{5^2}{6^5},\quad \cdots.$$

Call these numbers

$$a \quad b \quad c \quad \cdots.$$

Their sum should equal 1.

The probability that A wins is

$$a + (1-a)b + (1-a-b)c + \cdots,$$

the factor in the parenthesis in any term indicating the probability that B has not yet thrown three aces. This expression may be written

$$\begin{aligned} &a + b + c + \cdots \\ &- (ab + ac + bc + \cdots). \end{aligned}$$

Similarly the probability that B wins is

$$(1-a)a + (1-a-b)b + \cdots,$$

which may be written

$$\begin{aligned} &a + b + c + \cdots \\ &- (a^2 + ab + b^2 + ac + bc + c^2 + \cdots). \end{aligned}$$

Put

$$M = a + b + c + \cdots$$
$$N = a^2 + b^2 + c^2 + \cdots$$
$$P = ab + ac + bc + \cdots,$$

$$M = \frac{1}{2\cdot 6^3}\left[1\cdot 2 + 2\cdot 3\cdot\frac{5}{6} + 3\cdot 4\left(\frac{5}{6}\right)^2 + \cdots\right],$$

$$N = \frac{1}{4\cdot 36^3}\left[(1\cdot 2)^2 + (2\cdot 3)^2\frac{25}{36} + (3\cdot 4)^2\left(\frac{25}{36}\right)^2 + \cdots\right].$$

Let

$$f(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad (|x| < 1);$$

then

$$f''(x) = 1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + \cdots = \frac{2}{(1-x)^3}.$$

Hence putting $x = \frac{5}{6}$ we have $M = \frac{1}{2\cdot 6^3}\cdot\frac{2}{(1/6)^3} = 1.$

$$x^2f''(x) = 1\cdot 2x^2 + 2\cdot 3x^3 + 3\cdot 4x^4 + \cdots.$$

Differentiating twice,

$$2f''(x) + 4xf'''(x) + x^2f^{\text{iv}}(x) = (1\cdot 2)^2 + (2\cdot 3)^2x + (3\cdot 4)^2x^2 + \cdots.$$

This equals

$$2\frac{2}{(1-x)^3} + 4x\frac{6}{(1-x)^4} + x^2\frac{24}{(1-x)^5} = \frac{4}{(1-x)^5}(1 + 4x + x^2).$$

Hence putting $x = \frac{25}{36}$ we have

$$N = \frac{1}{4\cdot 36^3}\cdot\frac{4\cdot 36^5}{11^5}\left(1 + 4\frac{25}{36} + \frac{25^2}{36^2}\right) = \frac{36^2 + 4\cdot 25\cdot 36 + 25^2}{11^5} = \frac{5521}{11^5}.$$

$$M^2 = N + 2P = 1;$$

hence

$$P = \frac{1}{2} - \frac{5521}{2\cdot 11^5}.$$

Now the probability that A wins is $1 - P$ and the probability that B wins is $1 - N - P$. and so we may say, the probability that A wins is $N + P$ and the probability that B wins is P; These probabilities are

$$\text{for } A \quad \frac{1}{2} + \frac{5521}{2\cdot 11^5},$$

$$\text{" } B \quad \frac{1}{2} - \frac{5521}{2\cdot 11^5}.$$

All of these series are absolutely convergent power series, the parentheses may be removed and the terms rearranged as shown in the expressions for the two probabilities, and M may be squared to give the relation $M^2 = N + 2P$.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

Miss JESSIE M. SHORT, formerly of Carleton College but recently with the National Workmen's Compensation Service Bureau of New York, has been appointed instructor in mathematics at Reed College.

R. A. WELLS, dean and professor of mathematics at Park College, Missouri, has been appointed associate professor of mathematics in the Michigan State Normal College.

Dr. H. F. MACNEISH, instructor in mathematics at the College of the City of New York, has been promoted to an assistant professorship.

Mr. B. D. ROBERTS has been appointed professor of mathematics at Parsons College, Fairfield, Iowa.

Instructor C. D. EHRMAN, of the University of Wisconsin (*1920*, 88), has been appointed assistant professor of mathematics at Richmond College.

Mr. A. S. ADAMS, Mr. T. H. JOHNSON, and Mr. W. L. LUCAS have been appointed instructors in mathematics at the University of Maine.

Professor I. L. MILLER, of Carthage College, Illinois, has been appointed associate professor of mathematics at the South Dakota State College.

Miss FRANCES B. HATCHER has been appointed associate professor of mathematics at Westhampton College, Richmond, Va.

S. L. BOOTHROYD, associate professor of mathematics and astronomy at the University of Washington, is spending the year at the Lick Observatory. He has been appointed to succeed Professor O. M. LELAND as professor of astronomy and geodesy at Cornell University and his duties are to commence in September, 1921.

Assistant Professor HILLEL HALPERIN, of the University of Arkansas, has been appointed to an associate professorship at the Texas Agricultural and Mechanical College.

Dr. TOBIAS DANTZIG, of Johns Hopkins University, has resigned to undertake mathematical research for an engineering company in New York City.

Popular Astronomy announces that at Carleton College, Dr. H. C. WILSON has been relieved for the present college year of his duties as professor of mathematics and astronomy and expects to spend part of the time in research at Mt. Wilson Observatory. Dr. E. A. FATH, formerly of Mt. Wilson Observatory, and later professor of astronomy in Beloit College, has been appointed in his place; he is also associated with the editorial work of *Popular Astronomy*.

At Purdue University, Assistant Professor W. A. ZEHRING has been promoted to an associate professorship, instructors C. T. HAZARD, C. K. ROBBINS, and Dr. G. H. GRAVES have been promoted to assistant professorships, and Dr. E. M. BERRY, W. H. FRAZIER, W. R. HARDMAN, and J. A. NEEDY have been appointed instructors.

At the University of North Carolina, Professor WILLIAM CAIN has been retired from active teaching after thirty-one years of service, and Professor ARCHIBALD HENDERSON (*1919*, 165, 208) succeeds him as head of the department of mathematics; Assistant Professors A. W. HOBBS and J. W. LASLEY, Jr., have been promoted to associate professorships.

At the University of Kentucky, Mr. W. E. PAYNE has been appointed instructor in mathematics.

Professor D. A. ROTHROCK has been elected dean of the College of liberal arts of Indiana University.

Mr. H. H. CONWELL, associate professor of mathematics in the University of Idaho, has resigned to accept a similar position in Beloit College.

Professor C. B. RIDGAWAY, head of the department of mathematics at the University of Wyoming, has retired after twenty-four years of service.

Professor J. M. TAYLOR, of Colgate University, [*1920*, 91] has retired after completing fifty-one years as teacher of mathematics in that institution.

In the department of mathematics of the University of Michigan, the following promotions are announced: Professors W. H. BUTTS and T. R. RUNNING to full professorships; Professors C. E. LOVE and T. H. HILDEBRANDT to associate professorships; Dr. L. J. ROUSE, Dr. A. L. NELSON, Dr. W. W. DENTON and Dr. R. B. ROBBINS to assistant professorships. Dr. ROBBINS has been granted leave of absence for the year, Mr. NORMAN ANNING, Mr. S. E. FIELD, Mr. K. W. HALBERT, Mr. G. D. JONES, Mr. J. N. LANDIS, and Mr. H. A. SIMMONS have been appointed instructors.

WILLIAM RINCK, professor of mathematics and registrar of Theological School and Calvin College, Grand Rapids, Michigan, and his eleven year old son were instantly killed by the overturning of an automobile on November 11, 1920. Professor Rinck was born October 6, 1877, at Rotterdam, Netherlands. While yet in his early teens America became his home. He was a graduate student in mathematics at the University of Michigan, 1900–1904, and at the University of Chicago, 1905–1906. He had been connected with Calvin College since that time.

Professor E. W. STANTON, fifty years a member of the teaching staff of Iowa State College, died September 12, 1920, at the age of seventy years.

Dr. K. F. W. ROHN, professor of mathematics at the University of Leipzig since 1905, died on August 4, 1920, aged sixty-five years. He is, perhaps, best known to mathematicians through the three volume work (in its third edition, 1906) entitled *Lehrbuch der darstellenden Geometrie* (1893–1896; fourth edition, 1913–1916), which he prepared, after the appearance of the first volume, in collaboration with Dr. E. Papperitz.

Dr. R. H. WEBER, son of the late Heinrich M. Weber (affectionately remembered by American mathematicians), and ordinary honorary professor of applied mathematics at the University of Rostock, died in August, 1920, aged forty-six years. He was appointed extraordinary professor at the University of Rostock in 1907. It will be recalled that he was the author of the sections in Weber and Wellstein, *Encyklopädie der Elementar-Mathematik,* dealing with "Analytische Statik," "Dynamik," and "Elektrische und magnetische Kraftlinien."

ARTHUR SEARLE, Phillips professor of astronomy, emeritus, at Harvard University since 1912, died on October 24, aged eighty-three years. An Englishman by birth but brought to this country at an early age, he and his brother George Mary, also an astronomer (1839–1918), graduated from Harvard within a year of one another. He was assistant at the Harvard Observatory 1869–83, assistant professor of astronomy 1883–87, and Phillips professor of astronomy

1887–1912. He wrote numerous astronomical papers, and a book entitled *Outlines of Astronomy*, 1874 (second edition, 1875); most of several volumes of the *Annals of the Harvard Observatory* (8, 1876; 62, parts 1, 2, 1907, 1911; 65–66, parts 1910; 67; 1912; 77, 1914) were prepared under his direction or written by him. In his little volume *Essays I–XXX*, published in 1910, the discussions are philosophical; essay IV deals with "Consciousness," essay VIII with "Mind and matter," essays XII–XVII with "Space and time," and essay XXX with "Mental diversities."

In addition to the previously listed [*1920*, 440] sixteen doctorates with mathematics as a major subject, conferred by American universities during the academic year 1919–1920, the following should be listed: H. R. BRAHANA, Princeton, "Curves on surfaces"; E. S. HAMMOND, Princeton, "Periodic conjugate nets of curves."

At the meeting of the American Mathematical Society, held at Columbia University on October 30, 1920, twelve papers were presented, the following speakers being also members of the Association: A. A. BENNETT, R. L. BORGER, O. E. GLENN, T. H. GRONWALL, D. JACKSON, E. KASNER, ELIZABETH LESTOURGEON, J. LIPKA, H. S. VANDIVER.

At the meeting of the National Academy of Science, at Princeton University on November 17, the following papers were read by members of the Association: "Equipartition of Energy" by E. B. WILSON; "Einstein gravitational fields: orbits and light rays" by E. KASNER; "Knots and Riemann spaces" by J. W. ALEXANDER; "The map coloring problem" by PHILIP FRANKLIN.

At the meeting of the mathematics section of the Virginia State Teachers' Association at Richmond, Va., on November 24, 1920, T. McN. SIMPSON, professor of mathematics at Randolph-Macon College, was elected president for the ensuing year. He delivered a paper on "Mathematics the common denominator of the sciences" which was published in the *Texas Mathematics Teachers' Bulletin*, November, 1920.

At the eighteenth annual meeting of The Association of Teachers of Mathematics in New England, held at the Massachusetts Institute of Technology, December 4, 1920, members of the Association gave the following addresses: "The mathematics of insurance" by Professor C. H. CURRIER; "Graphical methods of computation" by Professor JOSEPH LIPKA.

The Society of American Field Service Fellowships for French Universities offers fellowships for 1921–22, not to exceed twenty-five in number, and each of the value of $200 plus 10,000 francs. The awards are to be made early in 1921. Further details may be obtained on application to the secretary, Dr. I. L. Kandel, 522 Fifth Avenue, New York City.

A Royal medal has been awarded by the Royal Society to Professor G. H. Hardy, of Oxford University, for his researches in pure mathematics, particularly in the analytical theory of numbers and allied subjects.

Professor Adolf Hurwitz's mathematical library, containing 420 volumes of mathematical works, 370 volumes of periodicals, and 5000 reprints, is offered for sale. Those interested may see a detailed catalogue on applying to Mrs. Hurwitz, Bächtoldstrasse, Zürich, or to the editors of *Acta Mathematica*, Stockholm.

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THE NOVEMBER MEETING OF THE MISSOURI SECTION.

The fourth regular meeting of the Missouri Section was held at the Kansas City Junior College, Kansas City, Missouri, on November 13. The single session was held in the morning, the chairman being Professor W. A. LUBY.

The attendance was twenty-four, including the following ten members of the Association:

A. C. Andrews, Minnie W. Caldwell, A. Davis, B. F. Finkel, R. R. Fleet, E. R. Hedrick, W. A. Luby, P. R. Rider, P. Robertson, Eula A. Weeks.

Professor E. R. Hedrick, chairman of the committee appointed[1] by the Missouri Section of the Association to consider the report of the National Committee, presented to the Section the report of the National Committee concerning the junior high schools. The opinions of the committee were, in general, favorable to the recommendations made by the National Committee but a considerable number of detailed remarks and suggestions were made which will be transmitted to the National Committee for their consideration. The only one of these which is far-reaching enough to deserve mention here is the recommendation to the National Committee that the work in demonstrative geometry ought not to be included in the junior high school. The committee was entirely in favor of leading up to demonstrative geometry but felt the work in the junior high schools should stop short of actual demonstrative work.

Mr. Davis stated to the meeting the objects of the National Council of Teachers of Mathematics organized at Cleveland, Ohio, February 24, 1920. The Section voted to become an institutional member of the Council.

The following new officers were elected: Chairman, Professor LOUIS INGOLD; Vice-chairman, Professor R. R. FLEET.

In accordance with previous arrangements the next regular meeting of the Section will be held at the University of Missouri, November 25–26, 1921, in connection with the meeting of the southwestern section of the American Mathematical Society.

The following four papers were read:

(1) "The relation of caustics to certain envelopes" by Professor O. DUNKEL;

(2) "A so-called Russian multiplication method" by Professor P. R. RIDER;

(3) "Sun-spot data and the methods of analysis applied" by Dr. D. ALTER, associate professor of astronomy, University of Kansas (invited);

(4) "The work of the National Committee on Mathematical Requirements" by Dr. EULA A. WEEKS.

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[1] The other members of the committee are: Professor W. H. Zeigel, Professor R. R. Fleet, Miss Zoe Ferguson, Mr. Alfred Davis, Mr. Percival Robertson.

In the absence of the author, the paper by Professor Dunkel was read by title only. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

(1) The object of this paper was to show how certain problems concerning the envelopes of circles may be easily solved without the calculus, and the relation of such envelopes to caustics. By aid of the caustic other facts may be obtained without the calculus such as the radius of curvature and an expression for the length of arc of the envelope.

(2) This paper gave an explanation of a curious multiplication method reputed to be used by Russian peasants. [Cf. this MONTHLY, *1918*, 139.]

(3) Sun-spot data are typical of a great mass of physical data, of which intensities are observed and plotted as ordinates against the time. The same methods of analysis apply to all. The mathematical theory of these methods is very simple.

This paper reviewed the work of Schwabe, Wolf and Wolfer, Newcomb, Shuster, Larmor and Yamaga, Lockyer, Clough, Dagobert and Turner very briefly. The main part was devoted to an exposition of the methods of Shuster and Turner.

The conclusion of the paper made reference to a new method of attack developed by the author in connection with a closely related problem.

(4) This report by a member of the National Committee summarized the work that has been done by the committee and outlined its plans for the future.

PAUL R. RIDER, *Secretary-Treasurer.*

ON THE CONSTRUCTION AND MODELLING OF ALGEBRAIC SURFACES.

By ARNOLD EMCH, University of Illinois.

1. **Introduction.** In a paper on a simple method of curve tracing, which appeared in this MONTHLY (*1917*, 168–172), I have shown how the method of generating curves by projective pencils could be utilized very effectively in many cases of graphic representation in which the customary method of plotting by computation from the equation is unpracticable, or, to say the least, very tedious.

It is obvious to anyone who is thoroughly familiar with all phases of curve tracing that, whenever practicable, the projective, or more generally the geometrographic method, which is either purely constructive, or makes use of a minimum of arithmetical work, is preferable to the method of plotting from the equation. Moreover the graphic method reveals at least something of the curve as a geometric organism, while the plot of the equation merely shows the form of the curve without any inner geometric content.

The graphic method should, of course, not reject taking into account those geometric properties which are more readily and rigorously revealed by an analytical discussion of the equation.

The same remarks apply to the construction and representation of surfaces which are defined either geometrically, or by their equations. In fact, there is no doubt about the superiority of the graphic method when a model of the surface is required.

In analogy with the method used in my paper on curve tracing referred to above, I shall make use of the theorem that every algebraic surface may be generated by two projective pencils of certain surfaces of lower order. If the equation of the surface is given, we may put it in the form

$$(1) \qquad P \cdot S - Q \cdot R = 0,$$

in which P, Q, R, S are polynomials representing surfaces of lower order. The surface may now be generated by either of the two sets of projective pencils

$$(2) \qquad \begin{aligned} P + \lambda Q &= 0, \\ R + \lambda S &= 0, \end{aligned}$$

$$(3) \qquad \begin{aligned} P + \lambda R &= 0, \\ Q + \lambda S &= 0. \end{aligned}$$

For every value of the parameter λ the corresponding surfaces of the two projective pencils of a set intersect in a generatrix curve of the given surface.

The effectiveness of the projective method for the construction and modelling of surfaces, when their equations are known, lies in the proper choice of the polynomials P, Q, R, S. In the following examples the applicability of this method will be demonstrated.

2. **Construction of a quintic surface with a given quartic nodal curve.** Let the quartic nodal curve D be defined as the intersection of the sphere

$$(1) \qquad x^2 + y^2 + z^2 - r^2 = 0$$

with the cylinder

$$(2) \qquad x^2 + y^2 - rx = 0,$$

so that D has an ordinary node at $U(r, 0, 0)$, Fig. 1. In this figure only the portion of D below the xy-plane is shown; it projects as a parabolic arc upon the xz-plane.

The equation

$$(3) \qquad r(x^2 + y^2 + z^2 - r^2)^2 - z(x^2 + y^2 - rx)^2 = 0$$

evidently represents a quintic surface S_5 with D as a nodal curve.

The problem is to construct S_5. With λ as a variable parameter the quintic may be generated by the two projective pencils

$$(4) \qquad \sqrt{r}(x^2 + y^2 + z^2 - r^2) + \lambda(x^2 + y^2 - rx) = 0,$$

$$(5) \qquad (x^2 + y^2 - rx)z + \lambda\sqrt{r}(x^2 + y^2 + z^2 - r^2) = 0,$$

of quadrics and cubics, respectively. In fact by eliminating λ between (4) and (5), we get (3). All quadrics (4) and cubics (5) pass through D. For every value of λ the quadric (4) and the cubic (5) intersect in a generatrix curve C^* of S_5 of the sixth order. But as D is common to the quadric and cubic, C^* degen-

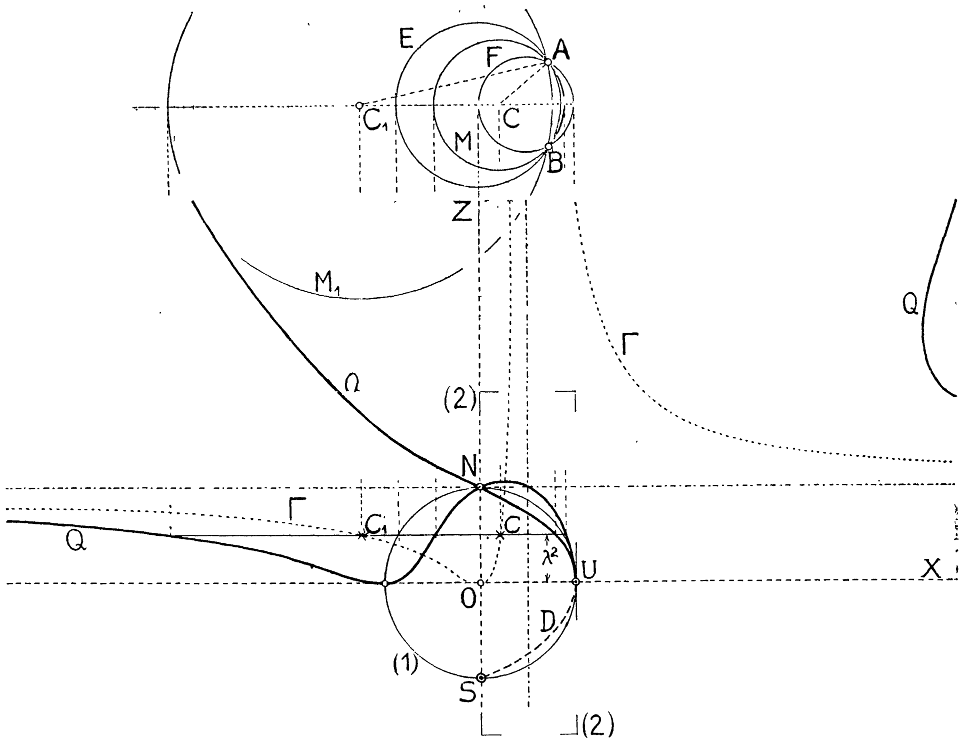

FIG. 1.

erates into D and a rest-curve M of order 2. To determine M, multiply (4) by λ and subtract from (5). After dividing the difference by the extraneous factor $x^2 + y^2 - rx$, the equation

$$(6) \qquad z - \lambda^2 = 0$$

remains which represents a plane parallel to the xy-plane. Thus M lies in this plane, and its equation is obtained by eliminating z between (6) and either (4) or (5). After some reductions this equation may be written in the form

$$(7) \qquad \left(x - \frac{\lambda r}{2(\sqrt{r}+\lambda)}\right)^2 + y^2 = \frac{4\sqrt{r}(\sqrt{r}+\lambda)(r^2-\lambda^4)+\lambda^2 r^2}{4(\sqrt{r}+\lambda)^2}.$$

The rest curve M is therefore a circle in the plane $z = \lambda^2$. By letting λ vary, the circle M generates S_5. But there is another circle M_1 in the same plane, which is obtained by replacing in (7) λ by $-\lambda$, so that this circle M_1 is determined by

(8) $$\left(x + \frac{\lambda r}{2(\sqrt{r} - \lambda)}\right)^2 + y^2 = \frac{4\sqrt{r}(\sqrt{r} - \lambda)(r^2 - \lambda^4) + \lambda^2 r^2}{4(\sqrt{r} - \lambda)^2}.$$

The centers of these circles lie on a rational cubic Γ in the xz-plane with the parametric equations

(9) $$x = \frac{\pm \lambda r}{2(\sqrt{r} \pm \lambda)}, \qquad z = \lambda^2,$$

or, with the Cartesian equation

(10) $$z = \left(\frac{2\sqrt{r}\cdot x}{r - 2x}\right)^2.$$

The radii ρ of the circles $M(+\lambda)$ and $M_1(-\lambda)$ are determined by

(11) $$\rho = \frac{\sqrt{4\sqrt{r}(\sqrt{r} \pm \lambda)(r^2 - \lambda^4) + \lambda^2 r^2}}{2(\sqrt{r} \pm \lambda)}.$$

By means of formulas (9) or (10), and (11), and the planes $z = \lambda^2$, it is a simple matter to determine and locate these circles, and, consequently, to construct and model the surface. To exhibit the two sheets of the surface through D, the circles M and M_1 may be traced on equidistant glass-plates (as transparent as possible). In a plaster model some portion of the surface would be hidden from view. Computations may be considerably reduced by making a graph of Γ (10) in the xz-plane. The plane $z = \lambda^2$ intersects Γ in two points C and C_1, the centers of M and M_1, and the quartic D in two points A and B. CA and C_1A are the radii of M and M_1. A and B are the intersections of the circles E and F in which the plane $z = \lambda^2$ cuts the sphere (1) and the cylinder (2). By this simple construction, which is shown in horizontal projection in the upper portion of Fig. 1, the computation for the complicated expression for ρ may be avoided for all values $\lambda^2 \leqq r$.

From the construction as outlined above, we may merely conjecture as to the singularities of the surface. However, in order to complete the model properly, these must be determined definitely by discussing the equation of the surface with reference to its singularities. In the first place, there are evidently no real points of the surface below the xy-plane except those on the loops of D. The surface S_5 presents therefore the peculiarity that it contains as a part an isolated real branch of a curve which does not lie on the real film of the surface. The plane $z = 0$ cuts S_5 in the curve $(x^2 + y^2 - r^2)^2 = 0$ and is therefore a trope[1] of the surface; *i.e.*, S_5 touches the xy-plane along the circle $x^2 + y^2 - r^2 = 0$. We may furthermore expect a singularity at the highest point $N(0, 0, r)$ of D. To determine its nature we transfer the origin to N, substitute $z' + r$ for z in (3), make the equation homogeneous, (replacing x, y, z' by $x/t, y/t, z'/t$, and multiplying through by t^5), and write the resulting equation in descending powers of t.

[1] Basset, *Treatise on the geometry of surfaces*, London, 1910, pp. 23–25.

The factor multiplying the highest power of t represents the nodal cone at N, if it exists. Treating (3) in this manner we get

$$t^3(4r^3z'^2 - r^3x^2) + \cdots = 0,$$

which shows that N is a binode with the biplanes

$$z - r = \pm \tfrac{1}{2}x.$$

Another singularity may be expected at the node $U(r, 0, 0)$ of the quartic D. Replacing x by $x' + r$ in (3) and proceeding as in case of the binode N, the equation becomes

$$t^3 \cdot 4r^2x'^2 + \cdots = 0,$$

which shows that U is an unode with $x = r$ as the uniplane. In a similar manner we find that S is a binode with imaginary biplanes.[1]

Figure 1, Q, shows the quintic cross-section of the surface with the xz-plane, its equation being

$$r(x^2 + z^2 - r^2)^2 - z(x^2 - rx)^2 = 0. \tag{12}$$

It has the line $z = r$ as an asymptote; N is an ordinary node, U a cusp. The dotted curve is the locus Γ of the centers of circles M and M_1. It has the lines $x = r/2$ and $z = r$ as asymptotes.

Another important question is whether it is possible that any of the circles M may degenerate into point circles. This will be the case when the expression for ρ vanishes, *i.e.*, when

$$4\sqrt{r}(\sqrt{r} \pm \lambda)(r^2 - \lambda^4) + \lambda^2r^2 = 0.$$

Introducing $\lambda^2 = z$ this equation reduces to

$$16z^5 - 16rz^4 - 24r^2z^3 + 31r^3z^2 + 8r^4z - 16r^5 = 0. \tag{13}$$

There are therefore 5 such circles. But only one of these is real, which is determined by the root

$$z = (1.0635 \cdots)r. \tag{14}$$

The abscissa of this point circle is $x = (0.255 \cdots)r$. The point circle is at the intersection of Γ with Q.

3. **Construction of a cubic cyclide.** Every cyclide may be generated by two projective pencils of spheres (of which one may degenerate into a pencil of planes). In case of a pencil of spheres and a projective pencil of planes the surface generated is a cubic cyclide. Making use of the fact that a pencil of spheres is projective to the point set formed by the corresponding centers of these spheres, it is very easy to generate a cubic cylide which may be constructed and modelled without difficulty.

---

[1] Basset, loc. cit., pp. 20–23.

Choose the line m ($y = 0$, $z = -c$) as the locus of the centers of the spheres (Fig. 2). The pencil shall be determined by the condition that its spheres S shall pass through the circle $(z + c)^2 + y^2 = c^2$ in the yz-plane. Let the centers M of S be determined by the parameter $M_0M = \lambda$, so that the equation of S becomes

$$x^2 + y^2 + z^2 + 2cz - 2\lambda x = 0. \tag{15}$$

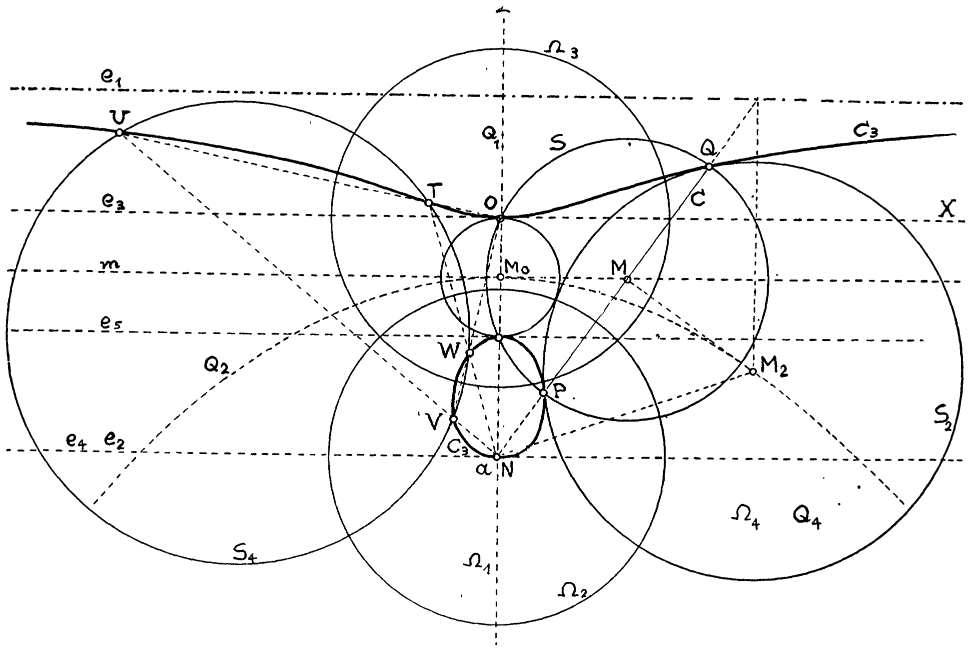

FIG. 2.

As the axis a of the projective pencil of planes choose the line $x = 0$, $z = -1/(2c)$, so that the pencil of planes through a and the centers of (15) becomes

$$(2c^2 - 1)x + \lambda(2cz + 1) = 0. \tag{16}$$

For the same value of λ a plane (16) and a sphere (15) intersect in a circle C which generates the cyclide when λ varies. Eliminating λ between (15) and (16), we get for the equation of the cyclide

$$2cz(x^2 + y^2 + z^2) + (4c^2 - 1)x^2 + y^2 + (4c^2 + 1)z^2 + 2cz = 0. \tag{17}$$

The construction of this cyclide is extremely simple. In Fig. 2 the circles C project as straight segments, like PQ, whose prolongation passes through N. The same figure also shows the cubic cross-section C_3 of the cyclide with the xz-plane. Fig. 3 is a picture of a plaster model of the cyclide.[1]

[1] There seems to be no model of a cubic cyclide (which is not a cubic Dupin cyclide) in the market. The model represented by the figure was constructed by the author.

The well-known properties of the cyclide may easily be shown on this surface. From equation (17) (making it homogeneous) follows that the infinite line i of the xy-plane lies on the cyclide. There must therefore exist five pairs of lines on the cyclide which cut i, *i.e.*, which are parallel to the xy-plane. In order to determine these, cut the cyclide by a plane $z=e$. As this plane passes through i, the rest curve of intersection with the cyclide is the conic

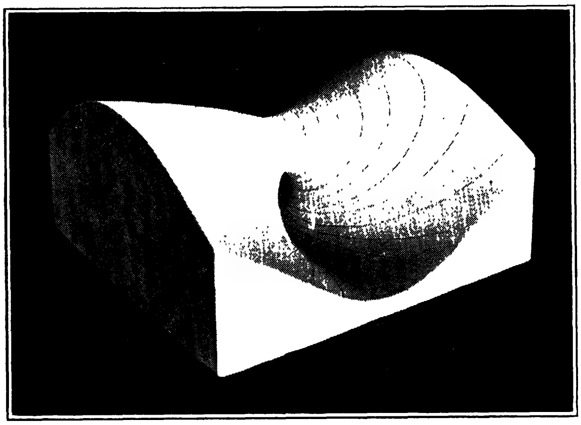

FIG. 3.

$$2ce(x^2 + y^2 + e^2) + (4c^2 - 1)x^2 + y^2 + (4c^2 + 1)e^2 + 2ce = 0,$$

or

$$(2ce + 4c^2 - 1)x^2 + (2ce + 1)y^2 + 2ce^3 + (4c^2 + 1)e^2 + 2ce = 0.$$

This conic degenerates into a pair of lines when its discriminant vanishes, *i.e.*,

$$\begin{vmatrix} 2ce + 4c^2 - 1 & 0 & 0 \\ 0 & 2ce + 1 & 0 \\ 0 & 0 & 2ce^3 + (4c^2 + 1)e^2 + 2ce \end{vmatrix} = 0,$$

or when either of the three equations $2ce + 4c^2 - 1 = 0$, $2ce + 1 = 0$, $2ce^3 + (4c^2 + 1)e^2 + 2ce = 0$ is satisfied. This gives for e the five roots

$$e_1 = \frac{1 - 4c^2}{2c}, \quad e_2 = -\frac{1}{2c}, \quad e_3 = 0, \quad e_4 = -\frac{1}{2c}, \quad e_5 = -2c.$$

Each of the five planes $z = e_i$; $i = 1, 2, 3, 4, 5$ cuts the cyclide in a pair of lines. Those in $z = e_1$ are imaginary and parallel to the x-axis. In fact the line $y = 0$, $z = (1/2c) - 2c$, is an asymptote of C_3, hence $z = e_1$ is the tangent plane to the cyclide at the infinite point of this asymptote. The line $x = 0$, $z = -(1/2c)$, the axis a, must be counted as two pairs. The other two real pairs are easily obtained as

$$e_3\begin{cases} z = 0, & \sqrt{1 - 4c^2}\cdot x + y = 0, \\ z = 0, & \sqrt{1 - 4c^2}\cdot x - y = 0, \end{cases}$$

$$e_5\begin{cases} z = -2c, & x + \sqrt{1 - 4c^2}\cdot y = 0, \\ z = -2c, & x - \sqrt{1 - 4c^2}\cdot y = 0, \end{cases}$$

and obviously cross each other orthogonally.

The two pencils of planes through each of these pairs of lines cut the cyclide in a system of circles which belong to one of the five systems on the cyclide. These circles are, in general, cut out by double-tangent spheres which cut a fixed sphere (plane) Ω orthogonally, and whose centers lie on a quadric (conic) Q. The cyclide is *self-inverse* (for which Moutard[1] has proposed the bizarre word "anallagmatic,"[2] which is now in general use) with respect to each of the spheres Ω. The five spheres Ω_i, $i = 1, 2, 3, 4, 5$, are mutually orthogonal and the quadrics Q are confocal. They may easily be determined by well-known methods.[3] For the sake of brevity I shall merely state the results:

$$z = e_1, \qquad \Omega_1 \equiv x = 0,$$

$$z = e_2, \qquad \Omega_2 \equiv x^2 + y^2 + \left(z + \frac{1}{2c}\right)^2 - \frac{1 - 4c^2}{4c^2} = 0,$$

$$z = e_3, \qquad \Omega_3 \equiv x^2 + y^2 + z^2 - 1 = 0,$$

$$z = e_4, \qquad \Omega_4 \equiv y = 0,$$

$$z = e_5, \qquad \Omega_5 \equiv x^2 + y^2 + (z + 2c)^2 + 1 = 0,$$

$$Q_1 \equiv x = 0 \text{ (plane)},$$

$$Q_2 \equiv cx^2 + 2(1 - 4c^2)(z + c) = 0 \text{ (parabola)},$$

$$Q_3 \equiv \frac{4c^2}{1 - 4c^2}x^2 - 4c^2y^2 + 4cz + 1 + 4c^2 = 0 \text{ (hyperbolic paraboloid)},$$

$$Q_4 \equiv y = 0 \text{ (plane)},$$

$$Q_5 \equiv 4c^2x^2 - \frac{4c^2}{1 - 4c^2}y^2 + 4cz + 1 = 0 \text{ (hyperbolic paraboloid)}.$$

For each of the five systems, the planes containing the circles of the system envelope quadric cones which have their vertices at the centers of the spheres Ω_i. In case of the cubic cyclide these cones degenerate into couples of pencils of planes through the corresponding pairs of lines on the cyclide. The spheres S_2, orthogonal to Ω_2 (and also to Ω_4), with their centers on the parabola Q_2, touch the cyclide along the circles C. The directrix of Q_2 is the line e_1, and its focus is N. The spheres S_4 orthogonal to Ω_4, with their centers on Q_4 (but not on Q_2) touch the cyclide in imaginary points. All spheres S_4 are also orthogonal to Ω_2. Among the ∞^2 S_4's there are ∞^1 spheres like the one shown in the figure, which are also orthogonal to Ω_3. Every sphere of this kind cuts the cubic C_3 in points T, U, V, W which, together with O and N, form a complete inscribed quadrilateral of the cubic. That there is a singly infinite number of such quadrilaterals through

[1] *Nouvelles Annales de Mathématiques*, vol. 22, 1864, pp. 306–309.

[2] (ἀ privatif, ἀλλασσω, je change.)

[3] Darboux, *Géométrie Analytique*, Paris, 1917, pp. 405–433, where a very clear and elementary account of the theory of cyclides in Cartesian coördinates is given.

O and N follows independently also from the fact that O and N form a *Steinerian couple*[1] on the cubic, *i.e.*, the points of tangency of two tangents from a point on the cubic (in this case the infinite point of C_3) to the same cubic. Like the cyclide itself, the cubic C_3 is anallagmatic with respect to O and N. In fact

$$OT\cdot OU = OW\cdot OV = 1; \qquad NV\cdot NU = NW\cdot NT = \frac{1-4c^2}{4c^2}.$$

In case of Ω_1 and Ω_4 the anallagmatic property reduces to symmetry with respect to these planes. In case of Ω_5 the anallagmatic constant (radius square of Ω_5) is -1.

A CURVE OF PURSUIT.

By F. V. MORLEY, New College, Oxford University.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The curve of pursuit is one of that class of problems so entertainingly described by Professor David Eugene Smith, which in their travel through the centuries have preserved traces of the times of their proposers. The problem in one dimension, of the pursuer following the pursued in line, is common since the time of Zeno's paradox[2]; but the curve of pursuit does not seem to have been studied till the 18th century. An attempt has been made to make Leonardo da Vinci responsible, among his other wealth of contributions, for the statement of the problem.[3] But although it is quite possible to read into Leonardo's passage the essence of the question, it is perhaps doubtful that he ever had a conscious formulation. And of necessity, careful consideration of the problem had to wait until the methods of the calculus were known.

At any rate, the problem of the curve of pursuit was stated by Bouguer in 1732.[4] Although the days of the buccaneers were numbered, it is characteristic of the times that he chose for his example a privateer and a merchant vessel. Bouguer considered only the simplest case, where the pursued point moves along a line, but in the same volume of the *Mémoires de l'Académie Royale des Sciences* is a generalization of the problem by the remarkable de Maupertuis. Since then the problem in various guises has appeared in texts and periodicals.[5] One simple variant, in which the pursued point moves along a circle and the pursuer starts from the center, was re-proposed by Professor A. S. Hathaway in this MONTHLY, *1920*, 31. It is this case which is considered in this paper.

[1] For the theory of Steinerian couples and quadruples on plane cubics see the author's *Introduction to Projective Geometry*, New York, 1905, pp. 197–204.

[2] See D. E. Smith, AMER. MATH. MONTHLY, Vol. 24, 1917, p. 64.

[3] Brocard, *Nouv. Corr. Math.*, Vol. 6, 1880, p. 211; cf. Loria, *Ebene Kurven*, 1902, p. 608.

[4] *Mémoires de l'Académie Royale des Sciences*, 1732.

[5] *E.g.*, *Math. Monthly* (ed. J. D. Runkle), 1, 1859, p. 249. There are also more elaborate papers such as "Sur les courbes de poursuite d'un cercle," by M. L. Dunoyer, *Nouv. Annales de Math.*, 4th series, Vol. 6, 1906, p. 193. [Compare page 91 of this issue.—EDITOR].

The problem suggested by Dr. Hathaway is illustrated in Fig. 1. A duck is swimming with constant speed around the edge of a circular pond; a dog starts from the center and swims always directly towards the duck, c (a constant) times as fast. Suppose that the radius of the pond is unity, and that at any instant the duck has traveled over an arc θ from its starting point; then the dog at the same instant will have traveled a distance s along its curved path, where

$$s = c\theta.$$

If as shown in the figure we use p and ω as normal coördinates of the line joining dog and duck (which is always tangent to the dog's path), the radius of curvature of the dog's path will be

$$p + \frac{d^2p}{d\omega^2} = \frac{ds}{d\omega} = c\frac{d\theta}{d\omega}.$$

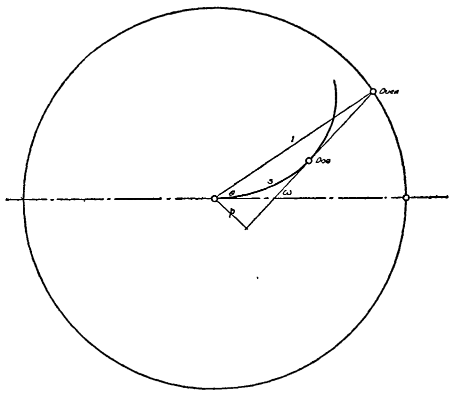

FIG. 4.

Now from the figure

$$p = \sin(\omega - \theta)$$

or

$$\theta - \omega = -\sin^{-1} p,$$

so that

$$\frac{d\theta}{d\omega} = 1 - \frac{\frac{dp}{d\omega}}{\sqrt{1 - p^2}}.$$

The differential equation then becomes

$$p + \frac{d^2p}{d\omega^2} = c\left[1 - \frac{\frac{dp}{d\omega}}{\sqrt{1 - p^2}}\right].$$

In this we make the substitution $dp/d\omega = u$. Then

$$\frac{d^2p}{d\omega^2} = u\frac{du}{dp}$$

and

$$p + u\frac{du}{dp} = c\left[1 - \frac{u}{\sqrt{1 - p^2}}\right].$$

Now let $1 - u^2 - p^2 = 1 - r^2 = v$, where $-v$ is the *power* of the point with respect to the circle; then

$$\frac{dv}{dp} = -2\left[u\frac{du}{dp} + p\right]$$

and finally[1]

[1] The equations of the normal and tangent to the dog's path may be written

$$x\cos\omega + y\sin\omega = \cos(\omega - \theta) - \rho$$

and

$$x\sin\omega - y\cos\omega = \sin(\omega - \theta),$$

$$\left(\frac{dv}{dp}\right)^2 + 4c\frac{dv}{dp} + \frac{4c^2v}{1-p^2} = 0.$$

This is the differential equation giving the dog's path, innocent enough in looks, but apparently not integrable by the ordinary methods. An approximate solution may then be sought by integration in series, or by graphical methods. Let us consider the latter.

Writing the differential equation in the more familiar variables x and y and using these as rectangular coördinates, we have to discuss the following:

$$\left(\frac{dy}{dx}\right)^2 + 4c\frac{dy}{dx} + 4c^2\frac{y}{1-x^2} = 0.$$

In the first place, at every point of the plane two directions are determined by this equation; namely, the roots of the quadratic in dy/dx. The locus of all points for which either of these directions is constant and equal to m, is given by

$$m^2 + 4cm + 4c^2\frac{y}{1-x^2} = 0.$$

This is a family of parabolas determined by the parameter m, and all passing through the points $(\pm 1, 0)$.

The equation thus determines two families of integral curves, such that each member of each family cuts every parabola in a definite and determinable direction. In order to pick out the particular integral curves in which we are interested, let us notice the initial conditions of the problem. When the dog is at the center of the circle, $p = 0$ and $v = 1$, so that in the new notation we want the particular integral curves which pass through the point $x = 0, y = 1$. And what we want to find is the value of p (or x) when v (or y) is 0. We therefore wish to trace the integral curves from the point (0, 1) as far as the x-axis, and to determine their intercepts.

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where ρ denotes the distance between the dog and duck. These equations are satisfied by the coordinates of the dog's position and may be regarded as expressing these coordinates in terms of ω, θ and ρ.

If we differentiate and divide by $d\theta$, remembering that $dy = dx \tan\omega$, and $dx \sec\omega = ds = cd\theta$, we shall get equations which reduce to

$$\frac{d\rho}{d\theta} = \sin(\omega - \theta) - c \qquad (1)$$

and

$$\rho\frac{d\omega}{d\theta} = \cos(\omega - \theta). \qquad (2)$$

These are the equations derived by Professor Hathaway in his solution of Problem 2801 given on pages 93–97, θ, $\omega - \theta$, ρ and c being the same as his s, θ, r and k.

Now $v = \rho[2\cos(\omega - \theta) - \rho]$, and $p = \sin(\omega - \theta)$, and these equations by aid of (1) and (2) lead directly to Mr. Morley's differential equation in p and v.

Dunoyer (l. c.) also derives equations (1) and (2) as expressing the components of the dog's velocity in the direction of the duck and at right angles to this direction.—EDITOR.

Let us then draw a few of the family of parabolas, lying between the x-axis and the parabola through (0, 1). The parabola through (0, 1) will be

$$y = 1 - x^2$$

and the directions at every point of this parabola will be the roots of

$$m^2 + 4cm + 4c^2 = 0$$

namely,

$$m_1 = -2c, \qquad \text{and} \qquad m_2 = -2c.$$

For this parabola the directions coincide.

Another parabola would be

$$y = \tfrac{3}{4}(1 - x^2),$$

cutting the y-axis at $(0, \tfrac{3}{4})$. Here the directions are given by

$$m^2 + 4cm + 3c^2 = 0,$$

and are

$$m_1 = -3c, \qquad \text{and} \qquad m_2 = -c;$$

and so for as many parabolas as we care to draw. Finally, on the limiting parabola $y = 0$ the directions are given by

$$m^2 + 4cm = 0,$$

and are

$$m_1 = -4c, \qquad \text{and} \qquad m_2 = 0.$$

Now to solve any particular case we have to attach to each parabola directions according to the value of c, and then the two particular integral curves in which we are interested (one will go with m_1 and the other with m_2) may be plotted without difficulty and to a considerable degree of accuracy, using the methods given by Runge.[1]

Let us for instance in Fig. 2 construct the integral curves for $c = 3$. For the curve defined by m_1 we start by drawing a line from (0, 1) with a slope of -6. Mark a point on this line about half-way between the two outer parabolas, and through that point draw a line with a slope of -8, which is the direction appropriate to the second parabola. Repeat the process, using a point on this line half-way between the second and third parabolas, etc. These construction lines are not shown in Fig. 2, but by continuing the process an approximation to the integral curve for m_1 is obtained. This is dotted in the figure, and in this case cuts the x-axis at $x = 0.10$. The curve for m_2 is drawn by the same process, though here the first approximation may not be sufficiently accurate. A second approximation may be obtained by successive differentiation and integration by graphical methods, as in Runge. When drawn, as in the full line of Fig. 2, the curve for m_2 cuts the x-axis at $x = 1$.

[1] C. Runge, *Graphical Methods*, Columbia University Press, 1912, p. 120.

In order to see the meaning of these solutions, let us draw in Fig. 3 the actual curve of pursuit. This may be very accurately drawn by a bracketing method. That is, by marking a series of small equal steps for the duck around the circle; one curve may be drawn where the dog is successively directed towards the

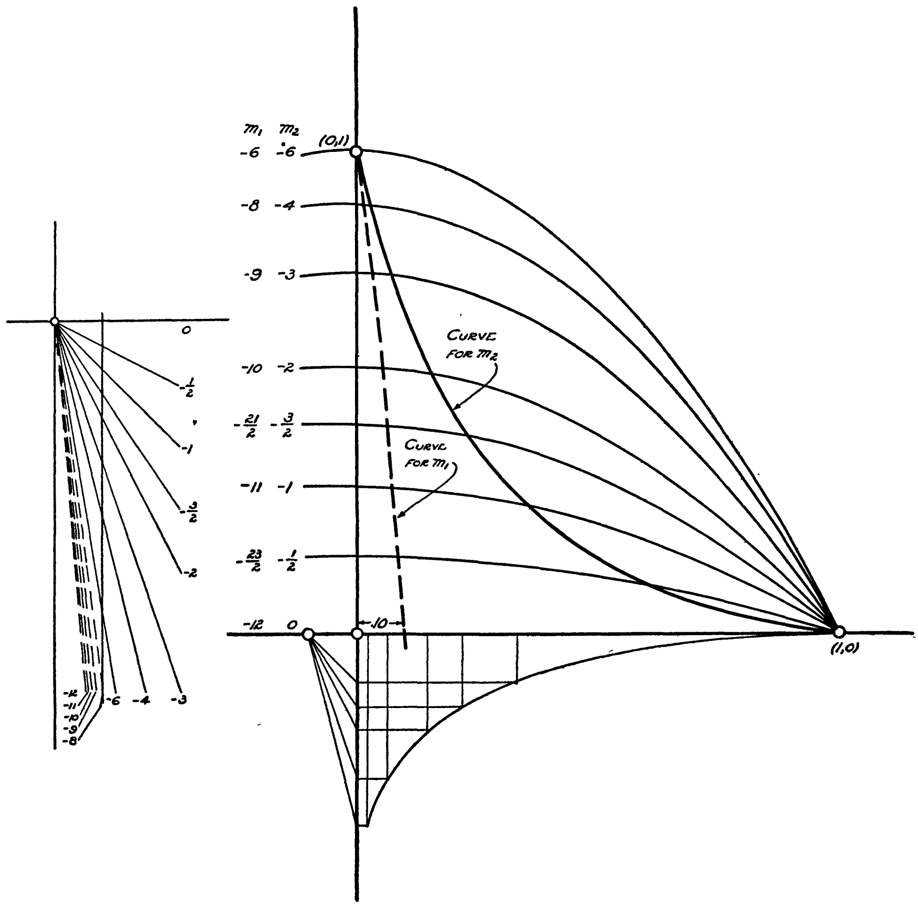

FIG. 2. Integral curves for $c = 3$.

beginning of each step, and another curve where the dog is successively directed towards the end of each step. This simple procedure is shown in exaggerated form in Fig. 4. The true curve of pursuit will lie between the two. When the true curve is drawn for the case of $c = 3$, it is seen that the value $x = 1$ when $y = 0$ means that the distance from the center to the tangent of the dog's path is 1 wherever the path cuts the circle. In other words, the dog comes up to the duck tangent to the circle. But if we had been tracing the dog's curve backwards from the center, as if he had been in flight instead of in pursuit, he would have reached the circle at a point where the tangent to his path is distant (by

measurement of Fig. 3) 0.093 from the center. The first approximation of the curve for m_1 thus checks reasonably with the curve of flight as drawn.

But let us notice there may be cases of difficulty in drawing the integral curves. In any case where c is less than 2, for instance, Runge's method breaks down, if followed blindly. If we follow the curve for m_2 we come to a point on one of the parabolas where the slope of the parabola is itself equal to m_2. The integral curve cannot there simply cross the parabola, yet it cannot turn upward. The presumption is that there the curve has a flex, and this we shall have to test.

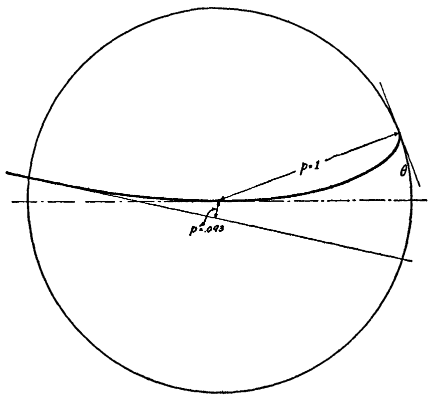

FIG. 3. Curve of pursuit, $c = 3$, $\theta = .058$.

First let us find the locus of those points where the direction assigned by the differential equation is the same as the slope of the parabola through that point. Call for brevity

$$\mu = \frac{y}{1 - x^2}.$$

The slope of the parabola is

$$\frac{dy}{dx} = -2\mu x = m.$$

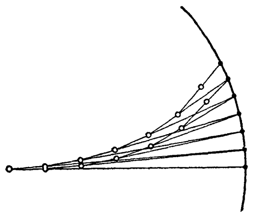

FIG. 4. Details of bracketing method for drawing the curve of pursuit.

Then

$$4\mu^2 x^2 - 8c\mu x + 4c^2\mu = 0.$$

If $\mu \neq 0$,

$$\mu x^2 - 2cx + c^2 = 0,$$

$$x^2 y - c(2x - c)(1 - x^2) = 0,$$

so that the cubic

$$y = \frac{c}{x^2}(2x - c)(1 - x^2)$$

is the locus of points for which the direction assigned is equal to the slope. This will cut the axis at $c/2$ and ± 1, and may be easily drawn.

Now the integral curve will have flexes where

$$\frac{d^2y}{dx^2} = 0, \qquad \text{or} \qquad \frac{dm}{dx} = 0.$$

But

$$2m\frac{dm}{dx} + 4c\frac{dm}{dx} + 4c^2\left[\frac{(1 - x^2)m + 2xy}{(1 - x^2)^2}\right] = 0.$$

Hence, there are flexes, when

$$m = -\frac{2xy}{1-x^2} = -2\mu x,$$

and this leads to the same cubic given in the preceding paragraph. Thus a flex does occur whenever the integral curve cuts the cubic.

Cusps will occur on the integral curve when

$$\frac{d^2y}{dx^2} = \frac{dm}{dx} = \infty$$

or when

$$m = -2c$$

and that is, on the parabola

$$y = 1 - x^2.$$

This additional information enables us to draw the integral curves with more ease and accuracy. There is very little difficulty about the integral curve for m_1, corresponding to the curve of flight, and we shall not consider this further. But the behavior of the curve for m_2 may be more complicated. This is the case in Fig. 5, drawn for $c = 1/2$. Fig. 6 shows the actual curve of pursuit drawn by the bracketing method. It is to be expected that the dog will pursue a path asymptotic to an inner concentric circle of radius 1/2. But the integral curve of Fig. 5 shows very nicely just how the path approaches the asymptotic circle. Following down the curve from (0, 1), there is first an intersection with the cubic, and a consequent flex, as shown. After this it cuts the cubic again, with another flex, and then meets the outside parabola at $x = 0.71$. Here it must have a cusp, which of necessity is of the rhamphoid type. Then it goes back, cutting the cubic twice more, and again meeting the outside parabola in a rhamphoid cusp at $x = 0.42$. The oscillations continue, growing smaller and smaller, and becoming asymptotic to the point where the cubic touches the outside parabola, namely (1/2, 3/4). A glance at Fig. 6 shows that the cusps of Fig. 5 indicate the apses in the dog's path.

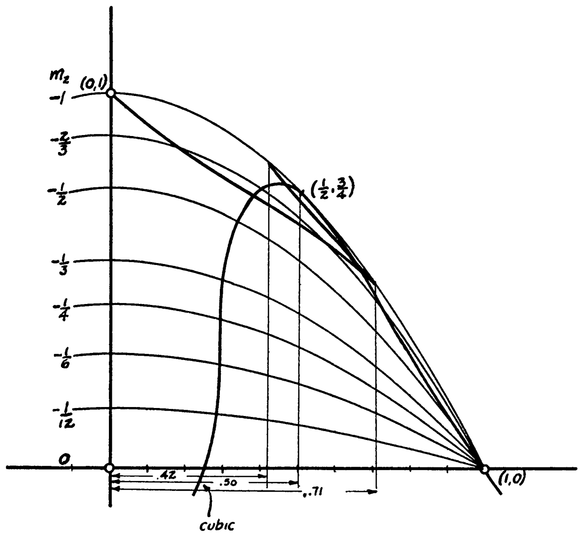

FIG. 5. Curve of pursuit, integral curves for $c = \frac{1}{2}$. Only curve for m_2 is drawn.

The results may then be summarized into three divisions, according to the value of c. When $c > 2$, the cubic does not interfere. When $1 < c < 2$, the cubic interferes, but does not affect the ultimate passage of the integral curve to (1, 0). When $0 < c < 1$, the cubic touches the outside parabola at the point

$(c,\ 1 - c^2)$, and the integral curve becomes asymptotic to this point by a series of cusps. In the first two cases the dog overtakes the duck tangentially, though with complete indetermination of his future course; and in the last one fails.

But although the differential equation has been solved as accurately as desired, the answer to the problem, namely the distance traveled by dog or duck, has not yet been obtained. The equation tells how to draw the curve of pursuit, but does not tell its length. For this an approximate formula may be derived from experiment. By measurement of the figures we know the corresponding values:

c	θ
less than 1	imaginary
1	∞
3/2	.136
3	.058
∞	0

(Figure not reproduced.)

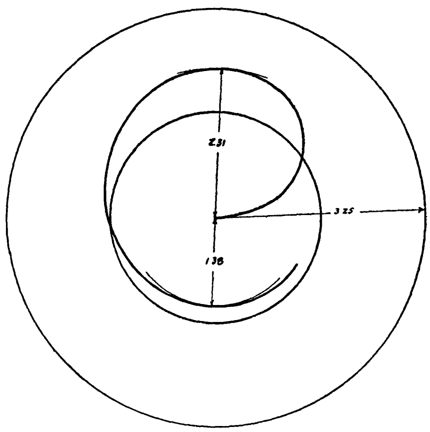

FIG. 6. Curve of pursuit, $c = \frac{1}{2}$; $p = (2.31/3.25) = 71$; $p = (1.38/3.25) = .42$.

Suppose that at a hazard we set up the empirical formula

$$\theta = \frac{A}{\sqrt{c^2 - 1}}\,.$$

This is satisfied when $c = 1$, and when $c = \infty$, independently of A. This is imaginary like θ when $c < 1$. For $c = 3/2$, $A = .150$, and when $c = 3$, $A = .162$. As a first approximation we might then use the formula

$$\theta = \frac{0.156}{\sqrt{c^2 - 1}}\,.$$

The curve of pursuit forms a good problem to test graphical methods of solving a differential equation,[1] since here the actual curve can be drawn easily, and the accuracy of the graphical solutions tested. This comparison shows graphical methods to be very satisfactory when used with care on such an equation as the above, and would lend confidence to cases where the actual curve may not be so easy to draw.

[1] For an interesting and elementary account of the graphical treatment of differential equations, see H. Brodetsky, *Mathematical Gazette*, October, 1919, and January, March, and May, 1920.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

EDITOR'S NOTE.—The large collection of autograph letters of famous mathematicians, now in the library of Professor Smith, includes many hundred valuable documents, most of them unpublished, and many of them either containing valuable historical information or giving such an intimate view of their writers as to be interesting to all who care for the mathematical sciences. On this account the editors have asked Professor Smith to prepare for the MONTHLY a series of brief articles under the above title, giving to its readers the opportunity of knowing something of the interesting letters in his collection.

1. DELAMBRE AND THE FOUNDER OF THE SMITHSONIAN INSTITUTION.

Among my autograph letters are upwards of twenty written by Delambre,[1] some of them containing the calculations made by him in the course of his survey for the metric system, and all of them giving evidence of the stirring times in which he lived. One of these letters possesses particular interest for American scientists, since it offers a subject for speculation as to the possible loss to our country of the Smithsonian Institution if it had not been written. The letter, in translation, is as follows:

PARIS, 16 April, 1809.

The Perpetual Secretary for the mathematical sciences, to His Excellency Monsieur le Comte d' Hunebourg, Minister of War.

Monsieur le Comte,

Permit me, in the name of the Class of the Mathematical and Physical Sciences of the Institute, to recommend to your benevolence M. Smithson, a member of the Royal Society of London and at the present time a prisoner of war at Hamburg.

Mr. Banks, president of the Royal Society and foreign associate of the Institute, has sent to us a very pressing letter in behalf of his friend, reminding us of the various reasons why M. Smithson is entitled to the esteem of savants. I take the liberty of joining him in his entreaty. Your excellency will readily see the strong reasons which prompt the Class to wish, on this happy and favorable occasion, to act as it has several times under similar circumstances, and to be able to reciprocate the protection and generous assistance which M. Banks has given to so many French savants in these unhappy times.

I beg Your Excellency to accept the assurances of my most respectful sentiments.

DELAMBRE.

The letter contains two official memoranda, one referring the case to the proper subordinate, and the other being a favorable recommendation.

The M. Banks mentioned in the letter was Sir Joseph Banks, a scientist of recognized standing, who had been for some years president of the Royal Society. He was an Oxford man, was made a baronet in 1781, and became an associate member of the Institute of France in 1802.

[1] Jean Baptiste Joseph Delambre (1749–1822) was a pupil of and a collaborator with La Lande, following his master as professor of astronomy in the Collège de France. His four histories of astronomy, *ancienne* (1817), *au moyen âge* (1819), *moderne* (1821), and *au dix-huitième siècle* (posthumous, 1827) are highly esteemed.

The M. Smithson was James Smithson, then a man of forty-four, the natural son of Hugh Smithson, later Duke of Northumberland. It may have been with his thoughts upon the bar sinister that he afterwards wrote:

"The best blood of England flows in my veins. On my father's side I am a Northumberland, on my mother's I am related to kings; but this avails me not. My name shall live in the memory of men when the titles of the Northumberlands and the Percys are extinct and forgotten."

Smithson spent most of his time on the Continent, and, evidently in the conquest of northern Germany, he had fallen into Napoleon's hands as a civilian prisoner of war. Delambre wrote this letter on the day that the emperor was hastening to the Battle of Ratisbon, where he defeated the Archduke Charles of Austria. This was seventeen years before Smithson made his will (October 23, 1826), and it is interesting to speculate upon the question of the founding of the Smithsonian Institution if Delambre had not written the letter.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTION.

42. In connection with the questions of Kakeya [*1920*, 256], Professor W. B. Ford is led to the following inquiry: A line-segment AB is to be moved in its plane to a new position $A'B'$. How should this be done in order that the area generated may, to the greatest extent possible, be passed over three times?

Professor Ford has proved that if the generated area is to be passed over, to the greatest possible extent, but *two* times, AB should be rotated about the intersection of the perpendicular bisectors of AB and $A'B'$.

DISCUSSIONS.

Professor Campbell considers below the conditions under which the expression $P(x, y)dx + Q(x, y)dy$ represents an exact differential. The ordinary form of the criterion is $\partial Q/\partial x = \partial P/\partial y$, and involves assumptions about the derivatives of P and Q. Professor Campbell gives a form of necessary and sufficient condition which is applicable even though these derivatives fail to exist. The condition which he derives involves forms which are usually explicitly used in the proof of the ordinary theorem; but it does not seem that his statement of the condition as an end in itself, is found in the literature. He gives also a generalization to the case of n variables. It seems that the restriction to a rectangular region, alluded to in a footnote, is essential for the accuracy of the proof.

Professor McKelvey contributes some remarks on a universally troublesome question,—the presentation of the theory of limits in secondary schools. While his indication that it is never of importance whether or not a variable reaches its limit may require occasional modification, such modification surely bears, not on the question of the general meaning of limit, but on the special problem

in hand. A more precise presentation of the idea of limit than is customary would greatly facilitate the use of the notion in college teaching.

Recent numbers of the MONTHLY have contained articles by Mr. Cheney [*1920*, 53] and Professor Lovitt [*1920*, 465] on geometric proofs of the law of tangents. A proof distinct from those as yet proposed is given in the last discussion this month by Professor Epperson.

I. ON EXACT DIFFERENTIALS.

By J. W. CAMPBELL, University of Alberta.

The criterion usually given that the differential $Pdx + Qdy$ shall be exact is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The functions P and Q are therefore assumed to be differentiable, and they are also assumed to have the other properties necessary for the application of Green's theorem. The purpose of this note is to suggest, in this case and in the case of n variables, an integral condition in which P and Q do not necessarily satisfy the hypotheses of Green's theorem.

THEOREM I.[1] *The necessary and sufficient conditions that*

$$Pdx + Qdy \tag{1}$$

shall be an exact differential are that P and Q shall be integrable with regard to x and y, respectively, and that

$$\int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy \equiv \int_{y_0}^{y} Q(x, y)dy + \int_{x_0}^{x} P(x, y_0)dx, \tag{2}$$

where (x_0, y_0) is an arbitrary fixed point in the vicinity of which P and Q are integrable.

For if (1) is exact it must be of the form

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy,$$

whence

$$\frac{\partial u}{\partial x} = P, \qquad \frac{\partial u}{\partial y} = Q.$$

Therefore

$$\begin{aligned} u &= \int_{x_0}^{x} P(x, y)dx + f(y) \\ &= \int_{y_0}^{y} Q(x, y)dy + g(x), \end{aligned} \tag{3}$$

where f and g are arbitrary functions of integration.

[1] It is assumed that the region under consideration is rectangular.—EDITOR.

The two expressions (3) for u must be identically equal in x and y, and therefore

$$(4)\qquad \begin{aligned} f(y) &\equiv \int_{y_0}^{y} Q(x_0, y)dy + g(x_0), \\ g(x) &\equiv \int_{x_0}^{x} P(x, y_0)dx + f(y_0), \\ f(y_0) &= g(x_0). \end{aligned}$$

The substitution of (4) in (3) gives (2), and therefore (2) is necessary.

It is also sufficient. For if $u(x, y)$ represents the common value of the two expressions in (2), then

$$P = \frac{\partial u}{\partial x}, \qquad Q = \frac{\partial u}{\partial y}$$

and therefore (1) is exact.

THEOREM II.[1] *The necessary and sufficient conditions that*

$$(5)\qquad \sum_{i=1}^{n} X_i dx_i$$

shall be an exact differential are that the functions X_i shall be integrable with respect to x_i and that

$$(6)\qquad \sum_{i=1}^{n} \int_{x_i^{(0)}}^{x_i} X_i dx_i$$

shall be invariant under any interchange of subscripts, where in the jth term of each sum so obtained the x_i with respect to which integrations have been made in the first $(j - 1)$ terms are replaced by $x_i^{(0)}$, $(j = 2, \cdots, n)$.

For if (5) is exact it must be of the form

$$\sum_{i=1}^{n} \frac{\partial U}{\partial x_i} dx_i$$

and therefore

$$X_i = \frac{\partial U}{\partial x_i}, \qquad (i = 1, \cdots, n).$$

Therefore

$$(7)\qquad U = \int_{x_i^{(0)}}^{x_i} X_i dx_i + Y_i, \qquad (i = 1, \cdots, n)$$

where Y_i is an arbitrary function of all the x_j except x_i.

Now let us suppose that the equations (7) imply the stated condition in the case of $n - 1$. Then on replacing the x_i successively by $x_i^{(0)}$ in (7) we readily show that the condition is implied in the case of n. And since the implication has been proved for $n = 2$, it follows by mathematical induction that equations (7) imply the necessity of the condition as stated.

[1] It is assumed that the region is a generalized rectangle, that is, that $a_i \leqq x_i \leqq b_i$.—EDITOR.

The proof of the sufficiency is similar to that for the case $n = 2$.

To these two theorems may be added a third related theorem.

THEOREM III. *The necessary and sufficient condition that*

$$\int_{(C)} Pdx + Qdy \tag{8}$$

shall vanish, where C is any closed contour in a region in which P and Q are continuous, is that $Pdx + Qdy$ shall be an exact differential.

For if the line integral is zero and C is arbitrary, then

$$\int_{x_0, y_0}^{x, y} Pdx + Qdy$$

is a function of x and y only, and does not depend on the path.

That is,

$$\int_{x_0, y_0}^{x, y} Pdx + Qdy = v(x, y).$$

Therefore

$$P = \frac{\partial v}{\partial x}, \qquad Q = \frac{\partial v}{\partial y}$$

and the diffential is exact.

And again if $Pdx + Qdy$ is exact, it must be of the form du where u is the common value of the two expressions in (2). But the u as there defined is continuous in x and y, and therefore the total algebraic variation about a closed contour is zero.

The form of the integral expressions appearing in (2) and (6) is usually given as the formula for the integral when the differential is exact, but so far as I have been able to find the invariance of these expressions under cyclic interchange of notation has not been given as a criterion for exactness.

II. THE TEACHING OF LIMITS IN THE HIGH SCHOOL.[1]

By J. V. MCKELVEY, Iowa State College.

The title of the present paper is to some extent either misleading or noncommittal. To make our purpose somewhat clearer, it may be stated that we hold no brief either for or against the teaching of limits in preparatory schools. We intend, rather, to state the results of several years' observation of high-school students during their early years in college particularly in regard to their understanding of limiting operations in the most elementary sense of the word. We open this discussion with whatever apologies may be necessary for saying some things that, perhaps, everybody knows.

To plunge rather abruptly into the midst of the question, we note that

[1] Read before the Iowa Academy of Science, April 24, 1920.

practically every high school graduate, except those who have avoided mathematics entirely, has heard of a limit and knows or thinks he knows what happens when a variable approaches a limit. If asked what is meant by a variable approaching a limit, he replies with the utmost assurance that if a variable approaches continually nearer and nearer to some constant to which it can never become equal, that constant is the limit of the variable. This is a most beautiful conception indeed. It would be a profoundly admirable one except for the fact that it is as totally unmathematical as anything could possibly be,—in the sense that the first part of the statement is insufficient and the second part is unnecessary. Nevertheless, the above reply may be taken as a fair composite statement of a college freshman's idea of a limit. This being the case, the question logically arises, where did he get it? The thousands of young people entering college each year are too nearly of one mind as to the definition of a limit for us to believe that their opinions are merely the result of accidental or spontaneous development in immature minds. Educational accidents do not happen with such regularity and persistence. If we discard the hypothesis that it just happened so, we are forced to the conclusion that somebody taught it to them.

In discussing the teaching proposition one is led to the consideration of both personnel and text-books. As regards personnel, we must with undisguised embarrassment admit that there are teachers of mathematics in high schools, normal schools, colleges and universities who either apologize for or openly accept the idea of a limit that we have just recognized as the almost unanimous choice of college freshmen. It is neither a student's place nor a scholar's to say that a definition is wrong but it is his privilege to believe that certain conceptions of fundamental ideas, used as definitions, are both useless and clumsy. The notion that a variable must of necessity regard its limit as a *sanctum sanctorum* into the privacy of which it dare not intrude is one that has neither defense nor excuse in mathematical argument. The writer has yet to learn of a single problem involving the theory of limits in which it is of the slightest consequence whether the variable reaches its limit or not. It is very difficult to understand why any definition of a limit that excludes the most favorable case in the limiting argument should find such wide acceptance among intelligent people. There seems to be abundant evidence in support of the statement that many persons attempt to teach mathematics who have no conception whatever of an infinitesimal except that it is some strange, unnameable sort of quantity but desperately small. This very unfortunate state of affairs is probably in large measure due to the fact that instructors exist who teach, letter by letter, the subject matter of their text-books with a reverence born of fear and uncertainty.

In discussing text-books, one must admit that many of them are in some respects a great handicap to the novice who takes them too seriously. Not long ago, the writer took occasion to examine a single shelf of about thirty elementary text-books in mathematics in regard to their definitions of a limit. A considerable number of them were designed for first and second year work and of course did not mention limits at all. Five of the remainder defined the limit as a

constant to which the variable could never become equal but such that the difference between the variable and the constant could become ever so small. A sixth, we blush to relate, merely defined zero as "nothing" or something less than "epsilon." Six text-books out of thirty contained these useless and confusing notions. The percentage should be stated somewhat higher than six out of thirty, for some of the books made no mention of limits whatever. The title pages indicate that the authors came from state normal schools, technical institutes, agricultural colleges and universities.

A plausible explanation of the persistence of these unfortunate conceptions is found in the fact that most students are introduced to the idea of a limit in terms of geometry. The time worn straight line illustration in which the point P moves from A toward B taking the positions P_1, P_2, P_3, etc., where P_i bisects the segment $P_{i-1}B$, is most popular. This may be because the illustration is easy, or perhaps because it is graphic. The one bit of information that the student gets out of this illustration and which eventually excludes every other feature of the argument is that the point P can never arrive at the point B. In this he is of course absolutely right, but so far as the limiting operation is concerned his information is just about as valuable as the discovery by a football coach that his star halfback had gray eyes instead of brown ones. In the above illustration of a limit the student should be taught that if C is a point on AB such that CB is arbitrarily small, then under the given law of motion P_i can be placed between C and B, *i.e.*, P_iB can be made less than CB. Sometimes the student's preparation may not be sufficient to make a rigorous proof of this fact either possible or desirable. In such cases, a few numerical examples will illustrate the principle so that the proper sequence of the operations may be understood. A satisfactory proof may be given when the student becomes acquainted with logarithms.

The various proportionality theorems, proofs concerning the areas and arcs of circles, together with a number of volume and surface problems which are discussed in our elementary plane and solid geometries constitute the subject matter from which the majority of students derive their notion of a limit. This is a particularly unfortunate circumstance because these are all cases in which the variables do not reach their limits. It is not surprising that the student thinks this ever present fact is of some consequence in the argument.

The idea that a point may take various positions on a straight line under such a law of motion that it can never reach the end of the line, or that the area of a polygon inscribed in a circle may be continually increased without being made larger than a certain fixed quantity, or in general the idea that any operation may continue indefinitely without reaching a specified goal so impresses or oppresses the beginner in mathematics that his reasoning powers seemingly cease to function so far as the essential argument in the case is concerned. Hence it is the writer's opinion that the above most vicious feature of the study of limits in geometry should be avoided by every means within the law. Can it be done? It can be done if, and only if, we give the student something else to think about.

The writer believes that limits should be taught entirely from the standpoint of inequalities. If a variable x assumes a sequence of values such that $|x - a|$ becomes and remains less than a pre-assigned positive number which is arbitrarily small, x is said to approach a as a limit. This is a quite generally accepted form of the definition. Its application depends absolutely and finally on the existence or non-existence of a certain inequality. The fact should be definitely emphasized that the positive number is assigned *first*. If after that the variable takes such values that the prescribed inequality exists, the variable has a limit, otherwise not.

This principle of the "order of choice" can not be over emphasized. It must be first the epsilon, then the variable. If this sequence is disturbed, the limiting argument breaks down completely. A variety of illustrations may be necessary to drive the principle home and make it stick. Numerical examples can be used to advantage. Instructive exercises may be given in finding the largest permissible numerical error in determining a required number so that the percentage error should be less than a specified value. Such examples will illustrate merely the skeleton of the argument. A constant effort should be made to induce beginners to waive temporarily any scriptural convictions they may have that the first should be last and the last should be first and to learn, in their mathematical reasoning at least, to put first things first and last things last.

III. Geometric Proof of the Law of Tangents.

By C. A. Epperson, Northeast Missouri State Teachers College.

Let $a > b$. Draw CD the bisector of the external angle at C (to meet BA produced at D) and CF the bisector of the angle C (meeting AB in F). Then CF is perpendicular to CD. Draw AN $(= w)$ and BM $(= y)$ parallel to FC, meeting DC in N and M respectively. $\angle BCM = \angle ACN = (A + B)/2$, and $\angle ADC = (A - B)/2$. Then

$$\frac{a}{b} = \frac{BD}{AD} = \frac{MD}{ND}.$$

By composition and division

$$\frac{a+b}{a-b} = \frac{MD+ND}{MD-ND} = \frac{MD+ND}{MC+CN} = \frac{(y+w)\cot\dfrac{A-B}{2}}{(y+w)\cot\dfrac{A+B}{2}};$$

$$\therefore \frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

RECENT PUBLICATIONS.

REVIEWS.

History of the Theory of Numbers.

History of the Theory of Numbers. Volume II:[1] Diophantine Analysis. By LEONARD EUGENE DICKSON. Carnegie Institution of Washington, 1920, publication number 256, Vol. II. 26 + 803 pages. Price, paper, $7.50; cloth, $8.00.

In one respect at least Diophantine Analysis is probably unique in the history of mathematics. Perhaps no other division of the whole field has at the same time furnished the subject of such numerous investigations for so many generations of mathematicians and yet has received so little systematic development. Some pleasing chapters are to be found in an exposition of the subject; and a few of the most beautiful theorems in mathematics belong to it. But trivial problems have been too often treated; and fragmentary and incomplete results are to be found dispersed throughout nearly the whole literature.

The nature of the subject has made it an easy prey to this evil. Two Diophantine equations which are much alike in external form may be totally different as regards the essential characteristics of their theory. One may be easy to treat and the other may be exceedingly difficult. A mediocre investigator can always find for himself some of these easy problems; and too frequently he has been willing to publish unimportant results. Some of the papers are of the character which would be produced if one who had failed in larger problems set out to find something of difficulty proportionate to his strength and then published whatever he found. Other papers are at the opposite extreme and have required for their production a command of a wide range of methods and the deepest insight on the part of the investigator. If this judgment concerning the less fortunate investigator seems to the reader to be ungenerous or even harsh he would probably have more sympathy with it after proceeding laboriously through some hundreds or thousands of pages of the more trivial articles and notes.

The immense number of disjointed elements brought to definite notice by a systematic and complete account such as that of the volume under review impels one to believe that the time has come for a change in the methods of developing the Diophantine Analysis. Diophantus himself and many of his followers have been content with special solutions of their problems obtained under restrictive hypotheses which are employed for no other reason than that they simplify the analysis. Such papers have at least the value of showing that the equations treated are not impossible. Moreover, these partial investigations have made clear the essential character of several types of equations which repeatedly recur as auxiliary to the solution of other problems.

But there seems to be no further need of the disjointed detail which is derived

[1] For a review of volume I see this MONTHLY, *1919*, 396–403.

by tentative methods of no wide range and which serves merely to cumber the literature with more uninteresting detail and separated fact answering no real question. "Since there already exist too many papers on Diophantine Analysis which give only special solutions, it is hoped that all devotees of this subject will in future refrain from publication until they obtain general theorems on the problem attacked if not a complete solution of it. Only in this way will the subject be able to retain its proper position by the side of other virile branches of mathematics" (p. xx). "A. Hurwitz's complete discussion (p. 697) of the positive integral solutions of $x_1^2 + \cdots + x_n^2 = xx_1 \cdots x_n$ furnishes a model for thoroughness which may well be imitated by writers on Diophantine equations, too many of whom seem to be content with a special solution of their problems" (p. xvii).

Ideas rather than computations are needed in this field. Some organizing force to bring order out of chaos is essential. If it is not supplied and if the accumulation of small detail continues there is danger that Diophantine Analysis will become an ungainly monster and a reproach to those who cultivate its acquaintance.

In criticizing the subject for the predominance of isolated problems one must not overlook the fact that unattached results of some sorts are of real use, namely, those which answer a real question. It seems not to be known, for instance, whether the sum of five fifth powers can itself be a fifth power just as there was a time when it was similarly unknown whether the sum of four fourth powers could itself be a fourth power. It was worth while to have the latter question answered affirmatively (p. 652); and it would likewise be of value to have the former (or any of its generalizations, pp. 648, 682) answered affirmatively by means of an example (if such exists).

Every one interested in Diophantine Analysis must have observed how it ties the ages together for the student of mathematics. A similar honor belongs to astronomy, logic, and geometry. In the case of Diophantine Analysis this connection has been maintained primarily by a continued interest in isolated Diophantine problems. In logic there has been an accumulating discussion of the method of reasoning. In astronomy observations have been made from time immemorial and these in more recent generations have been reduced to order in the theories of Celestial Mechanics. In geometry an elegant and satisfying statement was made by Euclid in a form to serve as a model even down to our own time; and the moderns have extended the subject into wide ramifications. But Diophantine Analysis has existed principally in a long chain of isolated problems and results. This, though the general fact, fails to be true of some topics of the subject.

A good example of the latter is that afforded by the remarkable theorem that every positive integer is a sum of four squares. Diophantus employed sums of four squares in three problems without naming any condition on a number in order that it shall be a sum of four squares while he did give such necessary conditions in similar cases for representations as sums of two or of three squares. On account of these facts Bachet (in 1621) and Fermat (in 1636) expressed the

judgment that Diophantus probably had a knowledge of the theorem that every positive integer is a sum of four squares; and the latter stated that he possessed a proof by infinite descent. For more than forty years Euler gave repeated attention to the theorem. He converted it into equivalent forms; but he was constantly baffled in his attempt to find a proof. "Not until twenty years after he began the study of the theorem did he publish in 1751 some important facts bearing on it, including his formula which expresses the product of two sums of four squares as such a sum" (p. x).

Lagrange, acknowledging his indebtedness to Euler's paper, published the first proof of the theorem in 1772; but the method is rather complicated. A much more elegant proof was given by Euler in the following year, a proof which has not been improved upon to the present time, though several others have been offered.

But the history of the theorem is not closed with the discovery of these proofs. The inevitable question arises as to the number of representations of a given integer n as a sum of four squares; and this was answered by Jacobi in a remarkable theorem (p. x) obtained by comparing two infinite series for the same elliptic function. Several elementary proofs of the theorem of Jacobi have been given (p. x), one of them as late as 1914, while a proof by means of theta functions was given in 1915.

Here we have, not isolated facts, but a general theorem of great beauty and interest which has served as an intellectual bond among mathematicians of several centuries. A similar connection is afforded, perhaps in an even more remarkable manner, by the theorem that every prime of the form $4n + 1$ is a sum of two squares and by its generalizations.

The book contains reports on more than five thousand writings. The method and point of view of the author in preparing these is briefly indicated in the following words from page xx of the preface: "While many of these papers are of minor importance, the aim has been to give an exhaustive account of the literature on the subject rather than a selective account reflecting the author's imperfect views as to relative importance. This work is intended as a source book not merely for the fastidious professional mathematician, but also for the larger number of amateurs who find endless fascination for the 'queen of the sciences,' whose rule began centuries ago and has continued without interruption to the present."

The table of contents contains an excellent and convenient classification of Diophantine problems and equations (with references to the parts of the text in which they are treated). A preface of twenty pages gives a most valuable outline of material contained in the whole volume and numerous illuminating remarks concerning the history of Diophantine Analysis. This introductory matter in the second volume is better prepared than the corresponding matter in the first volume. In fact, it is to be said that the author's experience in the preparation of volume I has been useful to him in the way of leading to several improvements in volume II. While the former excited our admiration for its remarkable excellences, the latter renews it and makes it keener.

The extent of the volume is too vast for the reviewer to undertake a summary (rendered unnecessary by the preface). Attention will be called merely to a few outstanding features of the book and its general subject matter.

The author believes that his chapter XXIII, on equations of degree greater than 4, will be more useful than any other in the volume since it contains reports on papers which offer general methods of attacking Diophantine equations; the principal methods referred to are mentioned on page xvii of the preface. A high degree of accuracy for chapters III, XXI–XXVI was especially desired since it is thought that they are the ones which will be most frequently consulted. They deal in order with the following topics: partitions; equations of degree three; equations of degree four; equations of degree n; sets of integers with equal sums of like powers; Waring's problem and related results; Fermat's last theorem (with certain closely related matters). Each of these chapters was checked with especial care by the author or by some other mathematician who gave especial attention to the single chapter.

After speaking of the inexhaustible store of interesting truths presented to us by the higher arithmetic and of the wholly unexpected ties which are often discovered among them, Gauss (as quoted in Moritz, *Memorabilia Mathematica*, p. 272) proceeds to add: "A great part of its theories derives an additional charm from the peculiarity that important propositions, with the impress of simplicity upon them, are often easily discoverable by induction, and yet are of so profound a character that we can not find their demonstration till after many vain attempts; and even then, when we do succeed, it is often by some tedious and artificial process, while the simpler methods may long remain concealed."

Several examples of the contrast between the ease with which empirical theorems are discovered and the difficulty attending a complete proof are afforded by Diophantine Analysis. The theorem that every positive integer may be represented as the sum of four squares is an interesting one whose history is instructive (see page x). A simpler case with a shorter history is that of the theorem that every prime of the form $4n + 1$ is a sum of two squares (p. ix). The author in his preface (p. xviii) singles out as a typical example of this sort the theorem of Waring that every positive integer is the sum of a limited number of mth powers and gives a brief summary of the history of the theorem.

Of the many interesting discoveries in the theory of numbers announced by Fermat all have now been proved with the single exception of his so-called "last theorem," which states that it is impossible to separate any power higher than the second into a sum of two powers of the same degree. Concerning this theorem Fermat said: "I have discovered a truly remarkable proof which this margin is too small to contain." The final chapter of forty-six pages is devoted to the history of this theorem. "The dignity of this famous theorem was injured by the offer of a very large prize in 1908" (p. xix). "Fermat's last theorem is not of special importance in itself, and the publication of a complete proof would deprive it of its chief claim to attention for its own sake. But the theorem has acquired an important position in the history of mathematics on account of its

having afforded the inspiration which led Kummer to his invention of his ideal numbers, out of which grew the general theory of algebraic numbers, which is one of the most important branches of modern mathematics" (p. xix). Kummer's restoration of law in the midst of the chaos in the theory of algebraic numbers was one of the chief scientific triumphs of the last century.

Fermat's famous method of descent, infinite descent, indefinite descent, as it has been variously called by many writers, beginning with Fermat himself and continuing with his successors down to the author of the volume under review, comes in for mention in many places, as one may see from the subject index. The learner may make an interesting and profitable study of the method by means of these references to it. The only instance of a detailed proof left by Fermat is one by the method of descent; it is reproduced in full on pp. 615–616. Another very interesting case of the method is outlined on p. 619. [In the name of this method the adjective "infinite," and perhaps even the adjective "indefinite," is somewhat misleading. Is it desirable to adopt the practice which seems to predominate in the volume under review and call it simply the method of descent?]

Every fragmentary result connected with a question of importance suggests a problem for further investigation. These are too numerous in this volume for summary and are not of a nature to make this desirable. But there are a few conjectured or empirical or unproved theorems (besides the last one of Fermat) to which attention should be directed. We list the following:

1. Every number of the form $8k + 3$ is the sum of an odd square and the double of a prime $4n + 1$ (p. 261).

2. The triple of any odd square not divisible by 5 is a sum of squares of three primes other than 2 and 3 (p. 266).

3. The double of any odd integer is a sum of two primes $4n + 1$ (pp. 282, 289).

4. Every prime $18n \pm 1$ or else its triple is expressible in the form $x^3 - 3xy^2 \pm y^3$ (p. 575).

5. The sum of n numbers each a kth power is never a kth power if $n < k$ (pp. 648, 682). It seems to be unknown whether we can have $n = k$ when $k > 4$ (see pp. 682, 683).

6. See also pages 633, 752, 767.

Where the material to be gathered is so vast it is impossible that nothing has escaped attention. The author urges his readers to supply him with notices of errata or omissions as well as abstracts of the few papers marked by the symbol * before authors' names to signify that the papers were not available for report. The errata discovered by the reviewer will not give the reader trouble except possibly in the case of the error of Lexell on p. 732 in taking as relatively prime two factors which are not shown to have this property, so that the suggested proof is inadequate. We may here record a few facts which we have not found stated at those places in the volume at which it seemed natural to expect them:

1. Several persons treated the equation $x^3 + y^3 + z^3 = 2u^3$ (cf. pp. 563, 604) in *The Mathematical Visitor*, 2, 1887, pp. 84–88, one of whom, J. H. Drummond, gave the following identity:

$$\{24ab^5(1 + 6b^6)\}^3 + \{72ab^9(1 + 4b^6) + a\}^3 + \{72ab^9(1 + 4b^6) - a\}^3 = 2\{72ab^9(1 + 4b^6) + 6ab^3\}^3.$$

2. O. D. Kellogg has stated (Carmichael's *Diophantine Analysis*, p. 115; cf. Dickson, pp. 688–691) that for the positive integral solutions $x_1, x_2, \cdots, x_n$ of the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1$$

the maximum value of an x which can occur in a solution is u_n where $u_1 = 1$ and $u_{k+1} = u_k(u_k + 1)$. [It seems desirable to have a complete theory of this equation developed.]

3. For the equation $x^2y^2 + x^2 + y^2 = t^2$ Carmichael (*Diophantine Analysis*, p. 106) gave the solutions $x = a$, $y = 2a^2$, $t = a(2a^2 + 1)$; $x = a$, $y = 4a^3 + 4a^2 + 3a + 1$, $t = 4a^4 + 4a^3 + 5a^2 + 3a + 1$; and also certain solutions not in general integral for integral values of the parameters. He applied these to the solution of certain problems of Diophantus and Fermat.

4. C. Störmer (*Bull. Soc. France* 27, 1899, p. 160) showed that all the integral solutions k, m, n, x, y of the equation

$$m \arctan\frac{1}{x} + n \arctan\frac{1}{y} = k\frac{\pi}{4}$$

are the following: 1, 1, 1, 2, 3; 1, 2, − 1, 2, 7; 1, 2, 1, 3, 7; 1, 4, − 1, 5, 239.

It may be profitable to state for investigation a few (apparently unsolved or incompletely solved) problems some of which are perhaps of such sort as to be of interest to amateurs:

1. Determine all polynomial solutions u_a and v_a of the functional equation $a^2u_a^2 + a^2 + u_a^2 = v_a^2$ and apply the results to the solution of a group of Diophantine problems. [Suggestions for dealing with this and certain similar problems are given in the last chapter of my *Diophantine Analysis*.]

2. Develop the theory of the equation $x^4 + ay^4 + bu^4 + abv^4 = t^2$ for constant values of a and b.

3. Find the general integral solution of the equation $t^3 = x^3 + y^3 + 1$. [One solution is afforded by the relation $9^3 = 8^3 + 6^3 + 1$.]

4. Determine the properties of the integer m such that the equation $x^3 + y^3 + z^3 - 3xyz = mt^2$ shall have solutions and solve it for such values of m.

5. Determine the integral values of a for which the equation $x^4 + y^4 + a^2z^4 = u^4$ has non-zero integral solutions, and develop methods for finding these solutions.

On pages xx to xxi of the preface the author takes the reader into his confidence in a remarkable passage a part of which we shall quote. He had initially planned to give his work the title "topical history of the theory of numbers"; but the word topical was omitted on the advice of a prominent historian, since it is inconceivable that any one would desire the vast amount of material in this

work arranged otherwise than by topics. Having said this, the author then continues:

"Conventional histories take for granted that each fact has been discovered by a natural series of deductions from earlier facts and devote considerable space in the attempt to trace the sequence. But men experienced in research know that at least the germs of many important results are discovered by a sudden and mysterious intuition, perhaps the result of subconscious mental effort, even though such intuitions have to be subjected later to the sorting processes of the critical faculties. What is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of those facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history. With such a view of the ideal self-effacement of the historian, what induced the author to interrupt his own investigations for the greater part of the past nine years to write this history? Because it fitted in with his conviction that every person should aim to perform at some time in his life some serious, useful work for which it is highly improbable that there will be any reward whatever other than his satisfaction therefrom."

It is refreshing and inspiring to find a man, when he pauses at a breathing place in the excellent performance of a great task, willing to set forth in a quiet way the fact that he has been moved by the highest and most unselfish ideal of duty.

R. D. CARMICHAEL.

Logarithmic and Trigonometric Tables. Revised edition. Prepared under the direction of E. R. HEDRICK. New York, Macmillan, 1920. 21 + 143 pp.

Preface: "The present edition of this book contains several tables not contained in the previous editions. The probability of the occurrence of errors has been minimized by using electrotype reproductions of the tables previously included, even when changes were made. Remarkably few errors existed in the original edition; what few have been discovered have been corrected.

"Minor changes only occur in the earlier pages. Care has been taken to preserve the page numbers of the principal tables up to page 114, so that older editions may be used in class-work without confusion, and texts which contain the principal tables may be used in the same class.

"Among the minor changes are the insertion of a condensed table of logarithms and antilogarithms (Table I*a*, p. 20), the insertion of a table of values of S and T for interpolation in logarithmic trigonometric functions (Table III*a*, p. 45), and the insertion on pages 1–19 of the logarithms of a few important numbers at appropriate points.

"The principal changes follow page 114. Tables VIII and IX (pp. 115–122) make reasonably complete the tables of hyperbolic functions formerly represented only by Table XII (pp. 112–114); These functions are of increasing importance, notably in Electrical Engineering.

"The table of haversines (Table X, pp. 123–125) will be welcomed particularly by those interested in navigation.

"The table of factors of composite numbers and logarithms of primes (Table XI, pp. 126–127) has obvious uses.

"Tables XII a, b, c, d, e, f, pages 128–132, are intended for work involving compound interest, annuities, depreciation, etc. They will be useful for statistics, insurance, accounting, and the mathematics of business.

"The same care has been exercised to eliminate errors in the new tables that resulted in so great a degree of reliability in the original edition of these tables."

NOTES.

The frontispiece of *Popular Astronomy* for October, 1920, is a group picture of members and visitors at the Northampton and South Hadley meeting of the American Astronomical Society, September 1–4, 1920. Among those in the group are the following members of the Mathematical Association: E. W. BROWN, W. J. HUSSEY, H. R. KINGSTON, J. A. MILLER, E. D. ROE, JR., L. SILBERSTEIN, A. B. TURNER, and R. E. WILSON.

The fourteenth volume of the remarkable edition of the *Oeuvres Complètes* of Christian Huygens appeared in 1920 (Nijhoff, The Hague, 5 + 557 pages). It contains his writings on the calculus of probabilities, and his work in pure mathematics, 1655–1666.

The geometrical proof of the law of tangents signed by Alex. D. Russell on page 58 of the last volume of the *Proceedings of the Edinburgh Mathematical Society* (published November, 1920) is identical with that given in this MONTHLY, February, 1920, by Mr. CHENEY.

We are glad to note that the editors of the second volume of *Journal of the Mathematical Association of Japan for Secondary Education* [compare *1920*, 25, 45] find in our MONTHLY considerable material of interest for their constituents. On pages 1–7 of the issue for March, 1920, there is an abridged translation into Japanese of Professor R. B. McCLENON'S "Leonardo and his Liber Quadratorum" [*1919*, 1–8]; in the issue for June, pages 85–107, 118, there are practically complete translations of Professor HUNTINGTON'S presidential address [*1919*, 421–435], and of Mr. W. F. CHENEY'S "New proof of the law of tangents" [*1920*, 53–54].

We have referred in earlier issues to the publication of six parts of *Materialien für eine wissenschaftliche Biographie von Gauss*, edited by F. Klein, M. Brendel and L. Schlesinger [*1919*, 160–161, 358]. Heft 7 (1919) by M. Brendel, *Ueber die astronomischen Arbeiten von Gauss*, contained the first part of "Theoretische Astronomie." Heft 8 (59 pages) by A. Fraenkel on *Zahlbegriff und Algebra bei Gauss* contains the first part "Mit einem Ahang von A. Ostrowski . . . Zum ersten und vierten Gausschen Beweise des Fundamentalsatzes der Algebra." This was published as "1920 Beiheft" to *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse.*

In *Revista Mathemática Hispano-Americana*, June, 1920, page 192, there is a note by G. A. MILLER on an incorrect definition of "simple group." The content of this note was contained in an article by Professor Miller published in this MONTHLY, *1919*, 290–291. The September number of the *Revista* has the following query by D. E. Smith on page 212: "In the arithmetic of Texada (1546) the sign *U* appears as abbreviation of the word thousand. Thus for example

c. lx. U462 q̃s. . ix U621

represents the number 160,462,009,621.[1] What is the origin of the symbol *U*,

[1] Cf. D. E. Smith, *Rara Arithmetica*, Boston and London, 1908, p. 242.—EDITOR.

probably a heritage from the Arabs?" Compare F. Cajori, "On the Spanish symbol *U* for thousands," *Bibliotheca Mathematica*, May, 1912, volume 12, pp. 133–144.

The following extracts from *Sylvanus Phillips Thompson, D.Sc., LL.D., F.R.S. His Life and Letters* by Jane S. Thompson [his wife] and Helen G. Thompson [his daughter] (London, T. Fisher Unwin, 1920) will give an idea of the style of treatment of the subject throughout the volume:

"In connection with his work on alternating electric currents, Thompson developed a lively practical interest in that branch of mathematics known as Harmonic Analysis. In 1904 he read a paper to the Physical Society which showed his familiarity with many of the various attempts of mathematicians to simplify the methods of this analysis, and he described in his paper, and later in *The Electrician* for the benefit of technical workers, 'A Rapid Approximate Method of Harmonic Analysis.' He continued to work at this for some years, and in 1911 presented to the Physical Society a second paper on what he called a 'New Method of Approximate Harmonic Analysis.' This method was also described in a paper read before a Swedish Society a few months later, and printed in the *Arkiv för Matematik, Astronomi och Fysik* of Upsala and Stockholm. The method is described in eight short pages, quite as obscure as Chinese to the lay mathematical mind, but evidently appreciated by those with sufficient training to follow its argument; and in June, 1914, he was requested to allow his method, with its scheduled forms, to be incorporated in the Handbook of an exhibition of forms for facilitating Harmonic Analysis, at the Napier Tercentenary Celebrations held at Edinburgh that summer. . . . Dr. Alexander Russell . . . wrote thus of Thompson's work in this field:[1]

"'He loved music and had an accurate musical ear. The valuable paper which he read to the Physical Society in 1910 on "Hysteresis Loops, and Lissajous' Figures" was a happy mixture of magnetism, sound, and mathematical theorems. In solving mathematical problems and inventing new mathematical theorems he took the keenest delight. He did most excellent work, for instance, in simplifying Runge's method of practical harmonic analysis. He was dissatisfied, however, with the accuracy obtainable by this method. He then invented a series method of harmonic analysis. The writer remembers how pleased he was when he first discovered it, and with what mutual pleasure we discussed it. He greatly appreciated the lectures which Dr. Kennelly of Harvard gave at the Institution some years ago. In proposing a vote of thanks to him he expressed himself, as usual, most happily. He said that he felt constrained to exclaim, "Great is the Hyperbolic Angle, and Kennelly is its Prophet!"'

"Thompson took a keen interest in hyperbolic trigonometry, and contemplated writing a little treatise on the subject, which was to have been a companion volume to the *Calculus made Easy*. He and his old student, Mr. Maurice Gheury, had already partly planned the work in 1914,[2] but like much else it was cut short by the war." [Pages 105–107.]

"In order to help his students to get a grasp of the Integral Calculus, a branch of mathematics absolutely essential for the training of a mechanical or electrical engineer, he invented a new way of presenting the subject which was used for many years in the college. At last, in 1910, he published this in the form of a small volume entitled *Calculus made Easy*, by 'F. R. S.' It was brought out by Macmillan's, and the secret of its authorship was faithfully kept until after the death of the author. It was written in a very amusing colloquial style, which raised the ire of some of the serious teachers of mathematics who objected to the subject being treated as a joke, but its tremendous success showed that it met the need of students. In the Prologue he says:

"'Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.'

"In the Epilogue he says:

"'There are amongst young engineers a number on whose ears the adage that, what one fool can do another can, may fall with a familiar sound. They are earnestly requested not to give the author away, nor to tell the mathematicians what a fool he really is.'

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[1] *Journal of Inst. E. E.*, vol. 55, p. 550.

[2] Such a work, *Exponentials made Easy*, was published by Mr. Gheury in 1920.

"The students who knew the secret kept it carefully. . . .

"Sir Oliver Lodge wrote:

"'My dear Silvanus,

"'You know that book *Easy Lessons in the Calculus*, I have concluded that the book is by John Perry, but recently I have heard it attributed to yourself. I do not in the least think that that is true, but perhaps you would not mind sending me a postcard either of denial or acceptance, for evidently the anonymity is not carefully preserved.

"'Yours ever,
"'Oliver Lodge.'

"After the death of the author the book was published in his name, and is still being largely used, both in this country and in America." [Pages 138–140.]

ARTICLES IN CURRENT PERIODICALS.

American Machinist, New York, volume 53, August 26, 1920: "Teaching machine shop mathematics" by G. Heald, 421.

Annals of Mathematics, second series, volume 22, no. 1, September, 1920: "On multiform functions defined by differential equations of the first order" by P. Boutroux, 1–10; "Hermitian metrics" by J. L. Coolidge, 11–28; "On the expansion of certain analytic functions in series" by R. D. Carmichael, 29–34; "Notes on the cyclic quadrilateral" by F. V. Morley, 35–42; "Note on the preceding paper" by F. Morley, 43; "Qualitative properties of the ballistic trajectory" by T. H. Gronwall, 44–64.

Athenæum. London, October 22, 1920: "James Clerk Maxwell" by S., 557–558 [First paragraph: "The place that will be held by James Clerk Maxwell in the history of physics is not easy to determine. That it will be a very high place is obvious, that he will emerge as the greatest of the physicists of the nineteenth century is probable, but the student of Maxwell must feel that this kind of ranking is somehow irrelevant, or likely to become irrelevant, to his peculiar effect. The unique impression produced by Maxwell's achievement is not adequately described by being referred to his "originality." There are different ways of being original; it is not a sufficiently penetrating term. A number of Maxwell's scientific contemporaries were original men, but one is conscious that they had more in common with one another than Maxwell had with them. An exception from this statement is found in W. K. Clifford, who, as has often been remarked, had a genius curiously akin to Maxwell's. Both men were exceptionally *independent* thinkers, both men resisted the attraction of the high road; both men, if the term may be permitted, had a personal and unique angle of approach to the problems of their time. But this, though true, is not a sufficient description. It is important that in neither case do we feel their individual quality to be an eccentricity; their work has a power, and, still more, a comprehensive serenity, which is never the product of mere oddity—the oddity, for instance, of a Samuel Butler. If we try to get closer to this elusive and important characteristic we do not meet with much success; but we may suggest that the ideas of these men have the effect of springing from an unusually rich, subtle and comprehensive *context*. The fundamental ideas of the science of their time were subtly modified by reception into these minds; they were connected in a personal and unusual web of implications."]

Bulletin des Sciences Mathématiques, volume 55, July, 1920: Review by A. Buhl of C. I. Lewis's, *A Survey of Symbolic Logic* (Berkeley, 1918), 154–155.

Chimie et Industrie, Paris, volume 3, May, 1920: "De l'influence des spéculations mathématiques sur les progrès de la chimie" by H. LeChatelier, 555–565 [translated into English, *Scientific American Monthly*, volume 2, November, 1920, pp. 229–234].—Volume 4, August, 1920: "A propos de la formation des chimistes" by E. Grandmougin, 252–254.

Engineering, London, volume 109, June 11, 1920: "Mathematics for the engineer," 795–796.

L'Enseignement Mathématique, volume 21, no. 2, September, 1920: "Charles Ange Laisant, 1841–1920" (portrait frontispiece) by A. Buhl, 73–80; "Extension du problème des triangles héroniens" by C. A. Laisant, 80–84; "Sur l'élimination algébrique" by C. Riquier, 85–105; "Table de caractéristiques de base 30030 donnant, en un seul coup d'œil, les facteurs premiers des nombres premiers avec 30030 et inférieurs a 901800900" by E. Lebon, 105–116 ["Extrait de l'introduction"]; "Sur la théorie des vecteurs, essai de calcul symbolique" by T. Rousseau, 117–131; "Calcul des racines reélles d'une équation algébrique ou transcendante par

more troublesome to the Marriners, and therefore the Tables last printed by Pitiscus were omitted as overtedious, his first being annexed herewith for the more ease in working. But if any man desire those Tables themselves, they may buy them apart in the Latin printed at Franckford, 1612.' In view of these observations it would seem that the tables of Pitiscus were not unknown in this country; and it appears unlikely that the mathematicians who used the decimal point in the English edition of the 'Descriptio' of 1616 were unacquainted with the tables of 1608 and 1612, in both of which it had been used."]

Tôhoku Mathematical Journal, volume 18, nos. 1 and 2, August, 1920: "Generalization of Bessel's and Gram's inequalities and the elliptic space of infinitely many dimensions" by K. Ogura, 1–22; "The irreducible cases of algebraic solutions" by C. E. White, 23–33; "Remarque sur un théorème relatif aux racines de l'équation $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ où tous les coefficients a sont réels et positifs" by G. Eneström, 34–36; "Theorems on convergent integrals" by T. Kojima, 37–45; "A proof of a theorem of Haskell's" [in Japanese] by T. Kubota, 46–48; "On the interpolation by means of orthogonal sets" by K. Ogura, 49–60; "On the interpolation by Legendre polynomials" by K. Ogura, 61–74; "On certain inequalities" by T. Hayashi, 75–89; "Sur les courbes orbiformes. Leur utilisation en mécanique "by G. Tiercy, 90–115; "On the passing of simple continuous arcs through plane point sets" by J. R. Kline, 116–125; "Einige Sätze über charakteristische Eigenschaften gewisser Flächen" by T. Kubota, 126–127; "On Bertrand curves" by M. Tajima, 128–133; "On continuous set of points, II" by K. Yoneyama, 134–186; Shorter notices and reviews, Miscellaneous notes, 187–203.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 51, nos. 7–8 (published July 22, 1920): "Fragen der Oberlehrerausbildung mit Beziehung auf angewandte Mathematik und Technik by R. Rothe, 177–190; "Die Genauigkeit des logarithmisch-trigonometrischen Rechnens" by A. Fischer, 191–195; "Die Einführung des Kraftbegriffes auf der Oberstufe" by K. Hahn, 195–198; "Bildliche Darstellungen gewisser Summenformeln" by K. Bochow, 198–203; [geometrical representations of $\sum_1^n n^2$, $\sum_1^n n^3$, and $\sum_1^n n^4$, together with their relations to each other and to $\sum_1^n n$;] "Zur Vielpassaufgabe" by H. Dostal, 204; "Aufgabenrepertorium," 204–207; [Discussion of the system of equations $ax + (a + n)y + (a + 2n) = 0$, $(a + 3n)x + (a + 4n)y + (a + 5n) = 0$] by P. Schulze, 216.

UNDERGRADUATE MATHEMATICS CLUBS.

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CLUB TOPICS.

18. Finite Geometries. By U. G. Mitchell, University of Kansas.

General analytic and synthetic definitions of finite (or modular) projective geometries were given by Veblen and Bussey, *Transactions of the American Mathematical Society*, Vol. 7 (1906), pp. 241–259. They used the symbol $PG(k, p^n)$, where k, p and n are integers and p a prime, to represent a finite projective space of k dimensions having $p^n + 1$ points on every line. As they pointed out (pp. 258–259) the finite geometries so defined included many new configurations[1] and many that were already well known. The $PG(k, p)$ had been defined

[1] The term "configuration" in its projective geometry sense seems to be due to Theodor Reye who used it first in 1876 in his *Geometrie der Lage*, Band I, 2 Aufl., S. 4, and who defined the term for the two-space and three-space in his article "Das Problem der Configurationen" in *Acta Mathematica*, Vol. 1, pp. 93–96. The general matrix definition now in use was given by E. H. Moore, "Tactical Memoranda," *American Journal of Mathematics*, Vol. 18 (1896), pp. 264–303.

synthetically by Fano[1] and analytically by Hessenberg[2] and its group properties had been studied by Moore.[3]

Veblen and Bussey (*l.c.*) developed a method for constructing all configurations which satisfy the definition and showed that they have a geometrical theory identical in most of its general theorems with ordinary projective geometry. They thus afford a treatment of finite linear group theory analogous to the ordinary theory of collineations.

Some of the geometric properties of the $PG(2, 2^n)$ have been given by the writer[4] and the three-space $PG(3, 2)$ was studied by Conwell.[5] Conwell showed that the group theory of the $PG(3, 2)$ furnishes a complete solution to Kirkman's school-girls problem[6] and is related to several functions which are of importance in the Galois theory of equations. Veblen[7] made use of finite geometries in considering the problem of map-coloring and H. H. Mitchell[8] determined the finite ternary and quaternary linear groups (previously determined by other methods) by geometrical methods suggested by finite geometries.

A systematic treatise (159 pp.) on trigonometry, logarithmic curves and the geometry of the straight line, circle and conics in a modular space of two dimensions was published in 1911 (Paris, Gauthier-Villars) by Gabriel Arnoux[9] under the title *Essai de Géométrie analytique modulaire a deux Dimensions.*

Dickson, in the *Madison Colloquium Lectures* (Amer. Math. Soc., 1914), discussed modular geometry and covariantive theory of a quadratic form in m variables, modulo 2 (Lecture IV), and a theory of plane cubic curves with a real inflexion point valid in ordinary and in modular geometry (Lecture V).

Much of the theory developed in Veblen and Young's *Projective Geometry,* (Boston, 1910 and 1918) is as valid in modular as in ordinary projective geometry.

A brief discussion of the $PG(2, 3)$ was given by Bennett in this MONTHLY for October 1920, pp. 357–360. His remarks on the significance of modular geometry (pp. 360–1 and footnote) should also be noted in this connection.

Finite geometries are useful in illustrating clearly the relation between

[1] "Sui postulati fondamentali della geometria projettiva," *Giornale di Matematiche,* Vol. 30 (1892), p. 106.

[2] "Ueber die projective Geometrie," *Sitzungsberichte d. Berl. math. Gesellschaft,* 1902–03, pp. 36–40.

[3] "Concerning Jordan's linear groups," *Bulletin Amer. Math. Society,* Vol. 2 (1895–96), pp. 33–43.

[4] "Geometry and Collineation Groups of the Finite Projective Plane $PG(2, 2^2)$," Lawrence, Kansas, 1913.

[5] "The Three-Space $PG(3, 2)$ and its Group," *Annals of Mathematics,* Vol. 11 (1910), pp. 60–76.

[6] For history of this problem and bibliography of literature relating to it, see Ball's *Mathematical Recreation and Essays,* fifth ed. (London, 1911), pp. 193–223.

[7] *Annals of Mathematics,* Vol. 14 (1912), pp. 86–94.

[8] *Transactions of the American Mathematical Society,* Vol. 12 (1911), pp. 207–242, and Vol. 14 (1913), pp. 123–142.

[9] The first sentence of the preface, however, gives credit to collaborators in the following language:

"Bien que le présent Livre sur la couverture mon nom seul, il a été en réalité le résultat d'une véritable collaboration entre M. Laisant, M. Gaston Tarry et moi."

euclidean and projective geometry. For example, if the finite euclidean plane geometry, modulo 3, be pictured it will be seen to consist of 9 points and 12 lines and that the addition of the 4 ideal points in which these lines meet and the addition of the ideal line on which the 4 ideal points lie gives the finite plane projective geometry, modulo 3, consisting of 13 points and 13 lines.

In working with modular geometries recently the writer observed that if the point $(x_n, x_{n-1}, \cdots, x_1)$ in the euclidean geometry $EG(n, p)$, be taken to represent the integer $x_n p^n + x_{n-1} p^{n-1} + \cdots + x_1$, where p is any prime and $x_1, x_2, \cdots x_n$ are marks of the Galois Field $GF(p)$, we have a geometric representation of the positive integral number system from 0 to p^{n+1}. The algebra of points in the $EG(n, p)$ would then be abstractly identical with the arithmetic of the corresponding integers and should afford a geometric attack on problems in number theory (*e.g.*, the problem of primes). In the correspondence thus set up the coordinates of a point become the ordinary positional notation for its corresponding number when the number is expressed in a scale whose radix is p. For example, if $p = 3$ the point (1, 2, 2) in the euclidean three-space $EG(3, 3)$ is the number 122 in the ternary scale or 17 in the decimal scale. Since the number of primes is infinite and such a correspondence can be set up for every prime we thus have an infinite variety of ways of representing the positive integral number system geometrically. Obviously other ordered correspondences can be set up for fields formed by adjoining other units. Hence it seems not unlikely that finite geometries may find important applications in the theory of equations, as, perhaps, might have been expected from the fact that they have found immediate applications in group theory.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I. [*1918*, 33; *1919*, 167; *1920*, 28, 223.]

The officers of the club for the year 1920–21 are:

Chairman, Professor Roland G. D. Richardson.

Committee on Program, Professor Raymond C. Archibald, Professor Robert W. Burgess, Rachel T. Easterbrooks Gr., Elsie E. Lord '21, Clarence M. Eddy '22, Everett L. Sweet '21.

Committee on Arrangements, Raymond L. Wilder, Assistant in mathematics, Katherine E. Colton '22, Margaret C. Packer '21, Allan A. Edgcomb '22, Francis L. Jones '23, Kenneth H. N. Newton '22.

Programs for the current year follow.

October 29, 1920: "Interpolation" by Albert A. Bennett '10, Associate professor of mathematics, University of Texas.

December 10: "Curiosities in numbers" by Margaret C. Packer '21; "Construction and use of mortality tables" by Philip M. Brown '22; "Stephen Leacock as biographer and epistemologist" [1] by Nellie C. Stokes '23.

[1] S. Leacock, "A, B, and C, the human element in mathematics" and "Boarding-house geometry" in *Literary Lapses*, London, 1912.

January 14, 1921: "Geometrical dissection of figures" by James B. Hobbs '18; "The problem of squaring the circle" by May B. Carter Gr.

February 18: "A new conograph" by Allen A. Edgcomb '22; "Incidents in the lives of mathematicians" by Constance W. Haley '21, Charles Hopkins '22, Elsie E. Lord '21, Edward S. Skillings '23, Helen F. Sheehan '22, Everett L. Sweet '21.

March 11: "Geometrical conics" by Julian L. Coolidge, Professor of mathematics, Harvard University. The club picture is to be taken at this meeting.

April 15: "Sir Isaac Newton" by Stuart H. Tucker '22; "The origin of our numerals" by Rose M. Finkelstein '22; "Gear ratios" by Kenneth H. N. Newton '22.

May (date to be announced later): Picnic.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF KANSAS, Lawrence, Kansas. [*1918*, 35, 450, 459; *1919*, 208; *1920*, 76.]

The officers of the club for the year 1920–21 are as follows: President, Nadene Weibel '21; vice-president, Vera Steininger '21; Secretary-treasurer, Lillie Strand '21; reporter, Sidonie Schafer '21; faculty adviser, Professor Ulysses G. Mitchell; program committee, Marie Shaklee '21, Marie McKinney '21, the secretary and the faculty adviser.

Programs for the year 1920–21 are given below.

October 13, 1920: "In times of rhymes," a review of an old textbook, by Professor Ulysses G. Mitchell.

October 27: "How elementary mathematics is used in astronomy" by Dinsmore Alter, Associate Professor of Astronomy, University of Kansas.

November 10: "Some peculiar graphs," by Assistant Professor Guy W. Smith.

December 8: "How elementary mathematics is used in chemistry" by Ralph Buffington Gr.

January 5, 1921: Review of Dudeny's "*Canterbury Puzzles*" by Vera Steininger '21.

January 19: "Some inherited problems" by Fern Smith Gr.

February 9: "The fourth dimension" by Lillie Strand '21.

February 23: "Card tricks" by Nadene Weibel '21.

March 9: "The great cryptogram,"[1] the argument for Bacon's authorship of Shakespeare's works, by Gladys Jones Gr.

March 23: "The arithmetic teaching of a hundred years ago" by Marie Brown '21.

April 13: "Reliability of teachers' marks" by Nina McLatchey, Instructor in mathematics.

April 27: "Arithmetical prodigies" by Ruth Strickler Gr.

May 11: "The story of Hypatia" by Jessamine Fugate '22.

May 25: Annual Picnic.

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[1] Cf. Ignatius Donnelly's book *The Great Cryptogram*, R. S. Peale & Co., Chicago, N. Y., London, 1888.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2876. Proposed by H. S. UHLER, Yale University.

Let a, d, n, and r denote respectively the refracting angle of a prism, the total deviation of the ray (in a principal plane) produced by the prism, the relative index of refraction of the material of the prism with respect to the single surrounding medium, and the angle of refraction at the first or incidence face of the prism.

(*a*) Deduce the (new) formula given below. (*b*) Use this formula to show (without the calculus) that the deviation has the least value when the ray is symmetrically situated with respect to the two refracting faces. (*c*) Show that, for minimum deviation, the formula reduces to the classical laboratory expression for determining the index of refraction.

$$\frac{\sin^2 \frac{1}{2}(a+d)}{\sin^2 \frac{1}{2}a} = n^2 + \frac{\sin \frac{1}{2}d \sin (a+\frac{1}{2}d) \sin^2 \frac{1}{2}(a+d) \sin^2 (r-\frac{1}{2}a)}{[\sin^2 \frac{1}{2}a \cos^2 \frac{1}{2}(a+d) \cos^2 (r-\frac{1}{2}a)+\cos^2 \frac{1}{2}a \sin^2 \frac{1}{2}(a+d) \sin^2 (r-\frac{1}{2}a)] \sin^2 \frac{1}{2}a}$$

2877. Proposed by J. B. REYNOLDS, Lehigh University.

A particle slides down the rough arc of a cardioid, $r = a(1 - \cos \theta)$, which lies in a vertical plane, the initial line being horizontal. If the coefficient of friction, μ, equals $\frac{1}{3}$, find θ for the point where the particle leaves the curve, if it starts at $\theta = 90°$.

2878. Proposed by R. S. HOAR, Fort Banks, Mass.

Consider the integers $0, 1, 2, 3 \cdots n-1, n$. Consider all possible permutations of combinations of these taken r at a time, allowing any integer to occur more than once. Select from these permutations all groups the sum of whose integers is n. Form the reciprocal of the product of the factorials of the r integers of each of these selected groups. Then the sum of all of these reciprocals will equal $r^n/n!$. Prove that this must be so.

Example. $n = 2$, $r = 3$.

$$\frac{1}{2!0!0!}+\frac{1}{0!0!2!}+\frac{1}{0!2!0!}+\frac{1}{1!0!1!}+\frac{1}{1!1!0!}+\frac{1}{0!1!1!}=\frac{3^2}{2!}, \text{ if } 0! = 1.$$

2879. Proposed by E. J. OGLESBY, Washington Square College.

Given the values of $U_{5:9}$, $U_{5:10}$, $U_{5:11}$, $U_{6:9}$, $U_{6:10}$, $U_{6:11}$, $U_{7:9}$, $U_{7:10}$, $U_{7:11}$ where $U_{h:k} = \sqrt{hk}$, find the value of $U_{6.2:9.3}$ by interpolation.

2880. Proposed by SIDNEY DORB, Detroit, Mich.

Solve the simultaneous equations: $xy = 2$,

$$\left(3 - \frac{6y}{x-y}\right)^2 + \left(3 - \frac{6y}{x+y}\right)^2 = 82.$$

2881. Proposed by E. B. ESCOTT, Oak Park, Ill.

If, in the polynomial $X^3 - 2$, we substitute $x^2 + x - 4$ for X, the given expression can be factored, that is, $X^3 - 2 \equiv (x^3 + 3x^2 - 3x - 11)(x^3 - 6x + 6)$. Find a substitution for X so that the polynomial $X^3 + pX^2 + qX + r$ may be factored.

2882. Proposed by E. T. BELL, University of Washington.

Solve each of the following systems of finite difference equations, giving x_n, y_n, z_n as explicit functions of x_0, y_0, z_0, n:

$$(1)\quad x_n = x_{n-1}(y_{n-1}^3 - z_{n-1}^3), \qquad y_n = y_{n-1}(z_{n-1}^3 - x_{n-1}^3), \qquad z_n = z_{n-1}(x_{n-1}^3 - y_{n-1}^3);$$

$$(2)\quad \begin{cases} x_n = x_{n-1}^6 y_{n-1}^3 + y_{n-1}^6 z_{n-1}^3 + z_{n-1}^6 x_{n-1}^3 - 3x_{n-1}^3 y_{n-1}^3 z_{n-1}^3, \\ y_n = x_{n-1}^3 y_{n-1}^6 + y_{n-1}^3 z_{n-1}^6 + z_{n-1}^3 x_{n-1}^6 - 3x_{n-1}^3 y_{n-1}^3 z_{n-1}^3, \\ z_n = x_{n-1} y_{n-1} z_{n-1}(x_{n-1}^6 y_{n-1}^6 + z_{n-1}^6 - y_{n-1}^3 z_{n-1}^3 - z_{n-1}^3 x_{n-1}^3 - x_{n-1}^3 y_{n-1}^3). \end{cases}$$

A solution of either would be acceptable. From the theory of plane cubic curves a geometrical construction for the solution may be given. Also both systems have solutions in terms of elliptic functions. Do solutions exist, and how are they found, without using elliptic functions?

PROBLEMS—NOTES.

4. On page 83 of *Revista de Matematicas y Fisicas Elementales*, published at Buenos Aires, August, 1920, Problem 2785 of this MONTHLY [*1919*, 366], and the solution [*1920*, 237] of its proposer, Professor W. H. ECHOLS, are translated into Spanish. The problem in question is: "If on the sides, as bases, of any closed plane polygon, there be constructed similar triangles similarly placed, all outward or all inward, then the centroid of the vertices of these triangles coincides with the centroid of the corners of the polygon."

5. In *The Ladies Diary: or the Women's Almanack*, for 1739, the following problem (207) was proposed: "There came three Dutchmen of my acquaintance to see me, being lately married; they brought their wives with them. The men's names were Hendrick, Claas, and Cornelius; the women's Geertruii, Catriin, and Anna; but I forgot the name of each man's wife. They told me they had been at market to buy hogs; each person bought as many hogs as they gave shillings for each hog: Hendrick bought 23 hogs more than Catriin, and Claas bought 11 more than Geertruii; likewise, each man laid out 3 guineas more than his wife. I desire to know the name of each man's wife?" Solutions were published in the *Diary* for 1740[1]; in one of them it is remarked that "the number of hogs, the three men and their respective wives bought, will be express'd by three pair of numb. the difference of whose squares must be 63."

6. Dr. OTTO KLOTZ, of the Dominion Observatory, Ottawa, suggested the following solution of the problem, to construct with ruler and compasses lines equal in length to the reciprocals of the positive integers: On AB, the unit of length, construct the square $ABCD$ (the reader is requested to draw the figure). Let the diagonals BD, AC intersect in E. Through E draw $EF \parallel DA$ and meeting AB in F; join CF meeting DB in G. Through G draw $GH \parallel DA$ and meeting AB in H; join CH meeting DB in I. Through I draw $IK \parallel DA$ and meeting AB in K; and so on. $BF = 1/2$; $BH = 1/3$; $BK = 1/4$; etc.

Dr. Klotz's construction would be equally valid if a parallelogram replaced the square $ABCD$. Indeed both of the constructions may be looked upon as

[1] These solutions were reprinted in *The Mathematical Questions proposed in the Ladies' Diary* . . . by Thomas Leybourn, London, 1817, volume 1, pp. 280–282; *The Diarian Miscellany* . . . by Cha. Hutton, London, 1775, volume 2, pp. 104–106. One of the solutions was reprinted in *The Diarian Repository; or Mathematical Register* . . . by a Society of Mathematicians, London, 1774, pp. 359–360.

particular cases of the following construction with ruler only, whenever a parallel to AB[1] is given: Join O, a point not on AB or the parallel, to A and to B meeting the parallel in D and C respectively. Let DB and CA meet in E, and OE meet AB in F. Join CF meeting DB in G, and let OG meet BA in H, etc. The points F, H, etc., are the same as those found before. This construction was given by Brianchon in 1818 in his *Application de la Théorie des Transversales*, page 37. He remarks: "Ce problème pourrait servir à se former, sur le terrain, une échelle de lever, si on n'avait pas à sa disposition une des mesures reçues, et qu'on connût d'ailleurs la longueur totale de la ligne prise pour échelle."

Lambert gave another construction in his *freye Perspective, oder Anweisung* . . . 1774, pages 173–174.

ARC.

PROBLEMS—SOLUTIONS

19 (Calculus) [1894, 165, 273–275]. Proposed by A. L. FOOTE, Merrick, N. Y.

A and B are in a circular room $2R = 30$ feet in diameter, A being at the center and B at the circumference. B runs around at the rate of $v = 600$ feet per minute and A pursues him at the rate of $u = 100$ feet per minute. How long will the race last, and how far will each have traveled till B is caught?

160 (Calculus) [1902, 271; 1903, 104–106]. Proposed by B. F. FINKEL, Drury College.

A dog at the vertex of a right conical hill pursues a fox at the foot of the hill. How far will the dog run to catch the fox, if the dog runs directly toward the fox at all times and the fox is continually running around the hill at its foot, the velocity of the dog being 6 feet per second, the velocity of the fox being 5 feet per second, the hill being 100 feet high and 200 feet in diameter at the base?

273 (Calculus) [1909, 76, 123–124; 1910, 221]. Proposed by J. SCHEFFER, Hagerstown, Md.

On one side of a circular pond a feet in radius is a duck. On the diametrically opposite side of the pond is a dog. Both swim at the same time, the duck swimming around the circumference of the pond at the rate of m feet a minute, the dog swimming directly towards the duck at the rate of n feet per minute. How far will the dog swim in overtaking the duck?

2801 [1920, 31]. Proposed by A. S. HATHAWAY, Houston, Texas.

A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $n : 1$, determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

I. REMARKS AND HISTORICAL NOTES BY R. C. ARCHIBALD AND H. P. MANNING, Brown University.

In 1732 Bourger read before the French Academy a memoir "Sur de nouvelles courbes ausquelles on peut donner le nom de Lignes de Poursuite"[2] in which he solved the following problem: "Trouver la courbe de poursuite, c'est-à-dire la courve par laquelle un vaisseau doit en poursuivre un autre qui s'enfuit par une ligne droit, en supposant que les vitesses des deux vaisseaux soient toûjours dans le même rapport."

Maupertuis gave[3] a briefer solution of this and formulated also the following more general

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[1] Or AA' bisected at B; with this given Lambert showed, in 1774, that a parallel to AB can with ruler alone, readily be drawn through any point.

[2] *Histoire de l'Académie Royale des Sciences*, 1732. Paris, 1735, Memoires, pp. 1–14.

[3] Idem, pp. 15–16.

problem in the solution of which he was led to "l'équation de la courbe en secondes différences": "La courbe CE étant donnée; trouver la courbe BM, telle que ses tangentes ME, coupent sur la courbe CE des arcs proportionnels aux arcs BM?"

The earliest reference, we have found, to the consideration of the curve CE as the arc of a circle is in J. Ficklin's problem in an anonymous article of *The Mathematical Monthly* (Runkle), volume 1 (1859), p. 249. In the problem the velocities of pursued and pursuer are supposed equal. For this case it is stated that "the curve of pursuit continually approaches the circle, to which it becomes an asymptote and meets after an infinite number of revolutions, when the two bodies will be together." In the course of the article (pp. 249–251) other velocity ratios $(m : n) \lesseqgtr 1$ are considered. For $m < n$, the pursuer will never overtake the pursued; "but its nearest approach is a point to be determined." And for $m > n$, the pursuer will overtake the pursued and if the pursuer's motion continues, his "path outside of the circumference will be a wave-like curve, the oscillations growing smaller as the pursuer's distance from the circumference becomes greater." No analyses or reasons are given in support of any of these statements.

The problem of determining the curve of pursuit in the case of the circle interested H. Brocard for several years before he proposed it for solution in *Nouvelle Correspondance Mathématique*, May, 1877, vol. 3, p. 175. No solution being forthcoming he asked for the differential equation of the curve in *Mathesis*, December, 1883, vol. 3, p. 232; this was given by Keelhoff in *Mathesis*, 1886, vol. 6, p. 135.

The problem appears in the following form in *Revue de Mathématiques Spéciales*, February, 1894, vol. 2, p. 272: "Un cheval se meut sur une piste circulaire avec une vitesse uniforme; un jockey parti du centre se dirige continuellement vers le cheval pour l'atteindre, animé lui-même d'une vitesse constante. Quel est la courbe décrite par le jockey?" No answer to this problem has been given in the *Revue*.

Another unanswered query was published in *L'Intermédiaire des Mathématiciens*, October, 1894, vol. 1, p. 183: "En supposant que la lumière d'une étoile mettre douze heurs a parvenir au centre du circle que l'étoile parcourt en vingt-quatre heurs, quelle serait l'équation de la trajectoire décrite par un oiseau qui, partant du centre, se dirigerait constamment, avec une vitesse donnée, vers le point où il voit l'étoile?"

And again in the first volume of this MONTHLY, August, 1894, 273–275 four contributors found for the problem enunciated above by A. L. Foote, that the solution of the differential equation "transcends," as one of the contributors remarks, "the present limits of mathematical genius." A slight variant of this problem appeared fifteen years later in J. Scheffer's problem.

Professor Finkel's form was first published in the *Educational Times*, 1888, problem 9448. "Solutions" were first given in the issue for July 1, 1903; *Mathematical Questions and Solutions*, new series, vol. 5, 1904, pp. 30–31. Professor Finkel made clear the equivalence of his problem with that of the old problem as follows: "Conceive the surface of the cone to be spread out on a plane. . . . This surface may be repeated a sufficient number of times to complete the race. We may conceive the surface as being an infinitely thin membrane and allowed to overlap; then when the race has been continued on the second round about the vertex we may conceive the dog in the surface beneath, and so on for any number of rounds. . . ." The concluding sentence is: "This differential equation has never been integrated so far as I know."

The differential equation of the curve of pursuit for a circle was discussed in print for the first time in an extensive paper by L. Dunoyer in *Nouvelle Annales de Mathématiques*, May, 1906, vol. 65, pp. 193–222. He considered the relative motion of two points, the one moving around the circle and the other starting anywhere in its plane, and showed that the problem led to a differential equation of the form $dx/X = dy/Y$; each of the functions X and Y is a polynomial of the third degree in x and y. The form of the integral curve in the vicinity of a singular point is studied according to principles of Poincaré set forth by Picard in his *Traité d'Analyse*, vol. 3, second edition, 1908, chapter 2.

Mr. F. V. Morley's graphical solution of the problem proposed by Professor Hathaway appears elsewhere in this issue of the MONTHLY (pages 54–61).

The only case when there is any difficulty in deciding whether or not the dog, starting from the center of the pond, will catch the duck is the case not considered by either Mr. Morley or Mr. Dunoyer, namely, when the dog and duck travel at the same rate. This case seems to have been solved by Professor Hathaway alone, see below, unless the anonymous author of the article in Runkle's *Mathematical Monthly*, referred to above, had proved the statement that he made.

To the case of problem 2801 when $n = 1$ an anonymous five-page pamphlet (about $3\frac{1}{4} \times 6\frac{1}{4}$ inches), without date, has been devoted. It is entitled *A Common Sense Solution of a Curve of*

Pursuit Problem that has been considered unsolvable by many eminent mathematicians.[1] The author is L. T. Houghton of Worcester, Mass.

Mr. Houghton has a relativity theory for his dog and duck problem. He says we might think of the duck as stationary and the pond as revolving in the opposite direction[2]; that it makes no difference in the dog's path through the water whether the pond revolves or the duck swims. If the duck is stationary and the dog moves along the radius to the duck, his path in the water will be carried around with the pond and will be curved. It is necessary for him to swim in the water in a direction oblique to the radius in order to overcome the current as well as to proceed along this radius.

It is the dog's path through the water that Mr. Houghton takes for his actual path when the pond is stationary and the duck swims. This path is characterized by the fact that the dog and duck are always on the same radial line. It is a circle of radius one-half of the radius of the pond, tangent at the center of the pond to the radius which passes through the starting position of the duck.[3] We can prove this by forming the equation of the dog's path, but it may be noted at once that the dog swimming on this circle at the same rate as the duck will always be on a line between the center of the pond and the duck, two positions of the line forming an angle which is at the center of the pond and inscribed in the circle. This makes the dog catch the duck when the latter has swum through an arc of 90°, instead of an arc equal to the diameter as Mr. Houghton supposes.

On this path, however, the dog does not swim directly towards the duck, for its tangent always points ahead of the duck. Mr. Houghton objects strongly to the "tangent method for tracing a pursuer's path," but this seems to be a question as to the meaning of "directly towards." We have simply two problems, Mr. Houghton's problem, where the pursuer is always on a straight line drawn from his original position to the pursued; and the "tangent problem," which is the problem under discussion, in which the pursuer's path is always tangent to the straight line joining the two. According to Professor Hathaway's solution, given below, the tangent problem must be answered in the negative.

II. Solution of Problem 2801 by the Proposer.

We shall show that: *When the dog starts from any point in the pond with equal or greater speed, he is invariably drawn into the outline of a leaf of a four-leaved clover fixed on the shoulder of the duck, remaining there and getting as near to the duck as one pleases to name, or eventually catching it.*

Let Q, P (fig. 1) be any positions of dog and duck. The tangents QP at Q and PT at P make angles ψ, ϕ, with a fixed line, and if C be the center and $\theta = \angle CPQ$, we have

$$\psi = \phi + \theta - \frac{\pi}{2}, \qquad d\psi = d\phi + d\theta. \tag{1}$$

If the arcs described from fixed points be S and s, we also have $dS = kds$, where k is the ratio of speeds. Further, $CP = a$, $QP = r$, and, drawing TR perpendicular to QP at R, we have the differential triangle PRT whose sides are

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[1] The following sentences taken from different parts of the pamphlet represent fairly well the position of the author:

"This curve of pursuit problem has estranged old friends and vexed eminent mathematicians. Many wagers have been made which professors of mathematics have been called on to settle, and their decisions have been in the negative [that is, the dog will not catch the duck], without one single line of proof to sustain their findings. A bare assertion is not satisfactory proof." . . .

"The 'nevers' have claimed that the pursuer's position [path] is always tangent to a straight line drawn to the corresponding position of the pursued. This is the modern theory and also a false one. Tracing the dog's path by setting off the circumference into short spaces, and setting off equal distances on the line from the dog to the duck [in its successive positions] will show that the pursuer is always on a straight line drawn from his original position to the pursued, and that the duck is caught when it has moved over a portion of the circumference equal to the diameter."

[2] This idea is virtually the same as the idea, which Professor Hathaway uses (in his solution of problem 2801), of a system of polar coördinates moving with the duck.

[3] See the "turn-table" problem, *Math. Visitor*, 1878, vol. 1, p. 37; also *Math. Quests. Educ. T.*, 1889, vol. 51, p. 157.

$$PT = ds, \qquad PR = dr + dS = dr + kds, \qquad RT = rd\psi = rd\phi + rd\theta.\text{[1]} \tag{2}$$

Since $\angle\, PTR = \angle\, CPQ = \theta$ (sides perpendicular),

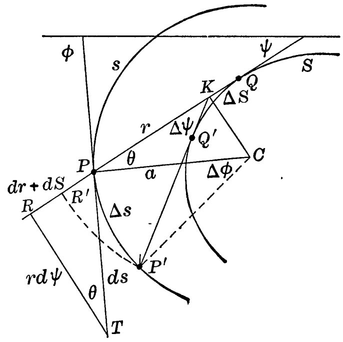

FIG. 1.

$$\sin\theta = \frac{dr}{ds} + k, \qquad \text{or} \qquad \frac{dr}{ds} = \sin\theta - k; \tag{3}$$

$$\cos\theta = r\frac{d\phi}{ds} + r\frac{d\theta}{ds} = \frac{r}{a} + r\frac{d\theta}{ds}, \quad \text{or} \quad \frac{d\theta}{ds} = \frac{a\cos\theta - r}{ar} \tag{4}$$

($\Delta s = a\Delta\phi$, therefore $ds = ad\phi$).

Eliminating ds we have

$$r\frac{d\theta}{dr} = -\frac{a\cos\theta - r}{ka - a\sin\theta}. \tag{5}$$

THEOREM 1. *If $k \geqq 1$, r decreases without a minimum; if $k < 1$, $\sin\theta = k$ determines a minimum or maximum value of r.*

Since ds is always positive (s always increasing), dr is always negative by (3) if $k \geqq 1$, but may be positive or negative if $k < 1$. Here $dr = 0$ is a max-min condition, but is not sufficient, namely, dr must change sign. This is not satisfied when $r = 1$, $\sin\theta = 1$. Therefore when $k \geqq 1$ r is always decreasing and has no minimum.

THEOREM 2. *The angle $\theta = \angle\, CPQ$ is increasing or decreasing according as Q is inside or outside of the circle on CP as diameter.*

Draw CK perpendicular to PQ at K. Then, if we take segments along PQ as positive in the direction from P to Q, we have $QK = a\cos\theta - r$, positive or negative, and therefore $d\theta$ positive or negative, according as Q is inside or outside of the circle CKP (r is always positive during the time of the present problem. Only after $r = 0$ may it be negative).

THEOREM 3. *QK is increasing or decreasing according as Q lies outside or inside of the leaves in the first and third quadrants of the four-leaved clover whose equation referred to pole P and axis PC is $kr = a\cos\theta\sin\theta$.*[2]

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[1] We follow Newton's definition. "Differentials are corresponding limits of equimultiples of vanishing differences." Also, "instantaneous state" is mathematically defined as that variation in which differentials are increments, just as in actual variation differences are increments. It is demonstrated, not assumed, that if a point be generating a curve, its instantaneous state generates the tangent, etc. Taking differences from a succeeding position, $Q'P' = r + \Delta r$ (Fig. 1), unwrap an inextensible string $QQ'P'$, from the arc QQ', describing the arc $P'R'$ always normal to the string.

$$\text{arc } PP' = \Delta s, \qquad PR' = \Delta r + \Delta S, \qquad \text{arc } R'P' = q\Delta\psi, \qquad r + \Delta r < q < r + \Delta r + \Delta S.$$

Extend PR' and PP' in the ratio $N : 1$ so large that $N \cdot PP'$ is as near as we please to PT, when $Q'P'$ is as near as we please to QP. The differences and figure $PR'P'$ are vanishing, but the finite figure, similar to $PR'P'$, approaches a limit, namely, the right triangle PRT. Hence the above differential values, by definition of differentials. See *Science*, July 11, 1919, and Feb. 13, Mar. 28, May 7, June 11, July 9, 1920, for correspondence on the early history and concepts of the calculus.

[2] This curve, so named by M. Simon (*Analytische Geometrie*, Leipzig, 1900, p. 316), and called also the four-leaved rose curve, and the quadrifolium, is one of the class of rose curves or rhodonees, $r = a\sin m\theta$, $r = a\cos m\theta$, discussed by Guido Grandi in letters to Leibnitz, 1713, in *Philosophical Transactions*, 1723, and in his book, *Flores geometrici ex rhodonearum et clœliarum curvarum descriptione resultantes*, 1728. These curves have been also called corollae by W. J. C. Miller (*Math. Questions with their Solutions*, London, vol. 19, 1873, p. 60) on account of their fanciful resemblance to the petals of an open flower. It was remarked empirically in 1752, by G. B. Suardi, that the corollae can all be generated as elongated or contracted epi- or hypocycloids, but the proof of this fact was first given in 1844 by L. Ridolfi. Bellavitis showed in 1852 that the pedal curve of any ordinary epi- or hypocycloid with respect to the center of its fixed circle is a rose curve; in particular the four-leaved clover is the pedal curve of the so-called four-cusped hypocycloid or astroid. It is also the orthoptic curve of the astroid (Lambiotte, 1877) and the inverse of the

These leaves are fixed on the duck, since the axis PC is so fixed. Also with that axis and pole (r, θ) represents Q. Then

$$\frac{d(QK)}{ds} = -a \sin\theta \frac{d\theta}{ds} - \frac{dr}{ds} = \frac{kr - a\cos\theta\sin\theta}{r}, \qquad [(3), (4)]$$

and $d(QK)$ is positive or negative with $kr - a\cos\theta\sin\theta$. Since r is positive, the latter is positive, not zero, in the second and fourth quadrants, and is zero only on the leaves in the first and third quadrants, positive outside, negative inside. This problem excludes the second and third quadrants, but in generalizing the start to any point on land, with the same ratio of velocities, approach may be from any quadrant.

THEOREM 4. *When $k \geqq 1$, the pursuing point Q can cross the clover leaf of the first quadrant only from outside to inside.*

Put $u = kr - a\cos\theta\sin\theta$, then by (3) and (4) we have, when $u = 0$ (Q on the clover leaf).

$$\frac{du}{ds} = \frac{\sin\theta - k}{\sin\theta} \,|\cos^2\theta + \sin\theta(k - \sin\theta)|\, < 0$$

since $\sin\theta > 0$. Therefore u is decreasing, and changes from positive to negative, or Q crosses from outside to inside.

If $k < 1$, then on the first quadrant leaf, entrances are up on the arc from P to $\sin\theta = \frac{1}{4}(k + \sqrt{k^2 + 8})$, and down from $\sin\theta = k$ to P; the arc between these rays contains exits only.[1]

THEOREM 5. *If $k \geqq 1$ the first quadrant clover leaf lies wholly inside of the semicircle CKP; if $k < 1$ the leaf intersects the semicircle on the chord $\sin\theta = k$.*

THEOREM 6. *The angle $CPQ = \theta$, and the distance $CQ = z$ are both increasing or both decreasing.*

From the triangle CPQ, $z^2 = a^2 + r^2 - 2ar\cos\theta$,

$$zdz = (r - a\cos\theta)dr + ar\sin\theta d\theta = kar\, d\theta$$

by (5); therefore

$$\frac{dz}{d\theta} = \frac{kar}{z} > 0.$$

The presence of Q in the circle CKP, if $k \geqq 1$, can now be shown, wherever it starts in the pond. When $k > 1$ we can say $dr/ds \leqq 1 - k$, a fixed number less than zero, and that r becomes zero for a finite value of s. When $k = 1$ we can say by theorem 2 that θ is always decreasing outside of the circle CKP. Suppose we start with the value θ_1, Q within the pond, so that $z = c < a$. As long as Q is outside of this circle θ will decrease, but will not decrease as far as $-\pi/2$, for z will also be decreasing. Then we have $dr/ds \leqq \sin\theta_1 - 1$, and therefore for a finite value of s, r becomes zero *or* Q comes into the circle. But in any case before r becomes zero z will be greater than c, and must sometime be increasing; then θ will be increasing, and Q inside of the circle CKP (theorem 2). Hereafter we start from this circle, dismissing $k < 1$ with the remark that r is increasing in the angle between the two rays $\sin\theta = k$ of the first and second quadrants, and decreasing outside of that angle, while θ is increasing inside of the circle CKP, and decreasing outside. There is a minimum r on $\sin\theta = k$ in the second quadrant, and one in the first quadrant inside of the circle CKP, and a maximum r on $\sin\theta = k$ in the first quadrant outside of that circle.

We found (5) to be the differential equation of pursuit as the duck sees it, that is, referred to the axis PC fixed on the duck. Let the plane of pursuit revolve about C with opposite the angular velocity of P (Fig. 2). No relative positions are changed, but an observer on the ground will see only a fixed point P and axis PC; and Q will appear to be moving on the ground, no longer towards P but in a direction QZ whose inclination to the radius vector QP is given by (5).

circular cross curve (Kreuzcurve)—concerning which considerable has been written since it was first conceived by Terquem in 1847.—EDITOR.

[1] In the leaf of the third quadrant, for all values of k, entrances are below, exits above, the ray of angle θ given by $\sin\theta = -2/(k + \sqrt{k^2 + 8})$ [180° to 225°], showing a drift round behind the duck.

QZ is constructed as follows: Produce CK to $CM = ka$ (in the positive sense, CK being equal to $a \sin \theta$, so that $KM = ka - a \sin \theta$). Then the direction of QZ is determined by revolving QM through a counter-clockwise right angle. For, the tangent of the angle which QM makes with the radius vector is $\dfrac{ka - a \sin \theta}{a \cos \theta - r}$, which by (5) is the negative reciprocal of the tangent of the angle made by QZ. By continuity, one case is a sufficient test of sense: when Q is on the circle CKP it moves towards P ($k \geqq 1$).

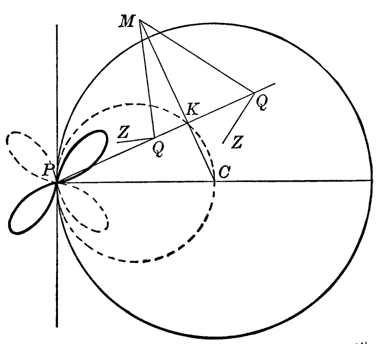

FIG. 2.

THEOREM 7. *Capture can take place only from directly behind or directly in front.*

If the chord QP of the curved path QP in Fig. 2 be vanishing, its angle with the tangent QZ must also be vanishing, and the equal angle KMQ. Therefore $QK = a \cos \theta - r$ is vanishing, and since $\lim r = 0$, $\lim \cos \theta = 0$, or the limit of θ is a right angle.

In the second and third quadrants PQ turns clockwise and all pursuit starting from there must capture from behind. Also all starting from the pond, or eventually in the circle CKP, must capture from behind. For, in the circle CKP QK is positive, and $\lim QK = 0$ (theorem 7), therefore QK must eventually decrease, and this is only true for Q in the clover leaf of the first quadrant (theorem 3) from which it cannot get out (theorem 4). This leaf is within the circle CKP (theorem 5), and θ must continuously increase to its limit $\pi/2$.

THEOREM 8. *The duck is caught or not caught according as* (*a*) $k > 1$, *or* (*b*) $k = 1$.

Let accents denote corresponding values. By the mean value theorem

$$(r' - r)/(s' - s) = \sin \theta'' - k, \qquad \theta < \theta'' < \theta' \quad (dr/ds = \sin \theta - k).$$

Therefore

$$(a)\ k > 1, \qquad s' - s = (r - r')/(k - \sin \theta'') < r/(k - 1).$$

Since any distance, $s' - s$, that the duck can go without being caught is less than a fixed distance it must be caught.

$$(b)\ k = 1, \qquad s' - s > (r - r')/(1 - \sin \theta).$$

Now if the duck is caught after going a finite distance, so that $\lim r' = 0$, we shall have eventually $s' - s > r/(1 - \sin \theta)$. But

$$\frac{r}{1 - \sin \theta} = \frac{a \cos \theta}{1 - \sin \theta} - \frac{a \cos \theta - r}{1 - \sin \theta} = \frac{a(1 + \sin \theta)}{\cos \theta} - a \tan \angle KMQ,$$

which may be made as large as we please, for we may start as near the duck as we please and take θ as near $\pi/2$ and $\angle KMQ$ as near zero as we please. This leads to a contradiction, and proves that the duck will not be caught in a finite time when $k = 1$.

We can employ the methods of this article when the path of the pursued is any curve, C being the center of curvature, a the radius of curvature, and $d\phi/ds = 1/a$ the curvature, but in differentiations a would not be constant.

The following dynamical development of the preceding is of interest. On the plane of pursuit draw the evolute of the path of the pursued, and let it roll on a fixed line carrying the plane of pursuit with it. When the pursued crosses the fixed line let the rolling motion begin and be always of velocity equal and opposite to that of the pursued about its center of curvature. Then the pursued remains in a fixed position on this line. For the actual motion of P is that of a fixed point on a line which rolls on the evolute, the point of contact C being its center of curvature at the instant. When the line rolls from C to C', bringing P to P', the arc CC' of the evolute rolls on the fixed line CP, C' falling at C'' on CP, and bringing P' back to P. All that varies is the position of the center of curvature from C to C'' on CP.

At the instant when C is the center of curvature and v the velocity of P the latter point has in the rolling plane a counter-clockwise angular velocity v/a about C, and this rolling plane, rolling to keep P stationary, has a clockwise angular velocity v/a about C. The apparent motion of the

pursuing point Q is the resultant of this clockwise angular velocity and a linear velocity kv towards P. This resultant is the same angular velocity about a new center M. Hence CM is perpendicular to PQ, and in sense so that, if turned a clockwise right angle about C it will lie in the direction PQ. Also $CM \cdot v/a = kv$, or $CM = ka$. Therefore the apparent velocity of Q is $MQ \cdot v/a$ in the direction QZ found by rotating QM about Q a counterclockwise right angle. From these considerations (3) and (4) may be derived.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

Mr. F. E. WOOD, who received his doctorate at the University of Chicago in December, has been appointed assistant professor of mathematics at the Michigan Agricultural College.

At Colgate University, Associate Professor A. W. SMITH has been made full professor and head of the department of mathematics as successor to Professor J. M. TAYLOR [*1921*, 43]; Professor T. R. AUDE, of the Carnegie Institute of Technology, has been appointed associate professor of mathematics.

Dr. L. A. POCHHAMMER, ordinary professor of mathematics at the University of Kiel for forty-three years, died on March 24, 1920, at the age of seventy-eight years. He was appointed extraordinary professor at Kiel in 1874, and was the author of a number of papers in *Mathematische Annalen* and Crelle's *Journal*.

At the meeting of the Mathematical Association of America, at the University of Chicago, December 28–29, the following officers were elected: President, Professor G. A. MILLER; Vice-presidents, Professor R. C. ARCHIBALD and Professor R. D. CARMICHAEL; Board of Trustees, to serve till January, 1924, Professor A. A. BENNETT, Professor FLORIAN CAJORI, Professor H. L. RIETZ and Professor D. E. SMITH. The Board chose Professor C. F. GUMMER to fill the vacancy on the Board caused by the election of Professor Carmichael as vice-president.

At the meeting of the American Mathematical Society, at Columbia University on December 28–29, 1920, the following officers were elected: President, Professor G. A. BLISS; Vice-presidents, Professor F. N. COLE and Professor DUNHAM JACKSON; Secretary, Professor R. G. D. RICHARDSON; Treasurer, Professor W. B. FITE; Librarian, Professor R. C. ARCHIBALD; Committee of Publication (to edit the *Bulletin*), Professor E. R. HEDRICK, Professor W. A. HURWITZ, and Professor J. W. YOUNG. The following members of the council were elected to serve until December, 1923: Professor T. H. GRONWALL, Professor O. D. KELLOGG, Professor FLORENCE P. LEWIS, and Professor A. D. PITCHER.

Announcement has been made at Brown University of the completion of the NATHANIEL FRENCH DAVIS FUND in honor of Professor Davis, now emeritus, who was for forty-one years a teacher of mathematics in the University. The Fund amounts to ten thousand dollars and the income is to supplement the regular library appropriations in purchasing mathematical books and periodicals for the mathematical seminary.

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

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I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars, to be known as the ....

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The Notes and News cover a wide range of interest and information both in this country and in foreign countries.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of mathematics for its own sake.

There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over twelve hundred individual and institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections hold one meeting and some cases two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

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FIFTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The fifth annual meeting of the Association was held at the University of Chicago on Tuesday and Wednesday, December 28 and 29, 1920, in affiliation with the American Association for the Advancement of Science and in conjunction with the western meeting of the American Mathematical Society. The Illinois Section met jointly with the Association in all its sessions and held a separate business meeting on Tuesday afternoon. There were 204 in attendance at the sessions, including the following 150 members of the Association:

KATHERINE S. ARNOLD, Milwaukee-Downer College.

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MYRTIE COLLIER, Southern Branch, University of California.
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H. H. CONWELL, Beloit College.
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ARNOLD DRESDEN, University of Wisconsin.

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SAINT MARY-OF-THE-WOODS COLLEGE, Sister Catherine Therese, official representative.
CREIGHTON UNIVERSITY, W. F. Rigge, official representative.

The meetings of the various scientific organizations were held in the buildings of the University of Chicago; thus there was readiness of access to a great variety of programs as well as to the three lectures on the general program of the American Association for the Advancement of Science, viz., "Twenty-five years of bacteriology—A fragment of medical research" by the retiring president, Dr. Simon Flexner, "The volcanic region of Katmai, Alaska" by Dr. R. F. Griggs, and "High power phosphorescence and fluorescence" by Professor R. W. Wood. Opportunity was afforded to visit on Wednesday afternoon the plant of Sears,

Roebuck and Company, and on Thursday afternoon the new Field Museum and the Newberry Library. The visiting ladies were pleasantly entertained on Thursday afternoon at Ida Noyes Hall by the director, Mrs. Goodspeed. Ample cafeteria service was afforded on the campus, and many small groups lunched each day at the Quadrangle Club.

On Wednesday evening the large number of 167 persons shared in the joint dinner of the Society, the Association, Section A of the American Association, and the American Astronomical Society. Professor D. E. SMITH acted as toastmaster, and brief speeches were made by Professor G. A. BLISS for the Society, Professor PHILIP FOX for the Astronomical Society, Professor G. A. MILLER for the Association, Professor D. R. CURTISS for Section A, Professor FLORIAN CAJORI for the newly organized Section L, and Mr. VINCENT W. BROWN, a representative of the Chamber of Commerce of St. Louis, who spoke of the necessity of a thorough training in the fundamentals and of a sound basis of mathematics for practically all lines of business. Professor C. I. PALMER exhibited a copy of the 1637 Descartes geometry, which Professor Cajori in his address of Wednesday had said he had never seen; this copy came to Professor Palmer from earlier possession by Professors Sylvester and George Bruce Halsted. At the close of the speeches Professor E. H. Moore read a reply from Professor Oskar Bolza, expressing appreciation of a joint greeting sent to him by those who attended the mathematical meetings last September.

It is gratifying to note here that the Association, as also the Society, has been invited to affiliate with the American Association for the Advancement of Science, and on this basis the Secretary-Treasurer took part in that association's Council meeting on Thursday morning, at which Professor E. H. MOORE was elected president of the American Association for the next year.

On motion of Professor Rietz at the final session, the Association adopted by a rising vote a resolution expressing appreciation of the courtesy and cordial welcome extended by the department of mathematics of the University of Chicago and of the work of the local committee that contributed so much to the very successful meetings.

The program for the various sessions continued the previous practice of the Association in including expository papers of a fairly elementary character and papers of historical interest. The latter was especially fitting at this time when Section L (Historical and Philological Sciences) of the American Association for the Advancement of Science was being organized. Professor Cajori presided at the Tuesday morning joint session with Section L, Professor Curtiss, retiring vice-president of Section A, at the Wednesday morning joint session, President D. E. Smith on Tuesday afternoon and incoming President Miller on Wednesday afternoon at the sessions of the Association. The following papers were given. Abstracts of most of these follow, the numbers corresponding to the numbers in the lists of titles:

Joint Session of the Association with Section L of the American Association.

(1) "Geometrical development of analytical ideas" by Professor L. C. Karpinski, University of Michigan.

(2) "The anharmonic ratio in projective geometry" by Professor E. B. Stouffer, University of Kansas.

(3) Introductory note on "The Association's ideal for expository papers" by Professor E. J. Wilczynski, University of Chicago.

(4) "The first work on mathematics printed in the New World" by Professor D. E. Smith, Columbia University.

(1) The purpose of this paper by Professor Karpinski is to show that many of the fundamental ideas of Greek geometry, and of all geometry up to the time of Newton, correspond closely to elementary analytical ideas. Even the proofs are frequently parallel to modern analytical proofs. The problem of the construction of the regular pentagon is closely connected with the historical development of a large portion of the first four books of Euclid; the problem reduces, of course, to the solution of a quadratic equation. The duplication of the cube, algebraically $x^3 = 2a^3$, was solved by the Greeks by two intersecting conics. Similarly the trisection of the angle, the regular seven- and nine-sided polygons were recognized by the Arabs as leading to cubic equations, and solved by conics. Even the squaring of the circle led the Greek Hippocrates to a problem on "application of areas," or quadratics. The problems of the infinitesimal and the theory of limits, with the Eudoxian "method of exhaustion" are also strictly analytical in statement.

(2) In line with the increased interest in the unifying concepts of mathematics, Professor Stouffer discussed the fundamental nature of the anharmonic ratio in projective geometry. The development of the anharmonic ratio concept from the time of Euclid was traced briefly and two theorems were stated which show that information concerning a projectivity is equivalent to information concerning anharmonic ratios. These theorems simplify the proof of many propositions of projective geometry. An illustration of this fact was given by the proof of an important theorem on conics. Several concepts of a fundamental nature in geometry were introduced by means of anharmonic ratios in order to show the value of the method.

(3) Professor Wilczynski's paper will appear in full in the April issue of the Monthly.

The Association was fortunate in having on the program four papers which were referred to by a number of members as excellent examples of the kind of expository papers which are possible and desirable.

(4) Professor Smith's paper was printed in the Monthly for January, 1921.

Session of the Association.

(5) "Rolle's theorem and its generalizations" by Professor A. J. Kempner, University of Illinois.

(6) "Some geometrical aspects of the theory of relativity" by Professor L. W. DOWLING, University of Wisconsin.

(7) Note on "The metric question from the historical standpoint" by Professor L. C. KARPINSKI, University of Michigan.

(5) Professor Kempner's paper dealt with extensions of the theorem that if a real polynomial equation, *i.e.*, an equation $f(x) = 0$, $f(x)$ a polynomial with real coefficients, has all of its roots real, then the roots of the derived equation are also all real and are separated by the roots of the given equation. He discussed the well-known Gauss-Lucas polygon theorem, according to which for any polynomial equation (with real or complex coefficients) the derived equation has all roots in the smallest convex polygon which can be described in the complex plane around the points representing the roots of the given equation. The extension of these theorems to other types of equations was considered, the various fields in mathematics in which Rolle's theorem is of importance were mentioned, and some outstanding problems were emphasized.

(6) The following is an outline of Professor Dowling's paper: I. *The Coördinate System.* (*a*) In primitive times. (*b*) In Ptolemy's day. (*c*) After Copernicus. (*d*) After Kepler and Newton. (*e*) After Lorentz;—the general quadratic form $g_{ik}\xi_i\xi_k$ and the associated bilinear form $g_{ik}\xi_i\eta_k$, together with the group of linear transformations which leave these forms invariant. A study of this group and its system of invariants and covariants constitutes the special theory of relativity.

II. *Minkowski Space-Time.* (*a*) The postulate of constant velocity of light; other known velocities less than or equal to that of light. (*b*) The formation of the "light-cone." (*x*) The quadratic form $\xi_1^2 + \xi_2^2 + \xi_3^2 - c^2\xi_4^2$ and the associated bilinear form $\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 - c^2\xi_4\eta_4$, together with the Lorentz transformation which leaves these forms invariant.

III. *Gravitation.* (*a*) The Principle of Equivalence. (*b*) A local gravitational field of force generated by the quadratic transformation $x = \bar{x}$, $y = \bar{y} + m\bar{x}^2$. (*c*) Quadratic Differential Forms $g_{ik}dx_idx_k$;—the necessary and sufficient conditions that two such forms shall be equivalent. (*d*) The Einstein Space-Time as compared with the Minkowski Space-Time;—their non-equivalence.

IV. The mathematical basis of the special theory of relativity lies in the work of Cayley, Sylvester, Salmon in England; Clebsch, Gordan, Aronhold in Germany; Hermite in France; Brioschi in Italy and many others.

The general theory of relativity rests upon the work of Gauss (1827), Lamé (1859), Riemann (1864), Christoffel (*Crelle* 1870), Ricci and Levi-Cività (*Math. Annalen* 1901), Lie (*Math. Ann.* 1884), Maschke (*Transactions A. M. S.* 1900–1903).

(7) This note by Professor Karpinski calls attention to the desirability that the Association actively enroll itself as supporting the meter-liter-gram system in the United States. A quotation from Simon Stevin, the first writer on decimal fractions, reveals the comprehensive appreciation on the part of Stevin of the great use of decimal fractions as applied to weights, measures, and money.

Joint Session of the Association with the American Mathematical Society and Sections A and L of the American Association.

(8) "A decade of American mathematics," Retiring address as chairman of Section A, Professor O. D. Kellogg, Harvard University.

(9) "Evolution of algebraic notations" by Professor Florian Cajori, University of California.

(8) The address of Professor Kellogg was a rapid survey of the distribution of the mathematical effort of the decade in America among the various branches of the subject. It contained pleas for greater development of mathematical physics through a more rational attitude of mathematician and physicist each toward the needs of the other; for the more general cultivation of a sense of values; for the development of a sense of obligation of the individual mathematician to support the publication of worthy American monographs, to produce to the best of his ability, and to give greater attention to making his own contributions and his science in general appeal as widely as possible. This address will be printed in *Science.*

(9) Professor Cajori exhibited slides made from early Italian, German, French and English books and manuscripts, as well as from seventeenth, eighteenth and nineteenth century text books, for the purpose of showing the struggle which has been going on between the rhetorical and the purely symbolic tendencies. From the experiences of the past, inferences were drawn which may serve as present and future guides to mathematicians on matters of algebraic symbolism.

Session of the Association.

(10) "General aspects of the problem of interpolation" by Professor Dunham Jackson, University of Minnesota.

(11) "Construction of double entry tables" by Professor A. A. Bennett, University of Texas, in charge of the U. S. Ordnance Ballistic Station, Baltimore, Md.

(12) "Certain general properties of functions" by Professor Henry Blumberg, University of Illinois.

(10) In Professor Jackson's paper, it was pointed out that the problem of interpolation is primarily that of determining a function of specified form, most often a polynomial, which takes on given values for a certain number of given values of the independent variable; but that it is important to generalize the notion, at least to the extent of including functions determined by a finite number of given values, whether coinciding absolutely with those values or not. There are indicated then some of the striking analogies, both formal and more profound, between the formulas of interpolation by polynomials and finite trigonometric sums on the one hand, and Taylor's and Fourier's series on the other.

(11) The double entry tables considered by Professor Bennett were tables with numerical entries, which may be regarded as the values of a function $F(x, y)$ for equally spaced intervals of x and equally spaced intervals of y. The original

data are supposed to be (1) not entirely reliable, (2) possible to obtain by observation or computation for preassigned values of the independent variables, but only with difficulty, (3) not subject to any known formal law. The implications of these conditions were discussed. The advantages of using throughout fourth order differences were pointed out in some detail. Finally the use of "central" differences was explained in the construction of double entry tables.

(12) The substance of Professor Blumberg's paper was largely drawn from his 1917 *Annals* paper and his communication to the American Mathematical Society, April, 1919. No technical knowledge beyond the calculus was presupposed from the hearers and the presentation was couched in expository form.

Meeting of the Board of Trustees of the Association.

Eight members were present at each session.

The following 73 persons and 3 institutions, on applications duly certified, were elected to membership:

To individual membership.

Anna H. Andrews, Ph.B. (Wesleyan). Teacher, Public High School, Hartford, Conn.

B. I. Baidaff, Licentiat in mat. (Univ. of Jassy, Roumania). Buenos Aires, Argentina.

J. P. Ballantine, A.B. (Harvard). Instr., Penn. State Coll., State College, Pa.

P. E. Basye, Univ. of Missouri, Columbia, Mo.

E. M. Berry, Ph.D. (Iowa). Instr., Purdue Univ., W. LaFayette, Ind.

Emile Borel, D.Sc. (Paris). Professeur à la Faculté des Sciences de Paris.

J. W. Calhoun, A.M. (Harvard). Asso. prof. of appl. math., Univ. of Texas, Austin, Tex.

R. H. Carpenter, A.M. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.

C. M. Cleveland, B.E. in Civil Engg. (Mississippi). Instr. in appl. math., Univ. of Texas, Austin, Tex.

R. P. Conkling, A.B. (Cornell). Head of dept. of math., Tech. School; asst. in math., Central C. and M. T. High School, Newark, N. J.

A. E. Cooper, E. E. (Texas). Instr. in appl. math., Univ. of Texas, Austin, Tex.

Odyne O. Cornell, A.B. (Nebraska). Mangum, Okla.

M. E. Cox, B.S. in M.E. (Clemson Coll.). Asst. prof., Texas A. and M. Coll., College Station, Tex.

A. S. Croom, A.B. (Louisville). Acting dean, Harper Coll., Harper, Kans.

Julia Dale. A.B. (Transylvania Coll.). Instr., Univ. of Missouri, Columbia, Mo.

Alice C. Dean, A.M. (Rice Inst.). Fellow and acting libr., Rice Inst., Houston, Tex.

Mary E. Decherd, A.M. (Texas). Instr., Univ. of Texas, Austin, Tex.

S. Dickstein. Prof., Univ. of Warsaw, Warsaw, Poland.

L. H. Dube, Ph.D., D.D. (Gregorian Univ., Rome); M.Sc. (Ottawa). Prof. of higher math., Ottawa Univ., Ottawa, Can.

J. R. Everett, A.M. (Wisconsin). Instr., Carnegie Inst. of Tech., Pittsburgh, Pa.

Lucy A. Feddersen, A.B. in Educ. (Wyoming). Instr., Univ. of Wyoming High School, Laramie, Wyo.

A. R. Fehn, Ph.B. (Baldwin-Wallace Coll.). Asso. prof., Univ. of Wyoming, Laramie, Wyo.

Florence E. Field, A.M. (Michigan). Acting head of dept., Park Coll., Parkville, Mo.

R. M. Foster, B.S. (Harvard). Dept. of development and research, Amer. Tel. and Tel. Co., New York, N. Y.

J. G. Fowlkes, A. B., B.O. (Ouachita Coll.). Head of dept. of math., Roger Ascham School, New York, N. Y.

H. J. Gay, A.B. (Harvard). Instr., Worcester Polytech. Inst., Worcester, Mass.

J. S. Gold, B.S. (Bucknell). Instr., Bucknell Univ., Lewisburg, Pa.

P. H. Graham, A.M. (Virginia). Instr., New York Univ., New York, N. Y.

H. H. Hammer, A.M. (Texas). Asst. to state actuary, Austin, Tex.

A. J. Hargett, A.M. (Transylvania Coll.). Head of dept. of math., Texas Christian Univ., Fort Worth, Tex.

A. S. Hathaway, B.S. (Cornell). Prof., (retired), Rose Polytech. Inst., Terre Haute, Ind. Houston, Tex.

T. B. Henry, A.B. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.

Mabel M. Heren, M.S. (Northwestern Univ.). Asst. prof., Knox Coll., Galesburg, Ill.

T. F. Holgate, Ph.D. (Clark), LL.D. (Illinois; Queens). Prof., Northwestern Univ., Evanston, Ill.

Helma L. Holmes, A.M. (Nebraska). Instr. in pure math., Univ. of Texas, Austin, Tex.

B. P. Hoover, A. B. (Baker Univ.), Asst. prof., Baker Univ., Baldwin, Kans.

Goldie P. Horton, Ph.D. (Texas). Instr. in pure math., Univ. of Texas, Austin, Tex.

C. G. Jaeger, A.B. (Missouri). Instr., Univ. of Missouri, Columbia, Mo.

L. M. Klauber, A.B. in E.E. (Stanford). Genl. supt., Cons. Gas and Electric Co., San Diego, Calif.

J. J. Knox, A.B. (Chicago). Instr., School of Eng. of Milwaukee, Milwaukee, Wis.

F. A. La Motte, M.S. (Chicago); A.M. (Wisconsin). Instr., Junior Coll., St. Joseph, Mo.

G. L. Lowry, B.S. (Bucknell). Instr., Bucknell Univ., Lewisburg, Pa.

Israel Maizlish, M.S. (Mass. Inst. of Tech.). Instr. in physics, Reed Coll., Portland, Ore.

C. E. Melville, A.B. (Northwestern). Asso. prof. and registrar, Coll. dept., Clark Univ., Worcester, Mass.

D. H. MENZEL, A.B. (Denver). Instr., Univ. of Denver, Denver, Colo.

A. D. MICHAL, A.B. (Clark Coll.). Fellow in math. and asst. in physics, Clark Univ., Worcester, Mass.

J. N. MICHIE, A.M. (Michigan). Adj. prof. of appl. math., Univ. of Texas, Austin, Tex.

PEARL C. MILLER, A.M. (Stanford). Asst., Washington Univ., St. Louis, Mo.

FANNIE S. MITCHELL, A.B. (N. Car. Coll. for Women). Teacher, High school, Gastonia, N. C.

ANNA M. MULLIKIN, A.M. (Pennsylvania). Instr., Univ. of Texas, Austin, Tex.

W. L. PHINNEY, B.S. (Dartmouth). Instr., Worcester Polytech. Inst., Worcester, Mass.

A. D. PIERSON, B.S. in Ed. (Missouri). Teacher, N. E. High School, Kansas City, Mo.

C. S. PORTER, A.B. (Amherst). Instr., Worcester Polytech. Inst., Worcester, Mass.

HUGH PORTER, A.M. (Texas). Asso. prof., N. Texas St. Normal Coll., Denton, Tex.

CAROLINE M. REAVES, A.M. (Oklahoma). Prof., Coker Coll., Hartsville, S. C.

N. B. ROSENBERGER, A.M. (Pennsylvania). Grad. student, Teachers Coll., Columbia Univ., New York, N. Y.

JEAN F. ROSS. Librarian, High school, Sacramento, Calif.

BERNICE SANDERS, A.B. (Wilberforce). Prof., Wilberforce Univ., Wilberforce, Ohio.

E. W. SCHREIBER, A.B. (Michigan). Head of dept. of math., Proviso Township High School, Maywood, Ill.

PINCAS SCHUB, formerly student in Turkey. Fellow, Clark Univ., Worcester, Mass.

JABIR SHIBLI, A.M. (North Dakota). Prof., Fargo Coll., Fargo, N. Dak.

F. L. SMITH, A.B. (Drury Coll.) Asst. in physics, Univ. of Missouri, Columbia, Mo.

R. F. SMITH, M.S. (New York Univ.). Asst. prof., Coll. of City of New York, New York, N. Y.

G. W. SNEDECOR, A.M. (Michigan). Asso. prof., Iowa State Coll., Ames, Ia.

MAY J. SPERRY, A.M. (Brown). Instr., Knox Coll., Galesburg, Ill.

LOUISE E. C. STUERM, A.M. (Columbia). Dept. of development and research, Amer. Tel. and Tel. Co., New York, N. Y.

J. A. SWENSON, A.B. (Columbia). Head of dept. of math., Wadleigh High School, New York, N. Y.

O. B. TROUT, A.M. (Denver). Prof. of math. and registrar, Univ. of Denver, Denver, Colo.

E. R. TUCKER, B.S. (Milit. Coll. of S. C.); A.B. (Texas Chr. Univ.). Asso. prof., Texas Chr. Univ., Fortworth, Texas.

F. N. WELLS, Instr., U. S. Naval Acad., Annapolis, Md.

Mrs. F. E. WOOD, A.B. (Baker Univ.). Fellow, Univ. of Kansas, Lawrence, Kans.

P. W. Wood, M.A. (Cambridge). Fellow, tutor, librarian and mathematical lecturer, Emmanuel Coll., Cambridge, Eng.

Frances W. Wright, A.M. (Brown). Instr., Elmira Coll., Elmira, N. Y.

To institutional membership.

Saint Benedict's College, Atchison, Kansas.
University of Cincinnati, Cincinnati, Ohio.
East Texas State Normal College, Commerce, Tex.

The Board granted authority to the Association members residing in Texas to organize a Texas Section. Professor H. J. Ettlinger has been elected chairman and Professor J. L. Riley secretary-treasurer of the temporary organization.

The nomination of Professor E. R. Hedrick, which had been made for one year from June 11, 1920, as the Association's representative on the Executive Committee of the Division of Physical Sciences, National Research Council, was extended to a term of three years from that date.

The Board voted to present the name of Professor E. H. Moore to the Council of the American Association for nomination to the office of president for 1921. Supported by similar action taken by other affiliated organizations, Professor Moore was elected to the presidency at the Thursday morning meeting of the Council.

At the meeting of the incoming Board of Trustees Wednesday afternoon, the following were appointed Associate Editors of the Monthly for the year 1921:

Albert A. Bennett,	Walter B. Ford,	Ulysses G. Mitchell,
Edward L. Dodd,	Cuthbert F. Gummer,	Charles N. Moore,
Otto Dunkel,	Henry P. Manning,	David E. Smith,
Benjamin F. Finkel,	Raymond B. McClenon,	Horace S. Uhler.

Other business was outlined in reference to the office of the editor-in-chief and with regard to the basis for exchange in the payment of dues by foreign members.

Annual Business Meeting of the Association.

The secretary-treasurer announced the names of those elected to membership by the Board. He reported also the death, in 1920 with one exception, of the following members:

Maelynette Aldrich, Professor of mathematics, Martha Washington College (February 22);

G. E. Fisher, Professor of mathematics, University of Pennsylvania (March 28);

Y. H. Ho, Fellow, graduate school, Cornell University (February 22);

R. S. Lawrence, Professor of mathematics, Hanover College (January 30, 1919);

O. A. Randolph, Associate professor of physics, University of Colorado (April 11);

William Rinck, Professor of mathematics, Theological School and Calvin College (November 11);

E. W. Stanton, Dean, Iowa State College (September 12).

On motion of Professor Karpinski the following resolution was voted:

"Resolved, that the Mathematical Association of America favors the national use of the meter-liter-gram system. The Association respectfully urges upon Congress that steps be taken to make the use of this system national in character and in particular that scientific departments of the government be required to use this system.

The election of officers for the year 1921 was conducted by mail and in person at this meeting, as provided in the By-Laws. The tellers (C. N. MOORE and W. L. HART) reported the result of the balloting, the noteworthy number of 454 ballots having been cast:

For President: G. A. Miller, 246 votes; E. J. Wilczynski, 208 votes.

For Vice-Presidents: R. C. Archibald, 307 votes; R. D. Carmichael, 247 votes; Elizabeth B. Cowley, 120 votes; Helen A. Merrill, 224 votes.

For additional members of the Board of Trustees (to serve until January, 1924): A. A. Bennett, 205 votes; W. H. Bussey, 163 votes; Florian Cajori, 358 votes; E. L. Dodd, 107 votes; C. F. Gummer, 175 votes; H. L. Rietz, 283 votes; W. H. Roever, 176 votes; D. E. Smith, 318 votes.

The following were accordingly declared elected:

President: G. A. MILLER, University of Illinois.

Vice-Presidents: R. C. ARCHIBALD, Brown University; R. D. CARMICHAEL, University of Illinois.

Additional members of the Board of Trustees: A. A. BENNETT, University of Texas; FLORIAN CAJORI, University of California; H. L. RIETZ, University of Iowa; D. E. SMITH, Columbia University.

Because of the election of Professor Carmichael as vice-president, the Board of Trustees, in exercise of its constitutional authority, appointed to the vacancy in the Board for the term ending January, 1923, Professor C. F. GUMMER of Queen's University, as a representative of our Canadian constituency.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to December 2, 1920. The report of the auditing committee (Mrs. MAYME I. LOGSDON, H. E. SLAUGHT, and F. M. MCGAW, chairman) was then made. The financial report is printed in full below.

If to the balance on 1920 business shown in this report, $3,234.37, there be added the amount of bills receivable, $165.00, and there be subtracted the estimated amount of bills payable, $2,040.00, there results an estimated final balance on 1920 business of approximately $1,360. It will be recalled that, of this surplus, about $1,000 was passed over to the association by the management of the MONTHLY when the Association was organized five years ago, and this fund is kept by the Board of Trustees of the Association as a reserve fund. If the estimated balance of $1,360 be compared with the corresponding figure of one

year ago, \$2,040 (See MONTHLY, March, 1920, p. 112), it will be seen that there is a probable deficit on the year's business of nearly \$700, a fact forecast in the report made at the summer meeting and noted in the MONTHLY for November.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 2, 1920.

RECEIPTS.

Balance Dec. 15, 1919		\$4,581.07
1917–19 indiv. dues	\$ 147.90	
1919 instit. dues	7.25	
1919 subscriptions	3.00	
1920 indiv. dues	3,171.66	
1920 instit. dues	359.30	
1920 subscriptions	517.85	
Initiation fees	324.00	
Sale copies of MONTHLY	104.92	
Sale reprints	1.95	
Advertising	632.52	
Exchange	.50	
Interest State Savgs. Bk.	78.20	
Interest Peoples Bk.	58.13	
Interest Liberty Bonds	32.49	
Total 1920 receipts		\$5,439.67
Total assets up to 1921 business		\$10,020.74
Total expenditures		6,786.37
Balance to the end of 1920 business		\$3,234.37
Received on 1921 business (including \$33 contributed to 1921 expenses)		547.39
Book balance Dec. 2, 1920		\$3,781.76

EXPENDITURES.

Publisher's bills (Sept. '19–Oct. '20)		\$4,720.10
President's office		46.16
Manager's office		22.19
Editor-in-chief's office		585.45
Other editors' postage		10.10
Secretary-Treasurer's office:		
Postage	\$ 99.77	
Bond	5.00	
Office supplies	24.48	
Express, telegrams, etc.	10.27	
Clerical work	217.75	
Printing 1919 Register	269.62	
Printing	215.76	
New York meeting	106.16	
Chicago summer meeting	43.00	
Paid to sections from initiation fees	59.31	
		\$1,051.12
Annals subvention		275.00
Services and fees for incorporation		76.25
Total expenditures		\$6,786.37
Cash on hand		\$ 20.64
Checking account		601.50
State Savgs. Bk. Co. account		1,247.48
Peoples Bkg. Co. account		912.14
Liberty Bond		500.00
Victory Bond		500.00
Bank balance Dec. 2, 1920		\$3,781.76

When the accounts were closed on December 2, 1920, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1920 the following items:

BILLS RECEIVABLE.

Advertising	\$ 80.00
1920 dues unpaid	75.00
Interest Liberty Bonds	10.00
	\$165.00

BILLS PAYABLE.

(Either paid in December or estimated.)

Publisher's bills (Nov.–Dec.)	\$1,250.00
December *Annals* subvention	50.00
Init. fees due to sections	100.00
President's office	50.00
Manager's office	20.00
Editor-in-chief's office	150.00
Secretary-treasurer's office	125.00
Printing prelim. and annual ballots, program, etc.	225.00
Additional postage	70.00
	\$2,040.00

On account of the financial outlook a special letter was sent to the members of the Association in October, announcing the new membership dues and the need for subsidiary funds. It is our hope that the appeal for special contributions for 1921 expenses will not be forgotten as members pay their dues, for it is chiefly on this that we rely to offset the decrease in the reserve fund of the Association. It is gratifying to note here that a generous gift of $45.50 from President Smith and of $10.00 from Teachers College covers the expense of an effective campaign for members in America and in Europe, an expense that would otherwise fall directly on the Association treasury. It should also be noted that of 400 members who had paid their dues for 1921 before January first, sixty have made special contributions ranging from one up to seventeen dollars, a total thus far of $150.

While doubtless some may feel that they cannot do more than to pay the increased dues, it is within the range of possibility for almost any member to secure at least one new member; this in itself is a very helpful contribution to the finances of the Association and a real service to those not yet affiliated with the Association.

W. D. CAIRNS, *Secretary-Treasurer.*

ACOUSTIC CIRCLES.

By H. M. DADOURIAN, Trinity College.

The determination of the position of enemy artillery by *sound ranging* is one of the most interesting examples of the application of science to modern warfare. This consists in observing the time of arrival of the report, or the muzzle wave, of the enemy gun at a number of observation posts of known positions and then in using the *phonotelemetric* data thus obtained to determine the position of the gun. One of the simplest methods used in sound ranging is the following, known as the *method of concentric circles.*

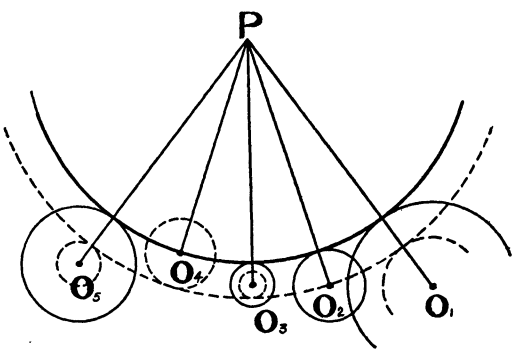


FIG. 1.

Let P, Fig. 1, denote the position of the enemy gun; O_1, O_2, etc., the positions of the observation posts at which the apparatus used for registering the arrival of the sound wave is placed; and T_1, T_2, etc., the times at which the sound wave reaches the posts O_1, O_2, etc., respectively. Furthermore suppose P, O_1, O_2, etc. to be in the same plane, with no intervening objects to obstruct the free propagation of the wave with uniform and constant velocity. Then it is evident that when a spherical wave originating at P reaches the nearest post O_i (O_4 in

the case represented by Fig. 1) its distances from O_1, O_2, etc. will be $V(T_1 - T_i)$, $V(T_2 - T_i)$, etc., respectively, where V denotes the velocity of propagation of the wave. Therefore the circle about P passing through O_i is tangent to circles drawn about O_1, O_2, etc., having radii

$$
\begin{aligned}
R_{1i} &= V(T_1 - T_i), \\
R_{2i} &= V(T_2 - T_i), \qquad (1) \\
&\cdots\cdots\cdots,
\end{aligned}
$$

respectively. This leads immediately to the method of concentric circles. On a drawing board points are located representing accurately the relative positions of the observation posts on a given scale, say 1 in 20,000; also a transparent sheet with closely drawn concentric circles is provided. When T_1, T_2, etc., are obtained circles are drawn about O_1, O_2, etc., with radii given by equations (1), the sheet with concentric circles is passed over the table until one of its circles becomes equally near tangency to all the circles about the points O_1, O_2, etc. Then the center of the concentric circles represents the position of the origin of the sound wave.

If in equations (1) O_i is not the post nearest P, then the right hand members of some of these equations become negative and the circle about P becomes tangent externally or internally to the circle about any post O_j according as $(T_j - T_i)$ is positive or negative, respectively. If in Fig. 1 O_2 is taken as the reference post, the circle passing through O_2 and having its center at P is tangent internally to the circles about O_3 and O_4, while it is tangent externally to the circles about O_1 and O_5 as is shown by the broken curves. Evidently the circle about the reference point has zero radius, consequently the tangency may be considered either as internal or as external or both.

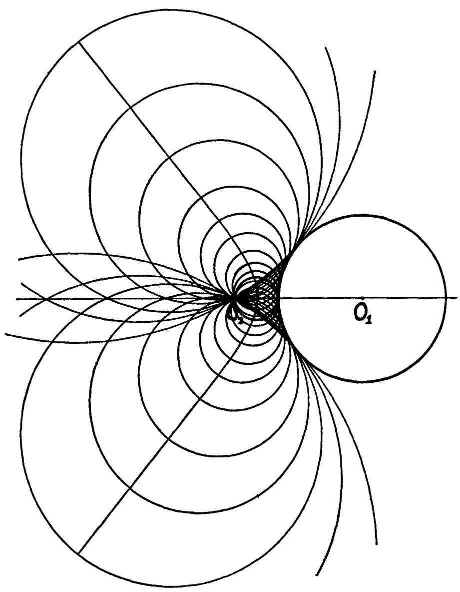


FIG. 2.

It is the object of this paper to show that the circle about P is common to a number of systems of circles which are determined by a set of phonotelemetric data. Consider two observation posts O_1 and O_2, Fig. 2, at which the arrival of a spherical sound wave is registered at the instants T_1 and T_2, respectively. Let r_1 and r_2 be, respectively, the distances of O_1 and O_2 from the origin of the wave; and t_1 and t_2 the intervals of time taken by the wave to reach the points O_1 and O_2, then

$$
\begin{aligned}
r_1 &= Vt_1, \\
r_2 &= Vt_2,
\end{aligned}
$$

and

$$r_1 - r_2 = V(t_1 - t_2).$$

But

$$t_1 - t_2 = T_1 - T_2.$$

Therefore

$$r_1 - r_2 = V(T_1 - T_2). \quad (2)$$

Since the right-hand member is constant, equation (2) is the well-known equation of a hyperbola of which

$$R_{12} = V(T_1 - T_2)$$

is the transverse axis, and O_1 and O_2 are the foci. Thus the origin of the wave may be anywhere on the hyperbola of which (2) is the equation. Therefore representing, by means of a circle, the wave which starts from any point of the hyperbola and which reaches the posts O_1 and O_2 at T_1 and T_2, respectively, we obtain a system of circles which pass through O_2, are tangent to the circle of radius R_{12} drawn about O_1 as center, and have their centers on the hyperbola of which (2) is the equation. Equation (2) may also be regarded as the *acoustic equation* of the system of circles, and the system of circles may be called *acoustic circles*. The origin of the wave then is at the center of one of the circles of the system of acoustic circles of which (2) is the acoustic equation. If $T_1 > T_2$, the circles of the system have their centers on the left-hand branch of the hyperbola as in Fig. 2 and are tangent to the circle around O_1 externally. If on the other hand $T_1 < T_2$, the circles have their centers on the right-hand branch and are tangent to the circle around O_1 internally, as in Fig. 3. If the position of the source of sound is not limited to a plane, the system of acoustic circles becomes a system of spheres which have their centers on the hyperboloid of revolution obtained by revolving the hyperbola about the line O_1O_2 and which are tangent to the sphere of radius R_{12} drawn about O_1 as center.

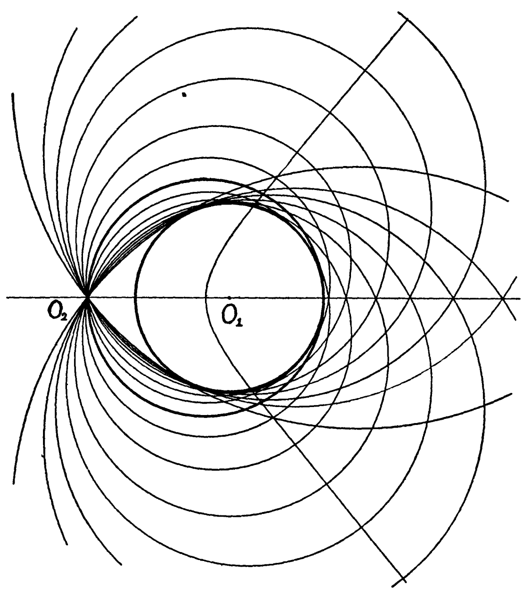


FIG. 3.

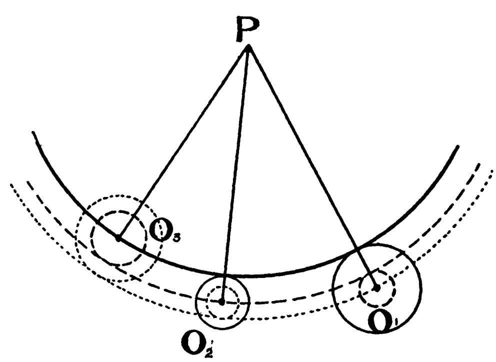


FIG. 4.

Now let T_3 be the time at which the wave front reaches another observation post O_3, then considering the data of two of the three posts at a time we obtain the equations

$$r_1 - r_2 = V(T_1 - T_2),$$
$$r_1 - r_3 = V(T_1 - T_3), \qquad (3)$$

and

$$r_2 - r_3 = V(T_2 - T_3),$$

which are the acoustic equations of three systems of circles and of as many hyperbolas. The origin of the wave is at the common center of three circles, Fig. 4, one from each system. Each of the three circles passes through one of the points O, is tangent to circles about the other two and has its center at the common point of intersection of the hyperbolas given by equations (3).

In general, if the times of arrival of a wave at n posts of known positions are given, $n(n-1)/2$ equations can be obtained from the data, each of which forms the equation of a system of acoustic circles. (The truth of this statement may be easily seen by considering the number of permutations and combinations of n things taken two at a time). Evidently there are two circles in each system whose centers are at the origin of the wave. Of the $n(n-1)$ circles having their centers at the origin of the wave $(n-1)$ pass through O_1 and consequently are coincident; $(n-1)$ pass through O_2 and are coincident; and so on to the $(n-1)$ which pass through O_n. Therefore in general there are n distinct circles in the $n(n-1)/2$ systems which have their centers at the origin of the wave. The circle which passes through any post O_i is tangent to a circle of radius $V(T_j - T_i)$ drawn about the post O_j as center. The tangency is external if $T_j > T_i$ and internal if $T_j < T_i$.

ON THE NUMERICAL VALUE OF i^i.

BY H. S. UHLER, Yale University.

The primary object of the present note is to place on record the values of $e^{-\pi/2}$ and of seven related powers of the same base which I have recently calculated to more than fifty decimal places.

In order to avoid the use of any table of mathematical constants the following series was employed

$$y \equiv e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \cdots \; [x^2 < 1]$$

$$= (1 + \sum_1^\infty t_{2k+1}) + (x + \sum_1^\infty t_{2k+2}).$$

Since the collections of series at my disposal did not contain a formula for the general term of the above series it was necessary to prove that

$$t_{2k+1} = (0^2+1)(2^2+1)(4^2+1)\cdots[(2k-2)^2+1]x^{2k}/(2k)!$$

and

$$t_{2k+2} = (1^2 + 1)(3^2 + 1)(5^2 + 1) \cdots [(2k - 1)^2 + 1]x^{2k+1}/(2k + 1)!,$$

where $k = 1, 2, 3, \cdots$. This was easily accomplished by the aid of Leibnitz's theorem in the manner illustrated in Williamson's *Differential Calculus*, 1889, pages 51–56; the fundamental equation being

$$(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n + 1)x\frac{d^{n+1}y}{dx^{n+1}} - (n^2 + 1)\frac{d^n y}{dx^n} = 0.$$

For $x = \pm 1/2$, $y = e^{\pm\pi/6}$.

With $x = 1/2$ the terms involving even powers of x were calculated independently of the odd-power terms. Since, in both sets, each term was derived directly from the preceding one, the arithmetical operations involved had to be checked very carefully by the inverse operations, for a single incorrect digit would have vitiated some or all of the succeeding terms in the set in which the error was committed. The total number of even terms used was the same as that of the odd terms, namely 85. Each term was computed to the nearest unit in the fifty-fourth decimal place, 5×10^{-55} being counted as 1×10^{-54}. $e^{\pm(\pi/3)}$ was obtained by squaring $e^{\pm(\pi/6)}$, $e^{\pm(\pi/2)}$ by multiplying together $e^{\pm(\pi/3)}$ and $e^{\pm(\pi/6)}$, and $e^{\pm\pi}$ by squaring $e^{\pm(\pi/2)}$. Let σ_1 and σ_2 denote respectively the numerical values actually found for the sum of the even terms and of the odd terms respectively when $x = 1/2$. The results of the calculation were:

$$\sigma_1 = 1.14023\,83210\,76428\,79214\,11319\,80379\,35089\,07668\,97667\,51132\,0551,$$

$$\sigma_2 = 0.54785\,34738\,88039\,80847\,57156\,47717\,43140\,33527\,83521\,77586\,1757,$$

$$e^{\frac{\pi}{6}} = 1.68809\,17949\,64468\,60061\,68476\,28096\,78229\,41196\,81189\,28718\,2308,$$

$$e^{-\frac{\pi}{6}} = 0.59238\,48471\,88388\,98366\,54163\,32661\,91948\,74141\,14145\,73545\,8794,$$

$$e^{\frac{\pi}{3}} = 2.84965\,39082\,26361\,49747\,41273\,19852\,90439\,39640\,06102\,78112\,6866,$$

$$e^{-\frac{\pi}{3}} = 0.35091\,98071\,78410\,96756\,57367\,15996\,95305\,83625\,73153\,62096\,1747,$$

$$e^{\frac{\pi}{2}} = 4.81047\,73809\,65351\,65547\,30356\,66703\,83312\,63901\,70874\,66453\,4901,$$

$$e^{-\frac{\pi}{2}} = i^i = 0.20787\,95763\,50761\,90854\,69556\,19834\,97877\,00338\,77841\,63176\,9614,$$

$$e^{\pi} = 23.14069\,26327\,79269\,00572\,90863\,67948\,54738\,02661\,06242\,60021\,16,$$

$$e^{-\pi} = 0.04321\,39182\,63772\,24977\,44177\,37171\,72801\,12757\,28109\,81063\,30854.$$

The reliability of the preceding numbers was tested by checks applied independently to the final results. Thus $s_1^2 - s_2^2 = 1$, where s_1 and s_2 are the exact values approximated above by the corresponding expressions in Greek letters, was used to check the values of those expressions. It is concluded that the calculated values of $e^{\pi/6}$ and $e^{-(\pi/6)}$ given above are certainly correct to 52 decimal places and they are probably in error by not more than two units in the fifty-

third place. It was also found that to fifty-two places of decimals neither $e^{\pi} \cdot e^{-\pi}$ nor $e^{\pi/2} \cdot e^{-(\pi/2)}$ differed from unity.

In this MONTHLY, *1917*, 237, it is stated that—for the value of $e^{-(\pi/2)}$ or i^i—"Mr. Escott using Steinhauser's 20-place tables, gets .20787957635076190854687 while Professor Reynolds, using Hutton's 20-place tables, gets .2078795763-4917907781." Comparison of these numbers with my datum given above shows that Mr. Escott's differs from it by only about one unit in the twenty-second decimal place, whereas Professor Reynolds's number is discordant at, and beyond, the eleventh place. Since this may mean that there is an error in Hutton's tables it would be helpful if Professor Reynolds would investigate and report the cause of the discrepancy.

Finally, in the seventh revised edition of the *Smithsonian Physical Tables*, 1920, page 55, a short table of values[1] of $e^{(\pi/4)x}$, $e^{-(\pi/4)x}$ and their logarithms is given. If it should ever become necessary to compute tables of this kind to a greater number of decimal places than can be effected advantageously by logarithms, my data given above may be used either as basic numbers or as independent checks.

HISTORICAL NOTES ON THE RELATION $e^{-(\pi/2)} = i^i$.

BY R. C. ARCHIBALD, Brown University.

In 1719 Count Giulio Carlo de'Toschi di Fagnano showed, in effect, that the arc of the quadrant of a unit circle ($\pi/2$) is[2]

$$2 \log.(1 - \sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1 + \sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}.$$

---

[1] For $x = 1, 2, 3, \cdots 20$. The part of the table without logarithms is also given on page 91 of J. B. Dale, *Five Figure Tables*. London, 1903. The values of $e^{19\pi/4}$ differ materially in these two sources.—EDITOR.

[2] *Opere Matematiche del Marchese . . . de'Toschi di Fagnano*. Milano, volume 2, 1912. On page 406 we find "$\int \frac{dt}{1+t^2}$ esprime l'arco di cerchio [radius unity], la di cui tangente é t" [if the equations of the circle are $x = \cos\theta$ and $y = \sin\theta$, $t = \tan\theta$]. On pages 422–423 we find the following:

$$\int \frac{dt}{1+t^2} = \log.\ (1 - t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1 + t\sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}$$

or

"(8) $$\int \frac{dt}{1+t^2} = \log.\ (A^2 - B^2).$$

[where $A = 1 + \frac{1}{2}t + \frac{1}{8}t^2 - \frac{7}{48}t^3 - \frac{43}{384}t^4 \cdots$, $B = \frac{1}{4}t^2\sqrt{-1} + \frac{1}{8}t^3\sqrt{-1} - \frac{3}{32}t^4\sqrt{-1} \cdots$].

"Queste due ultime equazioni manifestano una nuova, e bellissima proprietà del cerchio, ciascun arco del di cui quadrante à per suo elemento $\frac{dt}{1+t^2}$, quando la t denota la tangente dell'arco medesimo.

"Scolio IV.—Se l'arco di cerchio fosse equale al quadrante, allora la t diverrebbe infinita, e per avere il logaritmo equale al quadrante nulla gioverebbe l'equazione (8). In questo caso si divida per mezzo lo stesso quadrante, e la tangente dell'arco sudduplo di esso sarà eguale all'unità;

The relation $\pi/2 = -\sqrt{-1}\log\sqrt{-1}$ was known to Euler as early as 1728, since, on December 10 of that year he wrote as follows to Jean Bernoulli[1]: "Sit radius circuli a, sinus y, cosinus x, erit ex methodo tuâ quadraturam circuli ad logarithmos reducendi, area sectoris $= \dfrac{aa}{4\sqrt{-1}}\log\dfrac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$, et posito $x = 0$, habebis quadrans circuli $\dfrac{aa}{4\sqrt{-1}}\log(-1)$." Hence, Euler demonstrated that $\dfrac{\pi a^2}{4} = \dfrac{a^2}{4\sqrt{-1}}\log(-1)$, from which one readily deduces

$$\frac{\pi}{2} = -\sqrt{-1}\log\sqrt{-1}.$$

In his reply, Jean Bernoulli called attention[1] to the identity

$$\int_0^{\pi/4}\frac{a^2dx}{2\sqrt{a^2-x^2}} = \frac{a^2}{4}\frac{\log\sqrt{-1}}{\sqrt{-1}},$$

and observed at the same time that the integral is equal to one eighth of a circle of radius a. It follows immediately that

$$\frac{\pi a^2}{8} = \frac{a^2}{4}\frac{\log\sqrt{-1}}{\sqrt{-1}}, \text{ or } \frac{\pi}{2} = -\sqrt{-1}\log\sqrt{-1};$$

but Jean Bernoulli did not himself draw this conclusion because he believed that $\log\sqrt{-1}$ was zero.[2]

pongasi poscia 1 in luogo di t nel secondo membro dell'equazione (8), e si avrà il logaritmo eguale all'arco sudduplo del quadrante, e sarà log. (A^2-B^2). Quindi si avrà lo stesso quadrante $-2\,l.(A^2-B^2) = \log.(A^2-B^2)^2$."

In some notes written by Count Fagnano's son, G. F. Fagnano (1715–1797) in 1761, the following formula occurs (*Opere*, tome 3, 1912, p. 30):

$$\text{Quadrante} = 2\,L.\left(\frac{1-1\sqrt{-1}}{1+1\sqrt{-1}}\right)^{\frac{1\sqrt{-1}}{2}}.$$

It is shown that this reduces to

$$\text{"Quadrante} = 2\,L.(-1\sqrt{-1})^{\frac{1\sqrt{-1}}{2}} = \sqrt{-1}\,L.-\sqrt{-1}$$

la qual'Espressione con cede in bellezza alla Bernulliana." On page 34 is the following corollary:

"E facile dedurre da tale dottrina, che quantunque L. -1 abbia un'infinità di valori tutti immaginarj; siccome nell'Equazione v.g.

$$\frac{\text{Circonf.}}{\text{Diam.}} = \frac{\text{L.}-1}{\sqrt{-1}}$$

uno solo è il valore della Circonferenza, uno solo è il valore del Diametro, e uno solo il valore della radice di -1, così uno solo degli infiniti valori di L -1 salva la suddetta equazione."

[1] G. Eneström, *Bibliotheca Mathematica*, 1899, p. 46.

[2] This is discussed at length by Euler in "De la controverse entre Mrs. Leibnitz & Bernoulli sur les Logarithmes des nombres négatifs et imaginaires" (presented to the Berlin Academy in 1747), *Mém. de l'acad. d. sc. de Berlin*, vol. 5 (1749), 1751, pp. 146–148. On page 147 we find: "Or M. Bernoulli ayant si hereusement réduit la quadrature du cercle aux logarithmes des nombres

The discovery by Euler in this connection that log n has an infinite number of logarithms, which are all imaginary except when n is a positive number, is a striking indication of his clear thinking and genius.[1] In the course of his paper "Recherches sur les racines imaginaires des équations"[2] we find (pages 272–276) a discussion of the problem: "Une quantité imaginaire étant élevée à une puissance dont l'exposant est aussi imaginaire, trouver la valeur imaginaire de cette puissance." Assuming

$$(a + b\sqrt{-1})^{m+n\sqrt{-1}} = x + y\sqrt{-i}$$

he finds

$$x = c^m e^{-2\lambda n\pi - n\phi} \cos(2\lambda m\pi + m\phi + n \log c),$$

$$y = c^m e^{-2\lambda n\pi - n\phi} \sin(2\lambda m\pi + m\phi + n \log c),$$

where λ is a positive or negative integer, $c = \sqrt{(a^2 + b^2)}$ and $\cos(2\lambda\pi + \phi) = a/c$, $\sin(2\lambda\pi + \phi) = b/c$.[3] Euler's fourth corollary to this result is as follows:

imaginaires, si le logarithme de $\sqrt{-1}$ etoit $= 0$, toute cette belle découverte seroit fausse; par laquelle il a fait voir, que le rayon est à la quatrieme partie de la circonference, comme $\sqrt{-1}$ à $\log\sqrt{-1}$. Donc posant le rapport du diametre à la circonference $= 1 : \pi$, il sera $\frac{1}{2}\pi = \frac{\log\sqrt{-1}}{\sqrt{-1}}$, & pourtant $\log\sqrt{-1} = \frac{1}{2}\pi\sqrt{-1}$, ce qui seroit absurde s'il étoit $\log\sqrt{-1} = 0$. Il n'est pas donc vray que $\log\sqrt{-1} = 0 \cdots$."

In a summary Euler stated (*l.c.*, p. 175):

"Les valeurs de	seront celles-cy à l'infini					
$\frac{\log(+\sqrt{-1})}{\sqrt{-1}}$	$+\frac{1}{2}\pi$;	$+\frac{5}{2}\pi$;	$+\frac{9}{2}\pi$;	$+\frac{13}{2}\pi$;	$+\frac{17}{2}\pi$;	&c.
	$-\frac{3}{2}\pi$;	$-\frac{7}{2}\pi$;	$-\frac{11}{2}\pi$;	$-\frac{15}{2}\pi$;	$-\frac{19}{2}\pi$;	&c.
$\frac{\log(-\sqrt{-1})}{\sqrt{-1}}$	$+\frac{3}{2}\pi$;	$+\frac{7}{2}\pi$;	$+\frac{11}{2}\pi$;	$+\frac{15}{2}\pi$;	$+\frac{19}{2}\pi$;	&c.
	$-\frac{1}{2}\pi$;	$-\frac{5}{2}\pi$;	$-\frac{9}{2}\pi$;	$-\frac{13}{2}\pi$;	$-\frac{17}{2}\pi$;	&c."

[1] A discussion of his work in this connection may be found in F. Cajori's "History of logarithms" in this MONTHLY, *1913*, 75–84.

[2] *Mém. de l'acad. d. sc. de Berlin*, vol. 5 (1749), 1751, pp. 222–288. The memoir seems to have been presented to the Academy in 1746 (G. Eneström, *Verzeichnis der Schriften Leonard Eulers*, Erste Lieferung, 1910, p. 43).

[3] The proof that $(a + b\sqrt{-1})^{m+n\sqrt{-1}}$ is expressible in the form $x + y\sqrt{-1}$ is often referred to as "d'Alembert's theorem" [*e.g.*, *Annales de Mathématiques Pure et Appliquées* (Gergonne), July, 1913, p. 20]. He discussed the question in his *Reflexions sur la cause générale des vents*, Paris, 1747, p. 142, and in *Mém. de l'acad. de sc. de Berlin*, vol. 2 (1746), 1748, p. 192; cf. *Opuscules Mathématiques* . . . par M. d'Alembert, Paris, tome 1, 1761, p. 225 and tome 5, 1768, pp. 213–214. Part of this discussion was developed in L. A. de Bougainville, *Traité du calcul intégral, pour servir de suite à l'analyse des infiniment petits de l'Hôpital*, tome 1, Paris, 1754, pp. 42f. With Bernoulli, D'Alembert claimed that $\log\sqrt{-1} = 0$.

The "theorem" was also discussed by J. B. Labey in his notes to Euler, *Introduction à l'Analyse Infinitésimale*, tome 1, Paris, 1796, pp. 326–327; by Lagrange in his *Traité de la Résolution des Equations Numériques*, nouvelle édition, Paris, 1808, note IX; and by du Bourguet, and Gergonne, in *Annales de Mathématiques Pures et Appliquées*, tome 4, 1813, pp. 20–25. In *The Ladies' Diary*, 1833, p. 48, problem 1567 was, in effect: "When is $(a + b\sqrt{-1})^{m+n\sqrt{-1}}$ a real quantity? Determine whether all functions of $a + b\sqrt{-1}$ can be reduced to $A + B\sqrt{-1}$, or not." This was answered in the *Diary* for 1834, pp. 44–45. See also A. Cayley, *Proc. London Math. Soc.*, vol. 2, p. 54; *Coll. Math. Papers*, vol. 6, p. 68.

"Si $a = 0$; $m = 0$, & $b = 1$, il sera $c = 1$ & $\phi = \frac{1}{2}\pi$ d'où l'on tirera cette transformation:

$$(\sqrt{-1})^{n\sqrt{-1}} = e^{-2\lambda n\pi - \frac{1}{2}n\pi}$$

ou bien

$$(\sqrt{-1})^{\sqrt{-1}} = e^{-2\lambda\pi - \frac{1}{2}\pi},$$

qui est d'autant plus remarquable, qu'elle est réelle, & qu'elle renferme même une infinité de valeurs réelles differentes. Car posant $\lambda = 0$, on aura en nombres

$$(\sqrt{-1})^{\sqrt{-1}} = 0{,}2078795763507."$$

Euler gives a similar value in the last paragraph of a letter to Goldbach, dated June 14, 1746:[1] "Letztens habe gefunden, dass diese expressio $\sqrt{-1}^{\sqrt{-1}}$ einen valoren realem habe, welcher in fractionibus decimalibus = 0,2078795763, welches mir merkwurdig zu seyn scheinet."

But Euler's results were not generally known and accepted. For example, more than sixty-five years later we find Argand, notable for his geometrical interpretation of imaginary quantities,[2] stating that $\sqrt{-1}^{\sqrt{-1}}$ offers a simple example of a quantity which is irreducible to the form $x + y\sqrt{-1}$.[3] "He did not," as Hamilton remarks,[4] "anticipate De Morgan's theory of logometers."[5]

Among manuscripts published after Gauss's death (1855) were certain ones dealing with lemniscate functions. In one of these, values of $e^{-\pi}$, $e^{-(\pi/4)}$, $e^{-(9\pi/4)}$, $e^{\pi/2}$, and $11 \cdot e^{-(\pi/2)}$ are computed in an interesting manner (*Carl Friedrich Gauss Werke*, vol. 3, 1864, pp. 426–432). The values are as follows:

$$e^{-\pi} = 0.0432139182\ 6377224977\ 4417737171\ 7280112757\ 2810981063,$$

$$e^{-(\pi/4)} = 0.4559381277\ 6599623676\ 5921294728\ 0294194166\ 0436523820,$$

$$e^{-(9\pi/4)} = 0.0008514383\ 4280515803\ 5852453295\ 4846487994\ 1872486024\ 8176915,$$

$$e^{\pi/2} = 4.8104773809\ 6535165547\ 3044648993\ 1536, \text{ and}$$

$11 \cdot e^{-(\pi/2)} = 2.2866753398\ 58378$ which gives

$e^{-(\pi/2)} = .2078795763\ 50762$. On pages 418–419 (*l.c.*) Gauss sets down values for $2e^{-(\pi/4)}$ to 39 places of decimals, $2e^{-(9\pi/4)}$ to 27 places, $2e^{-\pi}$ to 40 places, $2e^{-(25\pi/4)}$ to 32 places, $2e^{-(49\pi/4)}$ to 28 places, $2e^{-4\pi}$ to 35 places, and $2e^{-9\pi}$, $2e^{-16\pi}$ to 27 places. The last five of these values are as follows:

---

[1] *Correspondance Mathématique et Physique de quelques célèbres géomètres du XVIIIème siècle* . . . publiée . . . par P. H. Fuss. Tome 1, St. Pétersbourg, 1843, p. 383.

[2] *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*, Pairs, 1806.

[3] *Annales de Mathématiques Pures et Appliquées*, tome 4, 1813, p. 146. Servois called attention to Euler's result (*l. c.* 1814, p. 231) in correcting this error. Details of the discussion in this connection are given in the volume of A. S. Hardy's English translation of Argand's *Essai* (Van Nostrand Science Series), New York, 1881.

[4] W. R. Hamilton, *Lectures on Quaternions*, Dublin, 1853, p. (56).

[5] See A. De Morgan, *Trigonometry and Double Algebra*, London, 1849, pp. 129–137; also R. B. Hayward, *The Algebra of Coplanar Vectors and Trigonometry*, London, 1892, pp. 119f.

$$2e^{-(25\pi/4)} = 0.0000000059\ 3851399312\ 9644497731\ 18$$

$$2e^{-(49\pi/4)} = 0.0000000000\ 0000003867\ 40505991$$

$$2e^{-4\pi} = 0.0000069746\ 8471241799\ 0983550387\ 96535$$

$$2e^{-9\pi} = 0.0000000000\ 0105109703\ 5201288$$

$$2e^{-16\pi} = 0.0000000000\ 0000000000\ 0295807$$

It will be observed that the values for $e^{-\pi}$ and $e^{-(\pi/2)}$ agree exactly with those given above by Professor Uhler, but that the values for $e^{\pi/2}$ differ—from the twenty-third decimal place on. Bastien's value, given below, appears to indicate that the value in Professor Uhler's paper is the correct one to twenty-eight places of decimals at least. By squaring Gauss's value for $e^{-(\pi/4)}$ Professor Uhler found the result to check to forty-seven places of decimals with his own value for $e^{-(\pi/2)}$.

Schellbach showed[1] in 1832 that many convergent series for π could be derived from such relations as

$$\pi = \frac{2}{i}\log i = \frac{2}{i}\log\frac{1+i}{1-i} = \frac{2}{i}\log\frac{(2+i)(3+i)}{(2-i)(3-i)} = \frac{2}{i}\log\frac{(5+i)^4(-239+i)}{(5-i)^4(-239-i)}$$

$$= \frac{2}{i}\log\frac{(10+i)^3(-515+i)^4(-239+i)}{(10-i)^3(-515-i)^4(-239-i)}.$$

Benjamin Peirce referred,[2] in 1882, to "the mysterious formula"

$$i^{-i} = e^{\pi/2} = 4.810477381.$$

The phrase "l'équation symbolique et mystérieuse

$$\frac{\pi}{2}\sqrt{-1} = \text{Log.}\,(\sqrt{-1})\text{''}$$

was employed by J. F. Francais[3] in September, 1813.

In the introduction to *Table d'Interpolation pour le calcul des parties proportionnelles faisant suite* aux *Tables de Logarithmes* . . . par L. Schrön precédé d'une introduction française par J. Houel (Paris, 1891) the following number is evaluated

$$e^{-\pi} = 0.043213918263772248 \cdots.$$

In 1919, Brocard suggested[4] that "la connaisance des nombres e^π and π^e donnerait peut-être l'indice de quelque relation entre e et π." E. Chanzy found such values[5] to be $e^\pi = 23.1406926327787 \cdots$, and $\pi^e = 22.4591577183 \cdots$;

[1] *Journal für die reine und angewandte Mathematik*, vol. 9, pp. 404–405.
[2] B. Peirce, *Linear Associative Algebra*, New York, 1882, p. 5.
[3] *Annales de Mathématiques Pures et Appliquées*, vol. 4, p. 67.
[4] *L'Intermédiaire des Mathématiciens*, vol. 26, p. 73, question 4935.
[5] *Sphinx-Oedipe*, August, 1920, pp. 127–128.

and L. Bastien, with the aid of his tables of logarithms to 32 places of decimals found[1]

$$e^{\pi} = 23.140692632779269005729086367 9,$$
$$e^{\pi/2} = 4.8104773809653516554730356667,$$
$$\pi^{e} = 22.459157718361045473427152204 5.$$

The symbol i for $\sqrt{-1}$ was first used by Euler in a "M.S. Academiæ exhibit. die 5 Maii 1777"[2] printed posthumously in 1794.[3] This notation was adopted by Gauss in 1801.[4]

The symbol π for the ratio of the circumference of a circle to its diameter was first used by W. Jones in 1706.[5] It was probably suggested to Jones by Oughtred who employed the symbol in a different sense.[6] Euler's first use of the symbol was in 1737[7]; up to that time he had used the letter p.

The symbol e for the number, defined by the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdots$, was first used by Euler in 1731.[8]

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

2. Dupin as Secretary of the Ionian Academy.

Among the most interesting but by no means most scholarly of the French mathematicians of the first part of the nineteenth century was that mélange of economist, politician, geometrician, and popularizer of science, Pierre-Charles-François Dupin (1784–1873). To the mathematician he is chiefly known as the favorite pupil of Monge and as his biographer; for his *Développement de géométrie*

[1] *L'Intermediaire des Mathematiciens*, vol. 27, p. 65, May–June (not published till October), 1920. In the value for π^e given above the number "1" has been inserted in the seventh place from the end where "r" appears in the original.

[2] L. Euler, "De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet."

[3] L. Euler, *Institutiones calculi integralis*, vol. 4, 1794, pp. 183–184. The symbol i is introduced on p. 184.

[4] C. F. Gauss, *Disquisitiones arithmeticae*, 1801, p. 337; *Werke*, vol. 1, 1870, p. 414.

[5] W. Jones, *Synopsis Palmiorum Matheseos*, London, 1706, p. 263.

[6] William Oughtred (1574–1660) in his *Clavis Mathematica* of 1647, etc., and in his *Theorematum in libris Archimedes de Sphaera et Cylindro Declaratio*, Oxford, 1652, frequently employs the symbol $\delta : \pi$ or $\pi : \delta$ (in modern notation) for the ratio of the semi-diameter to the semi-periphery or of semi-periphery to semi-diameter. It is noticeable that these letters are *never* used separately, that is, π is *not* used for "Semiperipheria," as Tropfke suggests (*Geschichte der Elementar-Mathematik*, vol. 2, 1903, p. 135). Oughtred states specifically in his "Theorematum": "$\frac{\pi}{\delta} R$, est semiperipheria circuli cujus Radius est R." In 1697 David Gregory used (*Philosophical Transactions*, vol. 19, p. 652) π/ρ to designate the ratio of the circumference to the diameter.

[7] L. Euler, "Variae observationes circa series infinitas," *Comment. acad. sc. Petrop.*, vol. 9 (1737), 1744, p. 165.

[8] *Corresp. math. et phys* . . . par Fuss, vol. 1, 1843, p. 58: "e denotat hic numerum, cujus logarithmus hyperbolicus est = 1."

(Paris, 1813); and for his contributions to the theory of curvature of surfaces, and the theory of light, the study of mechanics, curves of the third and fifth orders, and conjugate tangents. After having made a high record at the École polytechnique he entered the navy. His appointment to study the naval establishment of Great Britain led to the publication of his *Voyages dans la Grande-Bretagne* (Paris, 6 vols., 1820–1824). In 1819 he became professor of mechanics at the Conservatoire des Arts et Métiers, in 1824 he was created a baron by Louis XVIII, in 1837 he was made a peer, and in 1840 he became grand officer of the Legion of Honor. In the popular brilliancy of his career he showed much of the ability of his mother. It was largely she who directed the education of her three sons, all of whom became men of prominence, and it speaks well for her appreciation of their talents that she had placed upon her tomb the words, "Ci-gît la mère des trois Dupin." It was with reference to this inscription that Charles, who survived his brothers, was called in later life "the last of the Dupins."

Of the several letters of Dupin now in my collection perhaps the one that shows most clearly the great diversity of his interests relates to an incident in his life not generally known to those who study his contributions to geometry.

The letter to which I refer calls attention to quite a different line of interest and shows Dupin at the age of twenty-five in a capacity quite foreign to his mathematical interests. It reads as follows:

CORCYRE (CORFU), September 26, 1809; the 647th Olympiad, 2d year.
ACADEMY.

Μετὰ θάνατον ἀνίσαται

Ch. Dupin, Secretary of the Ionian Academy for the French language and the official correspondence, Engineer in the French Navy,

To Monsieur De Gérando, Member of the Consulta charged with organizing the Roman states, and Secretary of the National Institute of France, 3d Classe.

Sir: I have the honor of laying the program of the Prix Olympiadiques before the Secretary of the Institute, in the Classe which has antiquity for the special subject of its meditations. Nothing which relates to Greece can be foreign to you.

We present to you our homage . . . . You belong to the Consulta, created for the regeneration of Rome and Latium. We believe that we see in you the representative of the sciences and arts of France, sent to modern Rome for the purpose of repaying with interest the benefits which Europe received from ancient Rome.

Finally, Sir, it is pleasing to me that, as the representative of a learned society, I am allowed to pay my respects to an old friend of my father's (Dupin, de la Nièvre, ex-legislator, Rue du bacy, No. 998).

I earnestly hope that the program of our prizes may be so fortunate as to command your approval, and I could wish nothing more than that you would accede to the request which I have the honor to make.

Permit us to add a notable name to the list of associates who honor us, and among whom we are able to name Monti, Fabroni, Chaptal, and many other members of the Institute of France.

How great would be my happiness if I could be able to present to the Ionian Academy a present such as that of your fellowship in this institution, young and still feeble, but which recalls at least some of the glorious ideas of ancient Greece.

I have the honor, Sir, of saluting you with the highest consideration.

CH. DUPIN.

The letter thus presents a single interesting incident in a very interesting life,—a life that was probably more extended than that of the Academy which he was influential in establishing.

3. Picard[1] and Cassini.

Scientific letters more than two hundred years old are becoming rare.. They exist in the museums and great libraries, particularly in Europe, but every year it is becoming more difficult to secure them for private collections. One of the most valuable, of those which I have, came into my possession recently, and is of particular interest because of the men concerned and because it contains the results of certain important astronomical observations. The letter was written at Brest on October 2, 1679, by Jean Picard, and is addressed to "Monsieur Cassini, a l'observatoire Royal au faus-bourg St. Jaques. A Paris." Picard was then fifty-nine years of age, and this letter was written about three years before his death. His life had, as is well known, been actively devoted to scientific work. He was born in 1620 and became the foremost astronomer of his time, greatly improving the instruments of observation, making a very satisfactory measurement of a degree of a meridian (1669), and writing on the form of the earth. It was his measurements which enabled Newton to confirm his theory with respect to gravitation.

The Cassini to whom the letter was written was Giovanni Domenico (Jean-Dominique). He was for a time professor of astronomy at Bologna, but in 1669 the pope agreed to his acceptance of a call to Paris, where he became the first of four generations to fill the post of director of the great French observatory. He determined the motion of Jupiter's satellites, discovered four of Saturn's, made a close approximation to the parallax of the sun, corrected Kepler's value of the earth's eccentricity, and first described the ovals that bear his name.

It was very likely in connection with Cassini's observations on the satellites of Saturn that Picard, wishing to assist him, wrote the following letter:

> "I have already written you that, on the twenty-fourth, at 16 H. 15′ 56″ we observed the immersion of the first satellite in the shadow; this observation, which was very favorable, gives a difference of 27′ 33″ in comparison with the one that you made on the night of the twenty-sixth . . . I also made this last observation, but did not consider it valuable because of the humidity at that time. . . . We were more fortunate on the evening of the twenty-ninth, when the immersion of the second satellite took place at precisely 10 H. 16′ 47″. . . . We expect to leave on Tuesday, Oct. 4, or at the latest on Thursday, wishing to observe once more the immersion of the first satellite which takes place the night of the 3–4th.

The letter then proceeds to discuss more personal matters, relating in part to asking Cassini to write to "Mr. l'Abbé" about their departure. This M. l'Abbé was evidently an older scientist, for Picard remarks that he had been "pretending to make some dissections of fish, but an old man like him ought to be content with dissecting sardines,"—which shows that the greatest savants have their sense of humor.

[1] See the article in *Revue générale des Sciences* referred to on page 137 of this issue of the Monthly.—Editor.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

A number of questions have been standing in this department for some time with no replies or with replies which still leave something to be desired in the way of completeness or finality. Several of these are reprinted below, with the intention of directing our readers' interest to them again, and stimulating replies. To those which have already received some attention short notes are appended, indicating the extent to which they still remain open for consideration.

QUESTIONS.

15 [*1914*, 278; *1916*, 353; *1919*, 68; *1920*, 114, 361]. In the *Proceedings of the Royal Society of Edinburgh*, vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

A large number of replies to this question have been received; they are analyzed in the remarks by the editor, *1920*, 361. It seems necessary to emphasize again that what is desired is *not* a proof that if $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$—this is of course very easy; but a proof, by use of this theorem, that no two positive integers exist the sum of whose cubes is a cube.

21 [*1914*, 341; *1916*, 354; *1919*, 68, 239; *1920*, 114]. For the Diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{rrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given Diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

In a reply to this question published in the MONTHLY for June, 1919, E. B. Escott showed how to obtain an infinite number of rational solutions of the equation, and in particular all the integral solutions contained in the above list.

A number of references bearing on the question are given by Dickson, *History of the Theory of Numbers*, vol. 2, chapter 20, pp. 533–539. Of especial pertinence are the following:

Mordell, *Proceedings of the London Mathematical Society*, series 2, vol. 13, pp. 60–80; vol. 18, pp. v–vi.

Thue, *Crelle's Journal*, vol. 135, pp. 303–304.

30 [*1916*, 88, 354; *1920*, 114, 362]. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses

be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

Only one reply to this question has been received, *1920*, 362. It seems that more might be said on this question, and further replies are desired. Some idea of existing conditions may be obtained from the replies to another question in the MONTHLY, *1916*, 395.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

There has been a considerable amount of interesting discussion on this question, but its possibilities are by no means exhausted. A formulation of the sense in which the problem is to be interpreted was given by the editor, *1920*, 302. For continuous functions, analytic at the origin, the only solutions are cubic polynomials. It was shown by Gillespie, *1920*, 405, that the condition of analyticity at the origin may be replaced by the existence of six continuous derivatives near the origin. On the other hand remarks by Bennett, *1920*, 460, Weisner, *1921*, 20, and the editor, *1920*, 462, show that there exist solutions other than cubic polynomials, possessing continuous first derivatives. It would be of interest to know how little need be assumed beyond the continuity of the first derivative to restrict solutions to cubic polynomials.

The related question obtained by taking both limits of integration variable and suitably modifying the formula has reached a satisfactory answer with the proof by Ballantine, *1921*, 19, that the solution must be a cubic polynomial provided it is merely continuous.

35 [*1918*, 266; *1920*, 114]. Is the following theorem new or has it previously been published?

THEOREM. *If two parallel planes, π and π', cut sections from a cylindrical surface S and two spherical surfaces S_1 and S_2, and if the sum of the sections of S_2 is equal in area to the sum of the sections of S plus the sections of S_1, then the part of S_2 included between π and π' is equal in volume to the sum of the parts of S and S_1 included between π and π'.*

In communicating this question, E. O. Brower commented on the simplicity with which the theorem could be proved, which led him to wonder whether so simple a proposition had remained unnoticed.

39 [*1920*, 256]. There are certain problems in geometry which are simple in statement but can be reduced only to very complicated problems in transcendental analysis. Following are several examples of the type of problem in question.

1. What is the smallest plane area within which a given figure can be turned through a complete revolution? It is not implied that the figure should revolve about a fixed point, but merely that in the course of its motion the figure should have every possible orientation in the plane. The problem may be modified by considering only convex areas.

An interesting special case is that in which the given figure is a segment of a straight line. In this case it has been conjectured by Professors Osgood and Kubota that the smallest area may be bounded by a three-cusped hypocycloid; if we consider only convex areas, perhaps the result will be an equilateral triangle. I have no indication of a proof.

2. For every closed convex curve of area P there is an n-sided circumscribed polygon of least area Q and an n-sided inscribed polygon of greatest area R. For a fixed value of the integer n and for all closed convex curves, what is the upper limit of Q/P and what is the lower limit of R/P? I have succeeded only in proving that for the case $n = 3$, the upper limit of Q/P is 2.

3. Let the area of a given simple closed curve A be a. Remove from A the greatest possible area a_1 similar to another given simple closed curve B. From the remaining figure remove the greatest possible area a_2 similar to B. Continue this process indefinitely. Is it or is it not true that

$$a_1 + a_2 + a_3 + \cdots = a?$$

I have proved the statement to be true in the special case that A is convex and B is a circle.

4. Let a given closed convex curve K have the property that a given triangle whose angles are incommensurable with π can be revolved completely within K (see Part 1 of this question), always remaining inscribed in K. What may the curve K be? Can any other curve except a circle satisfy the conditions?

These questions were formulated by Professor S. Kakeya, who has treated some problems of similar nature in various papers in the *Science Reports* of the Tôhoku Imperial University. They involve considerations of decided difficulty in the analytic formulation of geometric relations. The special case mentioned in the first question has received some attention from W. B. Ford, *Bulletin of the American Mathematical Society*, vol. 27, p. 55.

40 [*1920*, 365]. How great emphasis is laid in freshman mathematics upon the elementary algebra of complex numbers? A recent paper by an eminent electrical engineer seems to indicate the need of a knowledge of this subject on the part of draftsmen and mechanics of very limited educational opportunities. The syllabus of the College Entrance Examination Board mentions the topic under the caption "Advanced Angebra," but the question papers call for only the slightest study of numbers of this type. Is Euler's theorem ($e^{i\vartheta} = \cos \vartheta + i \sin \vartheta$) usually presented in college algebra, or is it left to the calculus? Does the topic deserve greater emphasis than it usually gets, for the sake of applications in the field of periodic currents?

The discussion, "Complex Numbers in Advanced Algebra," by Mr. H. E. Webb, *1920*, 411, may be viewed as a reply to this question. There seems to be room for further discussion.

41 [*1920*, 365]. A reader asks for an elementary proof of the following two propositions in number theory, either of which can readily be obtained from the other:

Every positive integer of the form $8n + 3$ *is the sum of three odd squares.*

Every positive integer is the sum of not more than three triangular numbers.

Bachmann[1] states that these theorems have been proved only by the use of the theory of ternary quadratic forms, and refers to the discovery of the theorems by Gauss, and a comparatively simple proof by Dirichlet,[2] by means of this theory. A number of references are given by Dickson.[3] Apparently the theorem was first enunciated by Fermat.

42 [*1921*, 65]. In connection with the questions of Kakeya, Professor W. B. Ford is led to the following inquiry: A line-segment AB is to be moved in its plane to a new position $A'B'$. How should this be done in order that the area generated may, to the greatest extent possible, be passed over three times?

The questions of Kakeya here referred to are given as Question 39 above.

[1] *Niedere Zahlentheorie*, Leipsic and Berlin, 1910, Teil 2, p. 325.

[2] *Crelle's Journal*, vol. 40, p. 228; *Liouville's Journal*, series 2, vol. 4, p. 233.

[3] *History of the Theory of Numbers*, vol. 2, Chapter 1.

Professor Ford has proved that if the generated area is to be passed over, to the greatest possible extent, but *two* times, AB should be rotated about the intersection of the perpendicular bisectors of AB and $A'B'$.

DISCUSSIONS.

Doctor Morris gives a device for a perpetual calendar, with an account of the theory on which it is constructed. A similar explanation will hold for any of the many forms of such calendars. It is believed that this article will be of interest, since explanations of the theory of perpetual calendars are seldom given.

Professor Bennett gives a set of simple identities from which a four-place logarithm table can readily be calculated. The method is probably new. It bears some relation to the plan used by Briggs (cf. *Encyclopædia Britannica,* 11th ed., vol. 16, pp. 875–876); but Briggs needed first to calculate and use explicitly the modulus, while Professor Bennett does not make any use of the modulus.

I. The Theory of Perpetual Calendars.

By Frank R. Morris, University of California.

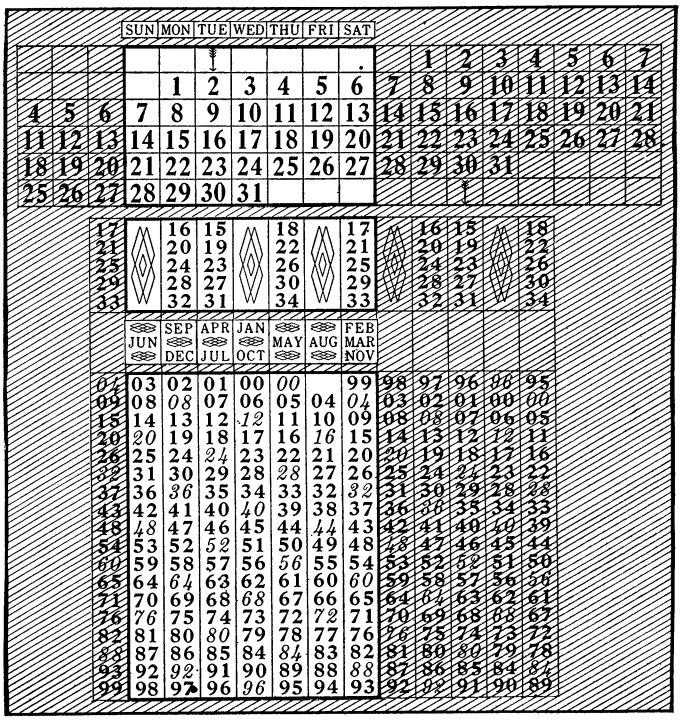
A short discussion of adjustable calendars is given by Irwin Roman[1] in the Monthly, *1915,* 241. He describes one of the several hundred mechanical devices used in presenting the calendar of a given month of a given year, but he does not give the mathematical theory upon which practically all perpetual calendars are based. It is the purpose of this article to present this theory.

There are approximately 365.2422 days in a tropical year. If each calendar year contained 365 days, in the course of 400 years an error of 96.88 days would accumulate. Then if 97 days be added for each period of 400 years the error will be very small. In any 400 consecutive integers there are 100 numbers which are divisible by 4, and 3 which are divisible by 100 but not by 400. Hence the rule: *Each year which is divisible by* 4 *but not by* 100 *and each year which is divisible by* 400 *is a leap year with* 366 *days. All other years contain* 365 *days.* Since this rule was first introduced by Pope Gregory XIII in the year 1582, no calendar based upon it need antedate the sixteenth century. The calendar may be extended into the future as far as one may choose. However, it becomes inaccurate after a few thousand years, because it accumulates a day in about 3,000 years, and also because the tropical year varies slightly as the years go by. Other calendars, which are used in some parts of the world or have been used in the past, are built upon other rules but I shall discuss only that calendar

[1] Mr. Roman's article begins with the statement, "So far as the writer has been able to learn, all perpetual or adjustable calendars are arranged so as to present the first day of the month as the first day of the week." He had evidently overlooked the most fruitful field for the study of perpetual calendars, viz., the *Official Gazette* of the United States Patent Office. During the past 25 years more than 100 calendars have been patented and more than a score of these present the days of the week in the natural order. For example see number 1048413, Dec. 24, 1912. [Doctor Morris's perpetual calendar was patented Nov. 26, 1918, number 1286058.—Editor.]

which results from the assumption of the above rule. The same general principles are used even though the rules are different.

Most perpetual calendars covering a period of several centuries contain five parts. These parts are *the days of the week, the days of the month, the months of the year, the years of the century* and *the centuries*. The days of the week are placed in regular order, beginning with any one of the seven days, and form an endless set of cycles of seven days each. The days of the month are the integers from 1 to 31 and are arranged in the usual manner. The months form an endless set of cycles of twelve months each. The years of the century, which occur in cycles of 100 years each, are indicated by the two right-hand digits of the numbers of the years; and the centuries are indicated by the numbers of the years after the years of the century have been displaced. They are the endless chain of integers which in our case begin with 15. The number which indicates the century is one less than its ordinal number. In the figure given herewith these five parts are shown from top to bottom in the order—days of the week, days of the month, centuries, months of the year, years of the century.



Let us start with the calendar for January, 1900, which is shown in the figure. Monday is the first day of the month. Since January has 31 days and $31 \equiv 3$ (mod. 7), *i.e.*, if 31 is divided by 7 the remainder is 3, February enters 3 days later in the cycle of days of the week than did January, which is Thursday. Hence the days of the week must be moved to the left 3 spaces or the days of the months to the right 3 spaces. Let us assume the latter movement and that the days of the week at all times remain in a fixed position. Also assume for the present that 00 is on a card which moves with the days of the month. It may be placed in any one of the seven columns. In the same column, but on the fixed card which contains the days of the week, place the month January. Now when the days of

the month are moved to the right 3 spaces 00 will be affected in the same manner. Then place February in the column containing the new position of 00, which is 3 spaces to the right of January. According to our rule February of 1900 contains 28 days. Thus the first day of March also falls on Thursday and March is in the same column as February. Since March has 31 days April is placed 3 columns to the right of March in the cycle of 7, which is one column to the left of January. This process may be continued until all the months of the year are located on the fixed card. When this is done the calendar for any month of the year 1900 is given when 00 is placed in the column of the desired month. The calendar shows 31 days for each month, but it is understood that only the proper number is to be used.

Now consider the year 1901. There are 365 days in 1900, and $365 \equiv 1 \pmod{7}$. Hence every day of the year 1901 falls one day later in the week than the corresponding day of the year 1900, and 01 should be placed one space to the left of 00, causing the days of the months to be shifted one space to the right. When 01 is moved to the column of any month the calendar is given for that month in 1901. Likewise 02 is located one space to the left of 01, and 03 one to the left of 02. However, the extra day of 1904, which is a leap year, must be accounted for. January and February run regularly, but the remaining 10 months of the year enter *two* days later in the week than they did in 1903. Hence we must have two symbols for a leap year. Let the italic figures *04* placed one space to the left of 03, which is in the cycle 3 spaces to the right of 00, be used for January and February, and let the usual figures 04 be placed two spaces to the left of 03 and used for the remaining 10 months of the year. Similarly the rest of the years of the century can be located.

We still have to consider the shift which must be made in passing from one century to the next. There are 36525 days from March 1, 1900 to March 1, 2000; and $36525 \equiv 6 \pmod{7}$. With the exception of the first two months, the months of the 21st century begin six days later in the week than did the corresponding months of the corresponding years of the 20th century. Hence in changing from 1999 to 2000 the days of the month must be moved 6 spaces to the right with respect to the years of the century. This can be accomplished if the days of the month are on an auxiliary slide which is mounted upon the main slide containing the years of the century. Now place 19 in any one of the columns on the main slide and 20 in the column 6 spaces to the right of 19. Before the auxiliary slide is moved from its original position place an arrow on it in the column containing 19. Then when the auxiliary slide is moved so that the arrow points to 20 the days of the month and the years of the century are in the correct relative position for all years of the 21st century. The first two months of the century are cared for by placing *00* in italics one space to the right of 00. There are 36524 days from March 1, 2000, to March 1, 2100; and $36524 \equiv 5 \pmod{7}$. Hence the days of the month must be shifted 5 spaces to the right relative to the years of the century, which means that 21 should be 5 spaces to the right of 20 on the main slide. Likewise 22 should be 5 spaces to the right of 21, and 23

five to the right of 22; but 24 should be 6 spaces to the right of 23. The shift is 6 spaces for all centuries divisible by 4 and 5 spaces for the other centuries. This may be continued to any desired extent and may be extended backwards to the beginning of the calendar.

It is easy to see from the figure that there must be at least 13 columns on the main slide of this particular form of the calendar. With the given position of the centuries it is necessary to have 18 columns on the auxiliary slide. This number might have been reduced by one; but there would then have been a loss in symmetry in two parts. Other forms of the calendar allow different parts to move. A common type has the parts on disks, which rotate with respect to one another. Still another type uses one or more of the parts as reference tables.[1] However, the theory which I have given is applicable to practically all types.

II. Note on the Computation of Logarithms.

By Albert A. Bennett, University of Texas.

The following approximate relations are easily verified by the use of a computing machine or by direct multiplication with paper and pencil, and without recourse to logarithms. (The writer checked these on a "Monroe" in about two hours.)

(1) $(1.024)^{214} = 160.02580 \cdots,$

(2) $3\,(9/8)^{18} = 24.99577 \cdots,$

(3) $2 \times 5 = 10,$

(4) $7\,(98/81)^4 = 14.998996 \cdots,$

(5) $((2.2)^3\,(2.1))^2 = 500.005377 \cdots,$

(6) $39\,(13/14)^6 = 25.000945 \cdots,$

(7) $25\,(17/14)^5 = 66.000071 \cdots,$

(8) $20\,(190/33)^2 = 662.9936 \cdots,$

(9) $13\,(11/10)^2\,(17/13)^4 = 45.9992752 \cdots,$

(10) $551\,(55/221)^2\,(23/39)^3 = 6.9997431 \cdots.$

If these relations be replaced by the approximate relations

(1′) $(1.024)^{214} = 160,$

(2′) $3\,(9/8)^{18} = 25,$

(3′) $2 \times 5 = 10,$

(4′) $7\,(98/81)^4 = 15,$

(5′) $((2.2)^3\,(2.1))^2 = 500,$

(6′) $39\,(13/14)^6 = 25,$

(7′) $25\,(17/14)^5 = 66,$

(8′) $20\,(190/33)^2 = 663,$

(9′) $13\,(11/10)^2\,(17/13)^4 = 46,$

(10′) $551\,(55/221)^2\,(23/39)^3 = 7,$

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[1] A very interesting example of this type is given by Augustus De Morgan in a volume of 88 pages entitled *The Book of Almanacs*. In addition to what is given by the ordinary perpetual calendar, his book includes saints' days, lunar calendars, several special calendars and discussions of historical interest.

and the logarithms to the base 10 be found formally, these reduce respectively, to the following:

(1″) $2136 \log 2 = 643,$
(2″) $37 \log 3 = 2 + 52 \log 2,$
(3″) $\log 5 = 1 - \log 2,$
(4″) $9 \log 7 = 1 - 5 \log 2 + 17 \log 3,$
(5″) $6 \log 11 = 11 - 7 \log 2 - 2 \log 3 - 2 \log 7,$
(6″) $7 \log 13 = 2 + 4 \log 2 - \log 3 + 6 \log 7,$
(7″) $5 \log 17 = -2 + 8 \log 2 + \log 3 + 5 \log 7 + \log 11,$
(8″) $2 \log 19 = -3 - \log 2 + 3 \log 3 + 2 \log 11 + \log 13 + \log 17,$
(9″) $\log 23 = -2 - \log 2 + 2 \log 11 - 3 \log 13 + 4 \log 17,$
(10″) $\log 29 = -2 + 2 \log 2 + 3 \log 3 + \log 7 - 2 \log 11 + 5 \log 13 + 2 \log 17 - \log 19 - 3 \log 23.$

These relations may be used as recursion equations to determine approximately the logarithms of the first few primes. When solved successively it is found that log 2 is given correctly to six places, so that its powers up to 1024 are correct to five places, and each of the other primes has its logarithm given correctly to at least five decimal figures.

These relations are found to be ample for completing correctly the construction of a four-place logarithm table, by use of linear interpolation when once the logarithms of all numbers of three significant figures or less, obtainable as powers and products of the above primes, have been written down. For example, between the logarithm of 23 and the logarithm of 29, we have without interpolation and by mere use of the fact that the logarithm of a product is equal to the sum of the logarithms of the factors, the logarithms of 24, 25, 26, 27, 28, as well as of such numbers, for example as, 25.6. With a little ingenuity (such as would be illustrated in determining the logarithm of 7, from the logarithms of 2, 3, and 5, by the relation that 49, lies between 48 and 50), a five-figure table with at most very few errors can be obtained from the above relations.

These relations have been found by an extension of the continued fraction algorithm, and for the accuracy desired simpler relations do not appear to exist.

This alternative method of constructing or checking logarithm tables is believed to be new.

RECENT PUBLICATIONS.

REVIEWS.

A First Course in Nomography. (Bell's Mathematical Series.) By S. Brodetsky. London, G. Bell and Sons, 1920. 12 + 135 pages. Price 10 shillings.

Preface: "In many branches of science, in engineering practice, in technology, in industry and in military science, Nomography is a recognized means of carrying out graphical calculations. The ballistic constant in gunnery, flame temperature in the research of coal-gas combustion, the angle of twist in a thread of given thickness with a given number of turns per inch, the conversion of counts in the textile industry, can all be calculated by means of nomograms. Nomographic

charts are simple and certain in use, so that calculations formerly entrusted to skilled and responsible computers can now be safely left to the care of a comparatively unskilled subordinate. It is the object of this First Course to offer a clear and elementary account of the construction and use of such charts.

"The method of treatment chosen is based on experience gained in the making of nomograms for various technological departments in the University of Leeds, and in other ways. It is a treatment that should be found useful by the reader who desires to become acquainted both with the theory of nomography and with its practical use. Chapter III begins the nomography proper, but the reader is advised to study Chapters I and II first, in order to see how the nomograms in Chapter III can be constructed. Special attention is directed to §§ 49–50 in Ch. IV, and to Chapter VIII. Answers have been purposely omitted, even where the examples lead to numerical results. . . ."

Contents: List of diagrams and index of nomograms, ix–x; Historical sketch, xi–xii ["When Descartes invented Coördinate Geometry, he put at the disposal of mathematicians a powerful weapon that has led to phenomenal advances in all branches of mathematical science. For the purpose of practical use in calculations by means of graphical representation on a plane, the number of variables that can be used is obviously restricted to two. This limitation was removed by Buache (1752), who introduced the method of contours—now incorporated in all atlases and surveys. By means of contours it is possible to deal with three quantities at once, whilst they are all represented on one plane, as, *e.g.*, in indicating the variation in the height of land or the depth of the sea. This sufficed for a time, but the extraordinary growth of railway systems all over the world led to important developments by Lalanne (1841), Massau (1884), and Lallemand (1886).

"The idea of using collinear points, which constitutes the chief beauty of the method of the present book, was developed by d'Ocagne (1884). It was d'Ocagne, too, who applied the name Nomography to this method, in his book *Les calculs usuels effectués au moyen des abaques* (1891). Since then further extensions have been made by d'Ocagne and others.

"In recent years the utility and convenience of nomography have been increasingly realized, and the subject has gained in importance and recognition, particularly in engineering practice. It is, in the main, a product of French mathematical genius. Articles have appeared in one or two English journals, and excellent accounts of the subject in English are to be found in Hezlett's (*sic*) *Nomography* (Royal Military Institution, Woolwich) and Lipka's *Graphical and Mechanical Computation* (Wiley, New York).

"But the reader who is interested in the subject cannot do better than read d'Ocagne's excellent *Traité de Nomographie* (G. Villars, Paris, 1899)"]; Introduction, 1–5; Chapter I: Nomograms for addition and simultaneous equations, 6–22; II: Generalized nomograms for addition and subtraction, 23–45; III: Nomograms for multiplication and division, 46–64; IV: Nomogram with two parallel scales. Quadratic equations, etc., 65–88; V: Generalized theory of nomograms with two parallel scales. Parallel coordinates, 89–99; VI: Nomograms with trigonometrical functions, 100–108; VII: Nomograms with intersecting scales, 109–117; VII: Practical and automatic construction of nomograms. Empirical nomograms. Practical details, 118–135.

Leçons sur l'intégration des équations aux dérivées partielles du premier ordre. Par EDOUARD GOURSAT. Deuxième édition revue et augmentée. Paris, Hermann, 1921 [published October, 1920]. 2 + 459 pages. Price 40 francs.

The first edition of this work appeared in 1891. In the present edition considerable change has taken place and the work as a whole is nearly one third larger. Goursat remarks in the preface:

"Dans cette nouvelle édition, on a conservé le plan général de ces Leçons. Seuls les premiers Chapitres, relatifs aux théorèmes d'existence et aux équations linéaires, ont été assez profondément remaniés. Dans la suite, on a simplement modifié quelques demonstrations et complété certains résultats.

C'est aussi pour ne pas bouleverser l'ordre suivi dans l'édition originale que je n'ai pas introduit dans ce Volume la méthode de Pfaff. Cette importante méthode sera exposée, avec les développements qu'elle mérite, dans un autre Ouvrage, spécialement consacré au *Problème de Pfaff* et à ses généralisations, et qui paraîtra, je l'espère, prochainement." The publisher expects that this new volume will be ready in September, 1921.

Contents—Chapter I: Théorèmes d'existence, 1–49; II: Equations linéaires, Systèmes complets, 50–102; III: Equations linéaires aux différentielles totales, 103–133; IV: Intégrals complètes. Méthode de LaGrange et Charpit, 134–161; V: Méthode de Cauchy. Caractéristiques, 162–201; VI: Etude géométrique des équations a trois variables. Courbes intégrales. Solutions singulières, 202–253; VII: Première méthode de Jacobi, 254–266; VIII: Seconde méthode de Jacobi. Généralisations de Mayer et de Lie, 267–308; IX: Théorie générale de Lie, 309–352; X: Transformations de contact, 353–409; XI: Groupes de fonctions. Méthode générale d'intégration, 410–454.

Pioneers of Progress: *Archimedes*. By T. L. HEATH. ("Men of Science" series edited by S. Chapman). London, Society for Promoting Christian Knowledge, 1920. 2 + 58 pages. Cloth. Price 2 shillings.

First two paragraphs: "If the ordinary person were asked to say off-hand what he knew of Archimedes, he would probably, at the most, be able to quote one or other of the well-known stories about him: how, after discovering the solution of some problem in the bath, he was so overjoyed that he ran naked to his house, shouting εὕρηκα, εὕρηκα (or, as we might say, 'I've got it, I've got it'); or how he said 'Give me a place to stand on and I will move the earth'; or again how he was killed, at the capture of Syracuse in the Second Punic War, by a Roman soldier who resented being told to get away from a diagram drawn on the ground which he was studying.

"And it is to be feared that few who are not experts in the history of mathematics have any acquaintance with the details of the original discoveries in mathematics of the greatest mathematician of antiquity, perhaps the greatest mathematical genius that the world has ever seen."

Contents—Chapter I: Archimedes, 1–6; II: Greek geometry to Archimedes, 7–23; III: The works of Archimedes, 24–28; IV: Geometry in Archimedes, 29–44; V: The sandreckoner, 45–49; VI: Mechanics, 50–52; VII: Hydrostatics, 53–56; Bibliography, 57; Chronology, 58.

An Introduction to String Figures. An Amusement for Everybody. By W. W. R. BALL. Cambridge, W. Heffer & Sons, 1920. 38 pages. Price 2 shillings.

"Prefatory Note" dated July, 1920: "The making of String Figures is a game common among primitive people. Its study by men of science is a recent development, their researches have, however, already justified its description as a hobby, fascinating to most people and readily mastered. The following pages contain a lecture on these figures and their history; to it I have appended full directions for the construction of several easy typical designs, arranged roughly in order of difficulty, and, for those who wish to go further, lists of additional patterns and references. The only expense necessary to anyone who takes up the pastime is the acquisition of a piece of good string some seven feet long; with that and this booklet to aid him, he will have at his command an amusement that may while away many a vacant hour."

Newton. (Profile N. 52). By GINO LORIA, Roma, A. F. Formiggini, 1920. 69 pages. Price 3.00 lire.

This is the latest volume in the dainty little series of booklets (4 x 6½ inches) among which A. Mieli's *Lavoisier* (no. 42), A. Favaro's *Archimedes* (no. 21), and *Galileo* (no. 10) have been published during the past ten years. Each volume is written with light touch by one thoroughly conversant with materials regarding the life of the subject, and an ample bibliography provides finger-posts directing the inquirer to more searching investigations. Most of the volumes of the series have a portrait frontispiece. In preparing his little volume Professor Loria discovered a discussion of Newton's laws in an eighteenth century manuscript which he described and reprinted in "Per la storia del newtonianismo in Italia," *Atti della Società italiana del Progresso delle Scienze*, Pisa, April, 1919, Rome, 1920.

NOTES.

Bulletin des Sciences Mathématiques for September, 1920, contains (pages 194–200) Picard's addresses at the opening and closing of the Strasbourg Mathematical Congress, September 22–30, 1920. They are also given in *Revue Scientifique,* November 13, 1920, pages 641–643.

In the *Quarterly Journal of Mathematics,* October, 1920, appears Sir Thomas Muir's seventh list of writings on determinants. It contains 264 titles, 32 of which are taken from this MONTHLY. Among the authors represented are 35 Americans.

The first 21 numbers (pages 1–254) of *Mathematical Notes, a Review of Elementary Mathematics and Science,* were published by the Edinburgh Mathematical Society, April, 1909–December, 1916. The next number of this publication did not appear until November, 1920, and then occupied pp. 51–64 of *Proceedings of the Edinburgh Mathematical Society,* volume 38.

Paul Appell's *Traité de Mécanique rationnelle* (Paris, Gauthier-Villars) has long been the leading treatise on the subject. The third edition ("entièrement refondue") of the third volume (*Equilibre et mouvement des milieux continus,* 674 pages, price 60 francs) and the first edition of a fourth volume (*Figures d'équilibre d'une masse fluide homogène en rotation sons l'attraction newtonienne,* 297 pages, price 30 francs) were published in November, 1920. Important additions to the third volume are the accounts of the work of Villat on movements of a fluid parallel to a fixed plane and of Bjerknes on "les fluides baroclines."

The concluding number of the first volume of *Archivio di Storia della Scienza* was published in August, 1920 [cf. *1920,* 474]. Aldo Miele continues his methodical bibliography (about 325 titles) of Italian works on the history of science, pages 397–420; there are also: (*a*) a brief review by A. Mieli of L. C. Karpinski, H. Y. Benedict, and J. W. Calhoun's *Unified Mathematics* (New York, 1918), 432; (*b*) "Elia Millosevich (1884–1919)" by G. Abetti, 446–447; (*c*) "H. G. Zeuthen (1839–1920)," with a list of his historical publications, by G. Loria, 447–451.

The following five mathematical periodicals have ceased publication (temporarily, at least): *Archiv der Mathematik und Physik* (last volume, 28, 1919); *Bibliotheca Mathematica* (last volume, 14, 1914–1915); *Journal de Mathématiques Pures et Appliquées* (Liouville, last volume, 84, 1919); *Nouvelles Annales de Mathématiques* (last volume, 79, 1920); *Zeitschrift für Mathematik und Physik* (last volume, 64, 1917). It has been announced that *Annales scientifique de l'Ecole Normale Supérieure* and *Bulletin de la Société Mathématique de France* will have to be discontinued unless many new subscribers are forthcoming at an early date.—Volume 81 of *Mathematische Annalen* (price, in Germany, 96

marks, 320 pages) was sent out by the new publisher, Springer, Berlin, in November, 1920. The editors plan to accept for publication articles on mathematics and all its applications, and not, as formerly, to limit publication to research papers in pure mathematics. Up to December 15, 1920, Teubner had published only the first number of volume 80.

The first number, June, 1920, of the second volume of *Bulletin de la Société Mathématique de Grèce* (*1920*, 314) opens with a list of the officers of the society and of its 123 members in 1919. This is followed by a portrait and brief sketch of "ΚΩΝΣΤΑΝΤΙΝΟΣ ΚΑΡΑΘΕΟΔΩΡΗ" (Constantine Carathéodory) recently appointed professor of mathematics at the University of Athens (*1920*, 337), but whom Vénizélos had expected would organize the new Greek university at Smyrna. The French articles in the number are: "Généralisation des formules de Combescure-Darboux" by N. Hatzidakis, 16–18; "Sur quelques propriétés des fonctions croissantes" by G. J. Rémoundos, 19–23; "Sur l'intégration de l'équation de Laplace entre deux sphères non concentriques" by D. Hondros, 24–28; "Sur la théorie de la flexion" by N. Sakellariou, 32–36.

The *Canadian Magazine* for July, 1920, contains an article by W. S. Wallace entitled "Some letters of Francis Maseres: 1766–1769." This interesting article furnishes new light on his life and on the period during which he was attorney general of Quebec.

Baron Maseres, historian, reformer and amateur mathematician, was born in London in 1731. He graduated from the University of Cambridge as senior wrangler in 1752 (according to Ball's *History of the Study of Mathematics at Cambridge*) and took up the practice of law. In 1773 he was appointed baron of the exchequer and he held this position until his death in 1824, "a length of tenure without parallel in the records of law." He was also for many years a senior judge of the sheriffs court in the city of London. "Homer he knew by heart, and Horace was at his fingers' ends. Lucian was his favorite next to Homer in ancient literature. . . . He spoke French fluently, but it was the language in idiom and expression which his ancestors had brought over to England." (*Dictionary of National Biography*).

Many of his numerous books, pamphlets, and reports, not referred to in '*D.N.B.*' are listed in H. J. Morgan, *Bibliotheca Canadensis*, 1867, and in P. Gagnon, *Bibliographie Canadienne*, Quebec, 1895. He wrote *Elements of plane trigonometry*, 1760, a meritorious treatise in two volumes on life assurance, 1783, and several works on algebra "which are valueless because he refused to allow the use of negative or impossible quantities." But his *Scriptores Logarithmici*, in six quarto volumes, 1791–1807, is a very useful work since it contains reprints of many of the older publications on logarithms.

Each of the three volumes of Wallis, *Opera Omnia*, 1699, in the Brown University library contains the autograph "F. Maseres. Aug. 27, 1774."

ARTICLES IN CURRENT PERIODICALS.

Bulletin des Sciences Mathématiques, volume 55, August, 1920: Review by M. Fréchet of E. B. Wilson's *Aëronautics* (New York, 1920), 169–172 [Quotation: "Son livre a tous les mérites des Ouvrages de langue anglaise: simplicité, recours constant à l'intuition visuelle, théories constamment appuyées d'exemples numériques et concrets. Mais en même temps (pourrais-je sans exagération en reporter un peu le mérite au séjour en France par lequel il a terminé ses études, ou serait-ce tout simplement un don propre de M. Wilson) on trouve dans son livre une clarté d'exposition, un souci de l'ordre qui sont généralement des qualités reconnues aux Ouvrages français. Ces qualités, j'avais déjà eu le plaisir de les signaler dans son excellent Ouvrage *Advanced Calculus*, avec d'autant plus d'assurance que je me sentais sur un terrain plus familier"].

Bulletin of the American Mathematical Society, volume 27, no. 2, November, 1920: "The twenty-seventh summer meeting of the American Mathematical Society" by A. Dresden, 49–65; "The Chicago colloquium" by W. A. Hurwitz, 65–71; "Note on velocity systems in curved space of n dimensions" by J. Lipka, 71–77; "Augustus De Morgan on divergent series" by F. Cajori, 77–81; "Russell's introduction to mathematical philosophy" by G. A. Pfeiffer, 81–90 [Review of Bertrand Russell's *Introduction to mathematical philosophy*. London, 1919]; Notes, 91–94; New publications, 94–96.

The Journal of the Indian Mathematical Society, volume 12, no. 1, February, 1920: "Multiplication of infinite integrals" by K. B. Madhava, 3–13; "Certain definite integrals and series" by C. Krishnamachary, 14–31; "Astronomical notes" by T. P. Bhaskara Sastri, 32–33; Questions and Solutions, 34–40—No. 2, April, 1920: "Functions of Legendre's type" by K. B. Madhava, 41–53; "Morley's theorem" by M. Bhimasena Rao, 54–58; "Note on the integration of $\int d\theta/(a - b \cos \theta)^n$" by K. B. Madhava, 59–60; "Note on the equation $a \sec \theta + b \operatorname{cosec} \theta = 1$" by M. T. Naraniengar, 60–61; Questions and Solutions, 61–80—No. 3, June: "The late Mr. S. Ramanujan, B.A., F.R.S." (portrait frontispiece) by P. V. Seshu Aiyar, 81–86 [Many details of his career and a complete list of his contributions to the *Journal of the Indian Mathematical Society*]; "In memoriam, S. Ramanujan" by R. Ramachandra Rao, 87–90 [Extracted from *Everyman's Review*, volume 5, nos. 7 and 8, 1920]; "S. Ramanujan, F.R.S." by G. H. Hardy, 90–91 [From *Nature*, June, 1920; compare this *Monthly, 1920*, 316, 338]; "Jean-Gaston Darboux, 1842–1917," by K. B. Madhara, 92–97; "On the set of points $\phi(n)/n$" by T. Vijayaraghavan, 98–99; "The generating planes of a quadric in five dimensions" by R. Vythynathaswamy, 100–101; Questions and Solutions, 101–119.

Mathematical Gazette, volume 10, October, 1920: "Imaginaries in geometry, and their interpretation in terms of real elements" by C. V. H. Rao, 129–133 [First paragraph: "The principal ideas are due to von Staudt and Lüroth. There is also a paper by Prof. Mathews, though it does not go far enough; and a recent treatise in English on the subject makes no reference at all to the question. It may be useful therefore to put down a brief account of the leading ideas of the two writers first mentioned, together with some simple developments."]; "Gleanings far and near," 133, 149; "The sound ranging problem" by G. Greenhill, 134; "Summation of harmonic progressions" by W. J. Ricketts, 135; "Some propositions relative to a tetrastigm" by J. H. Lawlor, 135–139; "The volume of a frustum of a sphere" by F. W. Russell, 139; "On the conic in polar coördinates," 140–141; "A method of finding the normal acceleration in circular motion," 142–143; "Brocard points for a quadrilateral" by R. W. Genese, 143–144; "Probability and athletic sports" by W. Hope-Jones, 144–145; "Some incidental writings of DeMorgan" (continued) 146–149; Reviews, 150–159 [pages 155–159: the first part of a review of A. Macfarlane's *Lectures on Ten British Physicists* (New York, 1919)].

Messenger of Mathematics, volume 50, no. 1, May, 1920: "4-tic & 3-bic residuacity-tables" by A. Cunningham and T. Gosset, 1–16.

Nature, volume 106, October 28, 1920: "Elementary geometry" [review of C. Godfrey and A. W. Siddons's *Practical Geometry* and *Theoretical Geometry: a Sequel to Practical Geometry* (Cambridge, 1920)], 273–274—November 18: "The surveyor's art" [review of G. L. Hosmer's *Geodesy: including Astronomical Observations, Gravity Measurements, and Method of Least Squares* (New York and London, 1919)] by E. H. H., 369.

Nouvelles Annales de Mathématiques, volume 79, September, 1920: "Surfaces de translation applicables l'une sur l'autre" (continued) by B. Gambier, 321–341; "Deux notes de

géométrie vectorielle" by R. Garnier, 341–347; "Un théorème général sur les complexes" by C. Servais, 347–355; "Démonstration géométrique du théorème de Liouville sur le groupe isogonal de transformations dans l'espace" by A. Lévêque, 356–358; "Chronique," 359–360—October: "Surfaces de translation applicables l'une sur l'autre" (conclusion) by B. Gambier, 361–372; "Sur une classe d'équations différentielles qui admettent des intégrales singulières" by E. Goursat, 372–395; "Sur des systèmes articulés" by R. Bricard, 395–400.

Pacific Review, University of Washington, volume 1, no. 1, June, 1920: "And still it moves" by E. T. Bell, 83–92 [First two paragraphs: "It is related that Galileo on rising from his knees after having 'abjured, cursed and utterly detested' the heresy of the Earth's motion, muttered in his beard 'And still it moves.' The story is an epitome of the scientific temper. For in one respect the truths of science are like murder: they will out, no matter to whom they are distasteful, costly or fatal.

"'And still it moves.' Those four words sum up the history of science. They remind each generation of a perennial truth which its predecessors learned and forgot, scientific progress is inevitable and to be arrested by none. It is not mere pessimism to hold that each generation, profiting little or not at all by the bitterly won victories of those that have gone before, has had to master this fact anew; it is plain fidelity to the undisputed records of scientific history. In short it seems to be a law of human nature that the radically new shall be violently resented, provided only that it is also true. Let us trust that our own generation, tolerating rather than obstructing the onward rush of science, will be the first but not the last to break this grand law of human stupidity"].

Proceedings of the Edinburgh Mathematical Society, volume 38, session 1919–20, November, 1920: "A geometrical proof of Professor Morley's extension of Feuerbach's Theorem" by H. W. Richmond, 2–5 [Morley's extension: "All curves of class three which (i) touch the six lines OP, OQ, OR, QR, QP, PQ joining four orthocentric points, O, P, Q, R, and (ii) pass through the circular points, also touch the common nine-points circle of the triangles PQR, OQR, ORP, POQ"]; "A proof of the binomial theorem, with some applications" by G. A. Gibson, 6–9; "The modified Bessel function $K_n(z)$" by T. M. MacRobert, 10–19; "Extension of Frenet's formulæ to a curve in flat space of n dimensions" by R. F. Muirhead, 20–23; "A version of Hagen's proof of the 'law of error'" by R. F. Muirhead, 24–26; "Note on the polynomials which satisfy the differential equation $x\dfrac{d^2y}{dx^2} + (\gamma - x)\dfrac{dy}{dx} - \alpha y = 0$" by N. McArthur, 27–33; "On direct and inverse interpolation by divided differences" by G. Smeal, 34–50; *Mathematical Notes, a review of elementary mathematics and science;* 51–64 [The last of the "notes" is "A link slide rule for the mechanical solution of quadratic equations" by G. D. C. Stokes, 61–64].

Proceedings of the National Academy of Sciences of the U. S. A., volume 6, no. 9, September, 1920: "On a differential equation occurring in Page's *Theory of Electromagnetism*" by H. Bateman, 528–529; "A new proof of a theorem due to Schoenflies" by J. R. Kline, 529–531; "On the structure of finite continuous groups with exceptional transformations" by S. D. Zeldin, 541–543.

Punch, London, volume 159, November 3, 1920: "Euclid in real life" by A. P. H., 346 [Illustrated by discussion of the problem: To find the center of a given circle, and the theorem: Any two sides of a triangle are together greater than the third side].

Quarterly Journal of Pure and Applied Mathematics, volume 49, no. 1, October, 1920: "An American tournament treated by the calculus of symmetric functions" by P. A. MacMahon, 1–36; "An approximation to $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^a e^{-t^2}dt$" by H. B. C. Darling, 36–39; "On the partitions into unequal and into uneven parts" by P. A. MacMahon, 40–45; "On class number relations and certain remarkable sums relating to 5 and 7 squares" by E. T. Bell, 45–51; "A seventh list of writings on determinants" by T. Muir, 51–73; "The Stieltjes' integral and its generalisations" by S. Pollard, 73–96.

Revue générale des Sciences, volume 31, September 15–30, 1920: "Le tricentenaire de l'abbé Picard" [1620–1684?] by E. Doublet, 561–564 [notable astronomer and professor at the Collège de France]. See page 123 of this issue of the Monthly.

School Science and Mathematics, volume 20, no. 8, November, 1920: "The mathematics of elementary physics" by P. F. Hammond, 714–722; "A geometric recreation" by Isabel Harris, 731–733; "The graphical solution of spherical triangles" by M. O. Tripp, 734–742; Problems and solutions, 743–745—No. 9, December: "Value of the history of mathematical

ignorance" by G. A. Miller, 813–817; "Solutions of cubic equations by straight line graphs" by M. G. Schucker, 818–820; "The parallel development of mathematical ideas, numerically and geometrically" by L. C. Karpinski, 821–828; "Prose problems of algebra" by J. A. Nyberg, 829–835; "A study in determinants" by C. M. Himel and C. A. Stone, 835–837; Problems and solutions, 854–858.

Science, new series, volume 52, October 29, 1920: "Galileo's experiments from the tower of Pisa" by F. Cajori, 409; "Jonathan Edwards on multidimensional space and the mechanistic conception of life" by J. M. C., 409–410; "Note on Einstein's theory of gravitation and light" by E. Kasner, 413–414—November 12: "Predilection and sampling of human heights" by E. G. Boring, 464–466—November 26: "The American Mathematical Society" by F. N. Cole, 518–519 [report of the meeting held in New York on October 30, 1920]—December 10: "The History of Science Section and the progress of science" by F. E. Brasch, 559–562.

Unterrichtsblätter f¨r Mathematik und Naturwissenschaften, volume 26, nos. 5–6, August 20, 1920: "Ueber die geometrische Bedeutung von $\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$" by E. Eckhardt, 57–61; "Verhandlungen über die Meraner Reform-Vorschläge," Mathematik, 61–65.

Yale Alumni Weekly, volume 30, December 3, 1920: "Teaching the freshmen—Mathematics in the freshman year" by W. R. Longley, 259–260 ["Three types of students elect mathematics in the Freshman year of Yale. Those of the first type choose mathematics for some reason best known to themselves but not because they expect to use it as a tool or because it is prerequisite for later work in the college course which they will probably take. Those of the second type need some mathematical training in Freshman year in order to follow successfully some of the later college courses for which one year of mathematics is prescribed. The students of the third type take Freshman mathematics as the first part of a more extensive course necessary for engineering study or specialized work in advanced mathematics, physics, or chemistry. . . . The topics covered are, necessarily, only those of an elementary and fundamental character, and the time is divided about equally between analytic geometry and calculus. As taught to Freshmen in Yale, and in most other colleges in this country, analytic geometry is no longer the exhaustive treatment of the multitudinous properties of conic sections that some graduates of twenty years ago may remember. The present course consists chiefly of methods for treating graphically problems of the kind arising in engineering, physics, chemistry, and other fields. Some practice is given in methods of calculation by numerical tables and the slide rule. In the part of the course devoted to calculus the aim is to present the fundamental ideas and concepts of the subject, rather than to develop any particular degree of skill in the manipulation of the machinery. . . . For those Freshmen whose entrance preparation in trigonometry and solid geometry is deficient, a course involving five exercises per week is provided. The instruction in analytic geometry and calculus in this course is identical with that in the shorter course and the extra time is used for trigonometry and solid geometry. The work is so arranged that any student deficient in only one of the entrance subjects mentioned is required to take the longer course for only one term. In the teaching of trigonometry and solid geometry the emphasis is laid upon the definitions and ideas necessary for later work and upon the use of formulas of a practical nature. The usual logical development of solid geometry is not followed; the obvious facts are merely stated and formal proofs are given only when necessary to establish mensuration formulas which can not be understood otherwise."]—January 14, 1921: "Death of Professor John E. Clark," 414 [Teacher of mathematics in Yale University 1872–1901].

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 51, nos. 9–10, published September 14, 1920: "Kopfgeometrie" by B. Kerst, 217–223; "Die Apollonische Berührungsaufgabe" by E. Salkowski, 224–230; "Das Apollonische Berührungsproblem in stereometrischer Behandlung" by A. Lanner, 231–239; "Atom- und Molekularwärmen fester Körper" by L. Müller, 239–246; "Kleine Mitteilungen," 246–248; "Bücherbesprechungen," 254.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

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PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2883. Proposed by ROBERT C. COLWELL, Geneva College, Beaver Falls, Pennsylvania.

Evaluate (*a*) $\int_0^\pi \cos^2\left(\frac{n\pi\cos\theta}{2}\right)d\theta$ and (*b*) $\int_0^\pi \frac{\cos^m\theta d\theta}{\theta^n}$.

2884. Proposed by E. H. MOORE, University of Chicago.

Consider an $m\times n$ array A of numbers a_{st} and an $n\times m$ array B of numbers b_{ts}. Show that the system of mn equations:

$$\sum_{tu} a_{st}b_{tu}a_{uv} = 0 \qquad (sv),$$

implies the equation:

$$\sum_{ts} a_{st}b_{ts} = 0.$$

The suffixes s, u have the range $1, 2, \cdots, m$ and the suffixes t, v have the range $1, 2, \cdots, n$.

2885.

If A, B, C, X, Y are given collinear points, construct Z so that $\{ABCX\}+\{ABCY\}=\{ABCZ\}$, where $\{ABCX\}$ denotes the cross-ratio of the points A, B, C, X. [From *the Math. Tripos Exam.*, Cambridge, Eng., 1905.]

2886. Proposed by S. A. COREY, Des Moines, Iowa.

If, in considering the purchase of bonds in the open market, x = interest yield to investor; i = interest rate named in bond; t = time in years to maturity; and q = quoted market price, the relation between these quantities is expressed by the equation, $xq-(x-i)(1+x)^{-t}-i=0$. Find value of x.

Also show that, if $a = i/q+(1-q)/qt$, we may, in practice, safely assume that very nearly,

$$x = \frac{at(a-i)+i[(1+a)^{t+1}-(1+a)]}{q(1+a)^{t+1}+t(a-i)-(1+a)}.$$

2887. Proposed by the late L. G. WELD.

A carpenter's square moves with its outer edges in contact with two round pegs of given equal diameters. Define the locus of the "heel" of the square.

2888. Proposed by J. A. BULLARD, U. S. Naval Academy.

Discuss the surface, $x^{1/2}+y^{1/2}+z^{1/2}=a^{1/2}$, by means of plane sections. (Cf. problem 2846, *1920*, 326.)

2889. Proposed by W. D. LAMBERT, U. S. Coast and Geodetic Survey.

Evaluate, $\int_c^1 \frac{\cos^{-1}x dx}{\sqrt{x^2-c^2}}$, where $0<c<1$.

PROBLEMS—NOTES.

7. The following problem is discussed in *Revista Matemática Hispano-Americana*, 1920, pages 190–191, 226–228: "What is the maximum number of spheres a decimeter in diameter that can be placed in a box in the form of a cube a meter along any interior edge?" It is found that this number is 1254. Professor R. P. Baker's problem, 501 (geometry), twice proposed in this MONTHLY [*1916*, 341; *1919*, 414], but not yet solved, may be recalled in this connection: "Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box." ARC.

8. Perfect Numbers. A number which equals the sum of its divisors other than itself is called perfect. Euclid [about 300 B.C.] proved that if

$$p = 1 + 2 + 2^2 + \cdots + 2^{n-1}$$

is a prime, $2^{n-1}\cdot p[= 2^{n-1}(2^n - 1)]$ is a perfect number. So far as known, the first six perfect numbers are: 6, 28, 496, 8128, 33550336, and 8589869056. A posthumous paper of L. Euler [1707–1783] contains the proof that every even perfect number is of Euclid's type.[1] He proved also that every odd perfect number must be of the form $r^{4\lambda+1}P^2$, where r is a prime of the form $4n + 1$. No odd perfect number is known. It has been proved that there is no odd perfect number less than two million, or with less than five distinct prime factors.[2]

Mersenne stated, in effect, in the preface of his *Cogitata Physico-Mathematica*, Paris, 1644, that the first eleven perfect numbers are $2^{n-1}(2^n - 1)$ for $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 167, 257$. Since p can be a prime only when n is a prime the problem of verifying the statement, so far as even perfect numbers are concerned, is equivalent to that of determining when p is prime for the 56 prime values of n less than 258. This has been effected for 45 of the prime values and the untested cases are for $n = 137, 139, 149, 157, 167, 193, 199, 227, 229, 241$, and 257. Mersenne's statement has been shown to be incorrect in at least five cases, namely when $n = 61, 89, 107, 127$, and 67. In the first four of these cases p has been shown to be prime. F. N. Cole found the two prime factors of p when $n = 67$, *Bulletin of the American Mathematical Society*, volume 10, page 137, 1903. America's further contribution to the discussion of perfect numbers was made through R. E. Powers who verified, in 1911, that p is prime when $n = 89$; proved, in 1914, that for $n = 107$, p is prime; and, in 1916, showed that for $n = 103$ or 109, p is composite.[3] According to *Sphinx-Oedipe*, July, 1914, and February, 1920, E

[1] The neatest proof of this was given in six lines by L. E. Dickson, in this MONTHLY, *1911*, 109.

[2] In a paper on perfect numbers by Benjamin Peirce, "Mathematical Instructor in Harvard University," *The New York Mathematical Diary*, no. 13, vol. 2, pp. 267–277, 1832, it is shown that there can be no odd perfect number "included in the forms a^r, a^rb^s, $a^rb^sc^t$, where a, b, and c are prime numbers and greater than unity." L. E. Dickson, in the work referred to on the opposite page, does not mention this paper. He does record: in 1844 "V. A. Lebesgue stated that he had a proof that there is no odd perfect number with fewer than four distinct prime factors." We now see that an American mathematician gave a *proof* of this theorem twelve years earlier.

[3] In his inaugural lecture before the University of Oxford (*Some Famous Problems of the Theory of Numbers and in particular Waring's Problem*, Oxford, 1920) G. H. Hardy remarked: "We have seen this ["tendency to exaggerate the profundity implied by the enunciation of a

Fauquembergue showed p prime for $n = 107$ and 127. Hence to sum up, p is known to be prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107$, and 127; and therefore 12 perfect numbers are known as such. A. Cunningham announced (*Report, British Association for the Advancement of Science*, 1895, p. 614) 12 other values of n ranging in value from $n = 317$ to $n = 132{,}019$, for each of which p is composite. Niewiadomski found a similar result for $n = 761$.

"It must always," wrote Sylvester in 1888, "stand to the credit of the Greek geometers that they succeeded in discovering a class of perfect numbers which in all probability are the only numbers which are perfect." (*Collected Mathematical Papers*, volume 4, 1912, page 589.) Sylvester referred also to the question of the non-existence of odd perfect numbers as "a *problem of the ages* comparable in difficulty to that which previously to the labors of Hermite and Lindemann . . . environed the subject of the quadrature of the circle" (*ibid.*, page 626).

The authoritative reference work which should be consulted by those desiring to learn what has been done in connection with perfect numbers is L. E. Dickson, *History of the Theory of Numbers*, volume 1, 1919. W. R. Ball's statements in the chapter on "Mersenne's numbers," and elsewhere in his *Mathematical Recreations and Essays* (fifth edition, 1911), should be carefully checked with Dickson's work. ARC.

9. In *Norsk Matematisk Tidsskrift*, the official organ of the Norwegian Mathematical Society, the following question was proposed in 1919, volume 1, page 80: "Usually x^y is greater or less than y^x; but for special values of x and y they are equal to one another, e.g., $2^4 = 4^2$. What are other values of x and y for which x^y and y^x are equal?" A reply by T. Nagel appears in the third number of the *Tidsskrift* for 1920, pages 93–94.

Mr. Philip Franklin, of Princeton University, has kindly informed me that in seeking the equations of all curves having the property that their evolutes are equal curves [compare *1920*, 303–306] he was led to a solution involving the roots of the transcendental equation $x = e^{ax}$. If x and y are two roots of this equation[1]

$$x^y = y^x = e^{axy}.$$

theorem"] even in the case of Fermat, a mathematician of a class to which Waring had not the slightest pretensions to belong, whose notorious assertion concerning 'Mersenne's numbers' has been exploded, after the lapse of over 250 years, by the calculations of the American computer Mr. Powers." Such a statement from such an authority, and on such an occasion, was extraordinary. So far as known Fermat made no erroneous statement whatever in connection with Mersenne's numbers. It is learned that the fanciful conjectures of Ball were Professor Hardy's only authority. *Mersenne's* inaccuracies and Mr. Powers's correction of *them* have been noted above.

[1] K. Schwering showed in 1878 (*Zeitschrift für Mathematik und Physik*, vol. 23, pp. 339–343) that solutions of the equation $x^y = y^x$ were obtained by taking any two solutions of the equation $a^x = x$. He showed also that for this latter equation there are an infinite number of solutions given by complex values for x. In 1896–1897, E. M. Lémeray found a number of results in connection with this equation (*Nouvelles Annales de Mathématiques*, vol. 16, pp. 548–556; and vol. 17, pp. 54–61). Among these were the following: when a is between 0 and 1, the equation $a^x = x$ has a real root between 0 and 1; when a is between 1 and $e^{1/e}$ there is a real root betwee 1 and e and a second real root between e and ∞ [$e^{1/e} = 1.444667\cdots$; for $a = \sqrt[3]{3} = 1.442249\cdots$ we have as roots of $(\sqrt[3]{3})^x = x$, $x_1 = 3$ and $x_2 = 2.478055\cdots$ (E. Heis, *Sammlung von Beispielen*

He showed in this MONTHLY, *1916*, 233–237, that the polar equation of $x^y = y^x$ is $r = \sec\theta\cdot(\tan\theta)^{1/(\tan\theta-1)}$ and that its parametric equations are $x = m^{1/(m-1)}$ and $y = m^{m/(m-1)}$.

But the discussion of the curve goes back much farther; indeed these parametric equations were given by Euler in his *Introductio in analysin infinitorum*, tome 2, Lausanne, 1748, p. 294 (French edition, 1797, p. 297). Setting $m - 1 = 1/u$ he gave also the form $x = \left(1+\frac{1}{u}\right)^u$ and $y = \left(1+\frac{1}{u}\right)^{u+1}$ and continued: "Thus the curve has, in addition to the line EAF, the branch RS which converges towards the lines AG and AH as asymptotes[1] and of which AF will be a diameter. Further the curve will cut the line AF at the point C where $AB = BC = e$, e denoting the number whose logarithm is unity. The equation furnishes also an infinity of separate points which with the line EF and the curve RCS exhaust those defined by it. There is, then, an infinity of numbers x and y which taken two and two satisfy the equation $x^y = y^x$; such are the following numbers, among these which are rational: $x = 2$, $y = 4$; $x = 9/4$, $y = 27/8$; $x = 64/27$, $y = 256/81$; $x = 625/256$, $y = 3125/1024$; etc." T. Nagel proved that: $x = 2$, $y = 4$, is the only pair of positive *integers* $(y > x)$ satisfying the equation. C. Herbst showed[2] in 1909 that, if $|y| > x$, $x = 2$, $y = 4$ and $x = -2$, $y = -4$ are the only integers satisfying the relation. These negative values were, apparently, overlooked by Daniel Bernoulli when writing to Goldbach[3]

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und Aufgeben aus ⋯ Arithmetik und Algebra. 75. Aufl., Köln, 1888, p. 371).] For $a = e^{1/e}$ there is a double root whose value is e. When $a = e$, Cauchy found (*Leçons sur le calcul différentiel*, Paris, 1829, leçon 14) $x = 0.3181317 \pm 1.3372357\sqrt{-1}$. By another method Stern got (Crelle's *Journal*, vol. 22, 1841, p. 59) $x = 0.318133 \pm 1.337238\sqrt{-1}$. The expression converging to the value of a root for a given a is $a^{a^{a^{\cdot^{\cdot}}}}$. This expression was studied by G. Eisenstein and F. Woepcke (Crelle's *Journal*, vol. 28, 1844, pp. 49–53; vol. 42, 1851, pp. 83–90). Certain errors of Eisenstein were corrected by P. L. Seidel who found, in effect, the Lémeray results noted above (*Abhandlungen der kgl. Baierischen Akademie der Wissenschaften*, zweite classe, vol. 11, 1870). The rôle that the equation $\omega^\xi = \xi$ plays in Cantor's theory of transfinite numbers will be recalled; see, for example, G. Cantor, *Mathematische Annalen*, vol. 49, 1897, pp. 242–246 (also English translation by P. E. B. Jourdain, Open Court Publ. Co., 1915, pp. 195–201).

Reference may be given also to: Hessel, (1) "Ueber die Bedingung unter welcher $a^x > x$ ist," (2) "Ueber das merkwürdige Beispiel einer zum Theil punctirt gebildeten Curve das der Gleichung entspricht: $y = \sqrt[x]{x}$," *Archiv der Mathematik und Physik*, vol. 14, 1850, pp. 93–96, 169–187; H. Scheffler, "Ueber die durch die Gleichung $y = \sqrt[x]{x}$ dargestellten Curven," *Archiv d. Math. u. Physik*, vol. 16, 1851, pp. 133–137; L. Oettinger, "Ueber den grossten Werth von $\sqrt[x]{x}$ und einige damit zusammenhängende Sätze," *Archiv d. Math. u. Physik*, vol. 42, 1864, pp. 106–113; and to L. Moreau: (1) *Analyse ou nombre de solutions et fixations des racines rémarquables de l'équation $a^x = x$.* (Brussels, 1900, 8vo, 16 pages); (2) "Variation du rapport a^z/z d'un nombre à son logarithme," *Journal de Mathématiques Spéciales*, vol. 25, pp. 170–173, 1900. Nomographic discussions of $a^x = bx$ and $a^x = x^b$ are given in S. Brodetsky, *A First Course in Nomography*, London, 1920, pp. 130–133.

[1] Euler's error here in stating that the coordinate axes are asymptotes (instead of $x = 1$ and $y = 1$) does not appear to have been previously remarked. The four asymptotes ($x = \pm 1$, $y = \pm 1$) were first correctly given in 1913 by A. M. Nesbitt, *Mathematical Questions and Solutions from 'The Educational Times,'* n.s. vol. 23, pp. 77–78, where the curve is plotted and discussed. Somewhat fuller considerations of a similar character were given by E. J. Moulton in this MONTHLY, *1916*, 233–237.

[2] *Unterrichtsblätter für Mathematik und Naturwissenschaften*, Jahrgang 15, pp. 62–63.

[3] *Correspondance Mathématique et Physique* (Fuss), vol. 2, 1843, p. 262.

June 29, 1728: "Je finirai par un problème qui m'a paru fort curieux et que j'ai résolu. Le voici: Trouver deux nombres inégaux x et y tels que $x^y = y^x$. Il n'y a qu'un cas ou ces nombres soient entiers, savoir $x = 2$ et $y = 4$ (car $2^4 = 4^2$), mais on peut donner une infinité de nombres rompus qui satisfont au problème. Il y a aussi d'autres espèces de quantités dont je ne dirai rien." [1] In reply to this on January 31, 1729, Goldbach wrote as follows[2] (*l.c.* pp. 280–281): "Je ne trouve pas la moindre difficulté à faire voir que, dans l'équation $x^y = y^x$, les nombres x et y ne peuvent être entiers à moins que l'un ne soit $= 2$, et l'autre $= 4$, et que, pour les nombres rompus, on peut donner une infinité de solutions. Voici comment je m'y prends: Je fais $y = ax$, donc $x^{ax} = a^x x^x$ et enfin $x = a^{1/(a-1)}$. Or, il est visible que x ne peut être un nombre entier que dans la supposition de $a = 2$; car si a est un nombre entier plus grand que 2, on voit d'abord que x devient irrationel; d'un autre côté, a étant un nombre rompu, toutes ses puissances seront autant de nombres rompus, et par conséquent x ne peut être un nombre entier; mais pour exprimer la valeur de x par des nombres rompus, il n'y a qu'à faire

$$x = f^{g/(f-g)} : g^{g/(f-g)}$$

où f et g soient des nombres entiers." [3] ARC.

PROBLEMS—SOLUTIONS

2791 [1919, 414]. A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom, the wine drips into the water, and the mixture drips out at

[1] Concerning this passage Cantor remarks (*Vorlesungen über Geschichte der Mathematik*, vol. 3_2, 1901, p. 610): "Man wird nach diesem Schlussworte wohl oder übel annehmen müssen, dass Bernoulli an complexe Auflösungen dachte." One must agree with Eneström (*Bibliotheca Mathematica*, vol. 13 (3), p. 270) that this surmise is "höchst unwahrscheinlich." The natural interpretation is that Bernoulli referred to the infinite number of irrational values of x and y which are obtained from Euler's equations given above, when u is not integral.

[2] The argument of Herbst is very similar.

[3] Other discussions of the relation $x^y = y^x$ are: T. Wittstein, *Archiv der Mathematik und Physik*, vol. 6, pp. 154–162, 1845; I. L. A. Lecointe, *Nouvelles Annales de Mathématiques*, vol. 11, pp. 187–189, 1852; M. Cantor, *Zeitschrift für mathematische und naturwissenschaftlichen Unterricht*, vol. 9, pp. 163–164, 1878; L. F. Marrecas Ferreira, *Jornal de Sciencias Mathematicas e Astronomicas*, vol. 2, pp. 165–166, 1880; M. Luxemberg, *Archiv der Mathematik und Physik*, vol. 66, pp. 332–334, 1881; D. Besso, U. Danielli, and L. Carline, *Periodico di Matematica per l'Insegnamento Secondario*, vol. 5, pp. 12–15, 115–117, 117–119, 1890; A. Flechsenhaar, and R. Schimmack, *Unterrichtsblätter für Mathematik und Naturwissenschaften*, vol. 17, pp. 70–73, 1911 and vol. 18, pp. 34–35, 1912; A. Tanturri, *Periodico di Matematica . . .*, vol. 30, pp. 186–187, 1915.

In Nouvelles Annales de Mathématiques, 1876, Moret-Blanc proved (pp. 44–46) that the only positive integral solutions of the equation $x^y = y^x + 1$, are $y = 0$, x arbitrary; $y = 1$, $x = 2$; $y = 2$, $x = 3$; also Meyl proved (pp. 545–547) that the only positive integral solutions of the equation $(x + 1)^y = x^{y+1} + 1$, are $x = 0$, y arbitrary; $x = 1$, $y = 1$; $x = 2$, $y = 2$. Landau showed (*L'Intermédiaire des Mathématiciens*, 1901, pp. 151–152) that the solutions found by Moret-Blanc for the equation $x^y = y^x + 1$ are the only positive rational solutions. This equation may be obtained from the simultaneous equations $x^y = 3$, $y^x = 2$ for which E. Heis gave the approximate solution $x = 2.23925113$, $y = 1.36280365$ (*Sammlung von Beispielen und Aufgaben aus . . . Arithmetik und Algebra*. 75. Aufl., Köln, 1888, p. 372; *Ausführliche Auflösuug der in Dr. Ed. Heis' Sammlung von Biespielen enthaltenen Aufgaben*, Dritter Theil, Bonn, 1880, pp. 386–388).

In *Messenger of Mathematics*, A. Cunningham discussed the factorization of $x^y - y^x$, x and $y > 1$, and x prime to y (April, 1916, vol. 45, pp. 185–192), and of $2^x - x$, x positive (May, 1917, vol 47, pp. 1–38).

$$\frac{dx_2}{dt} = \frac{q\left(1 - \frac{1}{e^{t/T}}\right)}{T} - \frac{x_2}{T}. \quad \text{Hence, } x_2 = q\left(1 - \frac{1 + \frac{1}{1!}\frac{t}{T}}{e^{t/T}}\right).$$

Similarly,

$$x_3 = q\left(1 - \frac{1 + \frac{1}{1!}\frac{t}{T} + \frac{1}{2!}\left(\frac{t}{T}\right)^2}{e^{t/T}}\right);$$

and so on.

For the *final* amounts in the successive cups, we have

$$X_1 = q\left(1 - \frac{1}{e}\right), \quad X_2 = q\left(1 - \frac{1 + \frac{1}{1!}}{e}\right), \quad X_3 = q\left(1 - \frac{1 + \frac{1}{1!} + \frac{1}{2!}}{e}\right), \cdots.$$

In general, we have

$$X_K = q\left(1 - \frac{e_K}{e}\right)$$

where e_K is the sum of the first K terms of

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \text{etc.}$$

As $K \doteq \infty$, $X_K = 0$.

Note.—Here also the rate of flow is not essential. If we take as the independent variable the amount of wine x which has been poured into the first cup, then the differential equations are

$$\frac{dx_1}{dx} + \frac{x_1}{q} = 1, \qquad \frac{dx_k}{dx} + \frac{x_k}{q} = \frac{x_{k-1}}{q}$$

and

$$x_k = q\left[1 - \left(1 + \frac{x}{q} + \frac{1}{2!}\left(\frac{x}{q}\right)^2 + \cdots + \frac{1}{(k-1)!}\left(\frac{x}{q}\right)^{k-1}\right)e^{-x/q}\right].$$

The final result is obtained by setting $x = q$.—Editors.

Also solved by W. D. Cairns, Alexander Knisely, L. C. Mathewson, and Arthur Pelletier.

2792 [1919, 414]. Proposed by B. J. Brown, Kansas City.

Solve the differential equation,

$$x^2(1 - x)\frac{d^2y}{dx^2} + 2x(2 - x)\frac{dy}{dx} + 2(1 + x)y = x^2.$$

Solution by C. P. Sousley, Pennsylvania State College.

This equation is exact and the first integral is,

$$x^2(1 - x)\frac{dy}{dx} + x(x + 2)y = \frac{x^3 + C}{3},$$

or

$$\frac{dy}{dx} + \frac{x + 2}{x(1 - x)}y = \frac{x^3 + C}{3x^2(1 - x)}.$$

Multiplying through by the integrating factor, $x^2/(1 - x)^3$, we have

$$\frac{x^2}{(1 - x)^3}\frac{dy}{dx} + \frac{x(x + 2)}{(1 - x)^4}y = \frac{x^3 + C}{3(1 - x)^4},$$

and on integrating, we have

$$\frac{x^2}{(1-x)^3}y = \frac{C+1}{9(1-x)^3} - \frac{1}{2(1-x)^2} + \frac{1}{(1-x)} + \log K.\sqrt[3]{(1-x)}.$$

Solved similarly by C. A. Isaacs, Gertrude McCain, and H. L. Olson.

2793 [1919, 458]. Proposed by J. L. Riley, Stephenville, Texas.

If a, b, and c, are complex, and α, β, and γ, real constants, the point

$$x = \frac{at^2 + 2bt + c}{\alpha t^2 + 2\beta t + \gamma}$$

traces a conic or a straight line when t takes all real values.

Discussion by A. F. Frumveller, Marquette University.

Since x is a complex number, let us put $x = u + iv$, $a = a_0 + a_1 i$, $b = b_0 + b_1 i$, $c = c_0 + c_1 i$, and clear of fractions. Separating the real and imaginary parts of this equation, we obtain the simultaneous set

$$(1) \qquad \begin{cases} t^2(\alpha u - a_0) + 2t(\beta u - b_0) + (\gamma u - c_0) = 0, \\ t^2(\alpha v - a_1) + 2t(\beta v - b_1) + (\gamma v - c_1) = 0. \end{cases}$$

The eliminant is $|p_0q_1| \cdot |p_1q_2| - |p_0q_2|^2 = 0$ (L. E. Dickson, *Elementary Theory of Equations*, New York, 1914, p. 155), where

$$|p_0q_1| = 2\begin{vmatrix} \alpha u - a_0 & \beta u - b_0 \\ \alpha v - a_1 & \beta v - b_1 \end{vmatrix}$$

$= 2\,[(\alpha b_0 - a_0\beta)v + (a_1\beta - \alpha b_1)u + (a_0b_1 - a_1b_0)]$ with similar expressions for the other two determinants.

The eliminant is, therefore, a quadratic in (u, v), *i.e.*, a conic, which under suitable conditions degenerates into straight lines. This conic in the plane uov (the plane of the complex number x) is in reality the projection of the actual path of the moving point in space as it spirals its way around the axis of t or a parallel line standing out at right angles to the lines $\overline{ou}$, $\overline{ov}$, in the x-plane.

Cf. an article on "The graph of $f(x)$ for complex numbers" (this Monthly, *1917*, 409), where many analogous examples are worked out and graphed in this rather unusual system of coördinates.

Also solved by Arthur Pelletier.

2796 [1919, 458]. Proposed by N. P. Pandya, Amreli, India.

Construct a triangle ABC having its centroid on a given ellipse, AB being a fixed diameter of the ellipse and C lying on one of the directrices.

Solution by Grace M. Bareis, Ohio State University.

Let O be the center of the given ellipse. Construct OM perpendicular to a directrix and meeting it at M. Determine P on OM so that $OP = \frac{1}{3}OM$. Through P draw a line parallel to the directrix and cutting the ellipse in N_1 and N_2. Draw ON_1 and ON_2 meeting the directrix in C_1 and C_2, respectively. Then ABC_1 or ABC_2 is a solution. It is to be noted that the points C_1 and C_2 are fixed points whatever diameter AB may have been chosen. The problem has four solutions, two corresponding to each directrix, if $e > \frac{1}{3}$; two solutions, one corresponding to each directrix, if $e = \frac{1}{3}$; no real solution when $e < \frac{1}{3}$.

Also solved by E. J. Oglesby, H. L. Olson, and Arthur Pelletier.

2797 [1919, 458]. Proposed by E. J. Oglesby, New York University.

Solve for x and y, the simultaneous equations,

$$x^3 + y^3 = 35 \qquad \text{and} \qquad x^2 + y^2 = 13.$$

SOLUTION BY H. N. CARLETON, West Newbury, Mass.

Let $x = u + v$ and $y = u - v$. Substituting these values of x and y in the original equations, we have after reducing

$$2u^3 + 6uv^2 = 35 \cdots \tag{1}$$

and

$$2u^2 + 2v^2 = 13 \cdots . \tag{2}$$

Equating the values of v^2 from (1) and (2) we have

$$4u^3 - 39u + 35 = 0, \quad \text{or} \quad (u - \tfrac{5}{2})(4u^2 + 10u - 14) = 0.$$

Whence, $u = 5/2, 1$, or $-7/2$. Hence, corresponding values of v are $\frac{1}{2}$, $\frac{1}{2}\sqrt{22}$, and $\frac{1}{2}\sqrt{-23}$, respectively. Hence, $x = u + v = 3$, $1 + \frac{1}{2}\sqrt{22}$, or $-7/2 + \frac{1}{2}\sqrt{-23}$ and the corresponding values of y are $u - v = 2$, $1 - \frac{1}{2}\sqrt{22}$, and $-\frac{7}{2} - \frac{1}{2}\sqrt{-23}$, respectively. By virtue of the symmetry of the equations, the corresponding values of x and y may be interchanged, thus making a total of six solutions.

Also solved by H. L. AGARD, NORMAN ANNING, GRACE M. BAREIS, H. C. BRADLEY, S. M. BERG, JOHN BIGGERSTAFF, G. A. BINGLEY, W. E. CLELAND, H. J. ETTLINGER, C. H. GROVE, OLIVE C. HAZLETT, ALEX. KNISELY, C. N. MILLS, J. Q. MCNATT, A. R. NAUER, H. L. OLSON, R. H. REECE, J. L. RILEY, EMETERIO ROA, M. M. SMITH, H. S. UHLER, W. W. WARNER, E. E. WHITFORD, and C. N. WUNDER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the genera interest of this department by sending items to the Editor-in-Chief.

Professor J. K. LAMOND, of Pennsylvania College, Gettysburg, has resigned to accept a position in the engineering department of the Bell Telephone Company at Philadelphia.

Dr. C. L. E. WOLFE, of the Junior College at Santa Rosa, has been appointed instructor in mathematics at the California Institute of Technology, Pasadena.

On page 187 of *Scientific Monthly*, February, 1921, there is a full-page portrait of Professor E. H. MOORE, of the University of Chicago, president of the American Association for the Advancement of Science.

At the close of the present academic year Dr. P. H. HANUS, professor of the history and art of teaching at Harvard University since 1901, will retire from active service. He was born in Silesia and came to the United States when he was four years old. He taught mathematics and science in a high school of Denver, Col. 1878–79, was professor of mathematics in the University of Colorado 1879–86 and was principal of another Denver high school 1886–90. His connection with Harvard dates from 1891. His first book, *An Elementary Treatise on the Theory of Determinants*, was published in 1886. There have been several reprints of his *Geometry in the Grammar School; an essay together with illustrative class exercises, and an outline of the work for the last three years of the grammar school* (4 + 52 pages, 1893).

JOHN EMORY CLARK, professor of mathematics, emeritus, in the Sheffield Scientific School, Yale University, died at Hartford, Conn., January 3, 1921. He was born in Northampton, N. Y., August 8, 1832. He graduated A.B. from the University of Michigan where he was an assistant professor of mathematics 1857–1859. He was professor of mathematics and astronomy at Antioch College from 1866 to 1872 when he became instructor in mathematics at Yale, and after a year was appointed to the chair of mathematics, which appointment he held until he was made professor emeritus in 1901. "Professor Clark will be remembered by his former students chiefly for his kindliness. Having himself an abstract and mathematical mind, he yet could endeavor to make his subject interesting to the members of his courses. He was perhaps not a great mathematician, but he was a great teacher, with patience and kindness beyond most, and his students held him in esteem and affection." (*Yale Alumni Weekly*, January 14, 1921.)

Doctor ALEXANDER PELL died at Bryn Mawr, Pa., January 26, 1921. He was born in Moscow, Russia, September 25, 1857. Entering the Artillery School at St. Petersburg in 1873, he graduated in 1878 as first lieutenant in the Russian artillery. He left the service in 1879 with the rank of captain and spent two years at the Institute of Civil Engineers in St. Petersburg. Forced by failure of revolutionary activities to leave his native country, he came to America in the early eighties. He was in a printing office of a Canadian village, and in a chemical factory of St. Louis, Mo., for sometime. Finally he entered The Johns Hopkins University as a graduate student in mathematics in October, 1895, was a fellow 1896–97, and received the degree of doctor of philosophy in June, 1897. His thesis entitled "On the focal surfaces of the congruences of tangents to a given surface" was published in *American Journal of Mathematics*, 1898. Dr. Pell's other mathematical papers were published in *Transactions of the American Mathematical Society*, 1900, in *Annals of Mathematics*, 1900, 1918, and in *Bulletin of the American Mathematical Society*, 1901. He was professor of mathematics and astronomy at the University of South Dakota, 1897–1908 (dean of the engineering school, 1906–1908), and assistant and associate professor of mathematics at the Armour Institute from 1908 to 1913. During the second semester of the year 1910–1911 he was incapacitated by a serious illness and Mrs. Pell acted as substitute. In one semester of 1915–1916 he was instructor in mathematics at Northwestern University. He married, for the second time, at Göttingen in 1907, Miss Anna Johnson, now associate professor of mathematics at Bryn Mawr College. A sketch of Dr. Pell's life by Professor Charlotte A. Scott was published in *The College News*, Bryn Mawr, February 9, 1921. This contains personal details not indicated above.

JOHANN MARTIN KRAUSE, ordinary professor in the Technische Hochschule, Dresden, since 1888, died March 2, 1920. He was born June 29, 1851. His article, "Ueber Systeme von Differentialgleichungen denen vierfach periodische Functionen Genüge leisten," in the first volume of *Transactions of the American Mathematical Society*, will be recalled. He was also the author of many other

articles, and of the well-known books: *Die Transformation der hyperelliptischen Functionen erster Ordnung nebst Anwendungen,* 1886; *Theorie der doppelt-periodischen Functionen einer veränderlichen Grösse,* 2 volumes, 1895–1897; and *Theorie der elliptischen Funktionen,* unter Mitwirkung von E. Naetsch, 1912.

Professors DUNHAM JACKSON, EDWARD KASNER, D. N. LEHMER, T. LEVI-CIVITA and H. L. RIETZ have been appointed associate editors of the *Bulletin of the American Mathematical Society* for 1921.

We welcome the first number of *Mathematics Teacher,* January, 1921, issued as the official organ of the National Council of Teachers of Mathematics, and devoted to the interests of mathematics in junior and senior high schools of the United States (compare *1920,* 474). It is edited by J. R. CLARK (editor-in-chief), E. R. SMITH (associate editor), J. A. FOBERG, (business manager), ALFRED DAVIS, H. D. GAYLORD, MARIE GUGLE, and J. W. YOUNG, with the cooperation of an advisory board consisting of W. H. METZLER (chairman), WILLIAM BETZ, W. E. BRECKENRIDGE, E. R. BRESLICH, J. C. BROWN, W. C. EELLS, G. W. EVANS, H. F. HART, W. W. HART, E. R. HEDRICK, H. M. KEAL, THEODORE LINDQUIST, W. A. LUBY, G. W. MYERS, J. H. MINNICK, W. D. REEVE, RALEIGH SCHORLING, H. E. SLAUGHT, D. E. SMITH, C. B. WALSH, and H. E. WEBB.

Professor FLORIAN CAJORI, professor of the history of mathematics at the University of California, addressed the Society of Sigma XI at Northwestern University, on December 13, 1920, on "Switzerland, the mother of American geodesy."

Dr. LUDWIK SILBERSTEIN, of the Eastman Kodak Company (*1920,* 474), delivered a series of six lectures at Cornell University, November 10 to Demceber 15, on the theory of relativity. An introductory lecture, on the experimental foundation for the theory, was given by Professors F. K. Richtmyer and E. R. Kennard on November 3.

We have referred elsewhere to addresses by Professors CAJORI, KARPINSKI, KELLOGG, and SMITH in joint sessions of the Mathematical Association of America, the American Mathematical Society and section "L" of the American Association for the Advancement of Science. The following papers in the history of science were also given on December 29–30, 1920, in separate sessions of Section L: "State of research in Egyptian science" by J. H. BREASTED, University of Chicago; "Early surveying and astronomical instruments in America" (illustrated) by Professor CAJORI, University of California; "Proposed periods in the history of astronomy in America" by W. C. RUFUS, University of Michigan; and "The history of physics" by Dr. H. A. BUMSTEAD, Yale University. (Dr. Bumstead died December 31, 1920).

In connection with the meeting of the American Mathematical Society in New York last December, the most notable event was the retirement of Professor F. N. COLE from his positions as secretary of the Society and as editor-in-chief

of the *Bulletin*. He served as secretary for twenty-five years, and as editor-in-chief for twenty-three years, with unfailing fidelity, tact, idealism, dignity and power. But he was much more than secretary and editor; the work which he carried on almost alone for years has now been distributed among four new officers of the Society.

The high position which American mathematics occupies in the world is due in no small measure to his quiet devotion, and sacrifice of leisure which, no doubt, he often coveted for continuing research investigations. Few of the younger mathematicians know of this work. His first paper, "The potential of a shell bounded by confocal ellipsoidal surfaces," was published in *Proceedings of the American Academy of Arts and Sciences*, in 1883. "A contribution to the theory of the general equation of the sixth degree" (his Harvard doctor's thesis), and a sixteen-page article, "Klein's Ikosaeder," appeared in *American Journal of Mathematics*, during the period 1885–87 that he was lecturer in mathematics at Harvard. This was just after his return from Germany "aglow with enthusiasm which Felix Klein inspired in his students," when he initiated at Harvard a new era[1] in graduate instruction in mathematics. While instructor and assistant professor at the University of Michigan, 1888–95, he published as follows: his English translation of Netto's *The Theory of Substitutions and its Applications to Algebra* (Ann Arbor, 1892); "The diurnal variation of barometric pressure," *Bulletin, U. S. Weather Bureau*, 1892; "On a simple group," *Mathematical Papers read at the International Mathematical Congress . . . 1893*, (New York, 1896); "The linear functions of a complex variable," *Annals of Mathematics*, 1890, 56 pages; "On rotations in space of four dimensions," *American Journal af Mathematics*, 1890; and a number of articles, one in collaboration with J. W. Glover, dealing with questions in the theory of groups. These included three in *American Journal of Mathematics*, two in *Bulletin of the New York Mathematical Society*, and two in *Quarterly Journal of Mathematics*.

He commenced to lay the foundation of the great monument of his life work shortly after his appointment in 1895 as professor of mathematics at Columbia University. His papers since that time have been: "On the factoring of large numbers" (where the often quoted factors of Mersenne's number, M_{67}, are given) in *Bulletin of the American Mathematical Society*, 1903; "The groups of order p^3q^β" in *Transactions of the American Mathematical Society*, 1904; and the memoir in collaboration with Professors White and Cummings, to which we have already referred (*1919*, 304), in *Memoirs of the National Academy of Sciences*, 1919. Some personal details regarding Professor Cole's life may be found in *National Cyclopædia of American Biography*, volume 13, 1906.

At the close of its sessions the Society testified to its appreciation of their retiring secretary and of his services, by the presentation of an address and a purse containing several hundred dollars.

Another event of the sessions, report on which will be of general interest, was the acceptance by the Council of the Society of the Bôcher Memorial Fund to

[1] For details the reader may, elsewhere in this MONTHLY (*1919*, 263), consult the report of a paper by Professor Osgood.

be held in trust, the income to be employed for the advancement of mathematical science. The council appointed a committee consisting of Professors W. F. OSGOOD (chairman), T. S. FISKE, C. N. HASKINS, O. D. KELLOGG, and E. B. VANVLECK to consider the most appropriate use to which the income of the Fund could be devoted.

The following four American doctorates in mathematics should be added to the lists already published (*1920*, 440 and *1921*, 44), making a total of twenty-two conferred in 1919–20: H. J. ETTLINGER, Harvard: "I. Existence theorems for the general real self-adjoint linear system of the second order. II. Oscillation theorems for the real self-adjoint linear system of the second order"; G. M. ROBISON, Cornell: "Divergent double sequences and series"; BIRD M. TURNER, Bryn Mawr: "Plane cubics with a given quadrangle of inflexions"; W. L. G. WILLIAMS, Chicago: "Fundamental systems of formal modular seminvariants of the binary cubic."

The *Bulletin of the American Mathematical Society* has published the following note (compare *1920*, 340): "Professor J. H. TANNER, of Cornell University, and Mrs. TANNER have given to the trustees of that institution fifty thousand dollars to establish a mathematical institute under the following stipulations. The money is to be allowed to accumulate without diversion for seventy-five years. At the end of that time one professor shall be appointed, whose duty it shall be to begin the formulation of plans for the proposed institute. At the end of each of the four succeeding periods of five years one or more additional professorships shall be established, the incumbents to collaborate in the same plans. The stipends of these professors shall be paid from the fund, but no other demands shall be made upon it until one hundred years from the date of the deed of gift (June, 1920), from which time the income of the entire sum shall be devoted to the maintenance of the institute; half of the expenditure of each year is to be applied to research in the mathematical sciences."

The National Research Council requested Professor H. L. RIETZ, of the University of Iowa, to call together a small group for a preliminary conference to discuss the desirability of a committee of the Council in the field of applications of mathematics to statistics, and to make appropriate recommendations. The group consisted of Professor J. W. GLOVER, Professor E. V. HUNTINGTON, Professor RAYMOND PEARL, and Professor W. M. PERSONS and the conference was held in Washington, January 21–22.

The Council of the American Association for the Advancement of Science chose Professor E. H. MOORE, of the University of Chicago, as president for the coming year. He will preside at Toronto in 1921 and deliver his address at Boston in 1922. *Science* refers to this "acknowledged leader of American mathematics": then adds: "It is now many years since that science which is fundamental to all others has supplied a president to the association, and it is fortunate that a representative could be selected with the unanimous approval of

all mathematicians." Professor W. D. CAIRNS and Professor G. A. MILLER were appointed members of the Council of the Association, and Executive Committee, respectively. In Section A, Professor OSWALD VEBLEN was elected vice-president and Professor W. H. ROEVER secretary for 1921.

We have already announced (*1921*, 44) the award by the Royal Society to Professor G. H. HARDY, of Oxford University, of a Royal Medal. Sir J. J. Thomson's statement in making the award is as follows (*Proceedings of the Royal Society*, series A, January 3, 1921):

"Prof. Hardy is well known both in this country and on the Continent for his researches in pure mathematics, particularly in the analytic theory of numbers and allied subjects. Immediately after taking his degree at Cambridge he engaged in a series of researches on the theory of functions of a real variable, from which results of the greatest importance and generality were obtained, at first by himself alone and later in collaboration with Mr. J. E. Littlewood. Perhaps his most remarkable work is his more recent work on Partitions, carried out in collaboration with the late Mr. S. Ramanujan, especially a paper on "Asymptotic Formulæ in Combinatory Analysis" (*Proc. Lond. Math. Soc.*), in which a system of approximations are obtained for $p(n)$, the number of unrestricted partitions of n and associated functions. The ingenuity of the method and the accuracy of the results are equally remarkable. For p (200) the authors' approximate formula gives the value 3,972,999,029,388.004, from which the true value differs by 0.004, an error of 1 in 10^{15}.

Among the more important researches of which Prof. Hardy is sole author may be mentioned papers on Dirichlet's Divisor Problem, on the representation of numbers as the sum of n squares, on the roots of the Riemann ζ-function, and on non-differentiable functions."

A similar award was made in 1919 to J. H. JEANS (*1920*, 46). Two Royal Medals are awarded each year. They were "instituted by George IV., and are awarded annually for the two most important contributions to science published in the British dominions not more than ten years nor less than one year from the date of the award."

There is a special call for copies of the MONTHLY for May, 1915. By an error in printing, the cover for that month read April, 1915, although the inside front page was marked correctly. Slips to be pasted over the incorrect date were sent out the next month, but a number of subscribers did not attend to this matter. As a result many libraries lack this one issue to complete their files. If anyone is willing to part with copies of the MONTHLY for May, 1915, such copies will be paid for by the Secretary of the Association at the rate of one dollar per copy, and will then be available for those desiring to complete their files. The same price will be paid for a limited number of copies for June and November, 1895, in volume 2.

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THE JANUARY MEETING OF THE KANSAS SECTION.

The seventh regular meeting of the Kansas Section was held at the Central High School, Topeka, Kansas, on January 22, in conjunction with a meeting of the Kansas Association of Mathematics Teachers. Two sessions were held, the first of which was a joint session with the Kansas Association of Mathematics Teachers. Professor Shirk presided at the morning session and Professor Garrett at the afternoon session.

The attendance was fifty-two, including the following twenty-one members of the Association:

C. H. Ashton, Florence Black, R. H. Carpenter, Lucy Dougherty, W. H. Garrett, W. A. Harshbarger, T. B. Henry, Emma Hyde, T. Lindquist, Anna Marm, U. G. Mitchell, H. S. Myers, P. S. Pretz (institutional representative), B. L. Remick, D. H. Richert, J. A. G. Shirk, G. W. Smith, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, Professor Shirk; Vice-Chairman, Miss Hyde; Secretary, Professor Stouffer. It was voted to hold the next meeting at Topeka in January, 1922, in conjunction with the meeting of the Kansas Association of Mathematics Teachers.

The following eight papers were read:

(1) "Report of the Kansas committee coöperating with the National Committee on Mathematical Requirements" by Professor U. G. Mitchell, University of Kansas;

(2) "Report on the preparation of college freshmen in entrance algebra" by Professor W. H. Garrett, Baker University;

(3) Question box:

(*a*) "What high school teachers want to know about college mathematics" by Professor C. H. Ashton, University of Kansas;

(*b*) "What college teachers want to know about high school mathematics" by Miss Lucy T. Dougherty, High School, Kansas City;

(4) "Mathematics and statistics" by Professor W. T. Stratton, Kansas State Agricultural College;

(5) "Hyperbolic functions" by Professor H. S. Myers, Southwestern College;

(6) "A problem in calculus" by Professor A. E. White, Kansas State Agricultural College;

(7) "Division of credit between college algebra, trigonometry, analytics and calculus" by Professor W. A. Harshbarger, Washburn College.

Most of the papers led to considerable discussion. Abstracts of the papers and discussions follow below, the numbers corresponding to numbers in list of titles.

1. The Kansas committee coöperating with the National Committee on Mathematical Requirements was first appointed in January, 1920, by the Kansas

Association of Mathematics Teachers at the request of the National Committee. Later the same committee was authorized to act for the Kansas Section of the Mathematical Association of America. The committee consists of Emma Hyde, Kansas State Agricultural College, Lucy T. Dougherty, Kansas City, Kansas, High School, Theodore Lindquist, Kansas State Normal School, and U. G. Mitchell, University of Kansas, Chairman.

As an aid in making clear his report Professor Mitchell brought mimeographed summaries of the work of the National Committee and of the Kansas committee and had them distributed. He reviewed briefly some of the reasons leading to the appointment of the National Committee, the scope of the Committee's work and the principles formulated for its guidance. The reports already released by the National Committee were discussed and a list of those to appear later given. The Kansas committee was reported to have held three meetings and planned four reports to be sent to the National Committee. One of these reports had been completed and sent in, two others had been discussed and formulated but not yet put into final form and the fourth had been discussed but not yet formulated.

2. Professor Garrett presented further results of an investigation in which he has been engaged since 1912. The data secured from over five hundred papers written in September, 1920, by freshmen students in Baker University, Kansas State Agricultural College, University of Kansas and Washburn College, was presented in graphic form and compared with the results secured previously. The measure used was a set of fifteen problems in elementary algebra, first used by Professor Garrett in 1912. The results show a slight improvement over 1912 in the number of correct solutions to three of the problems, no improvement in three others, and much poorer results in the remaining nine problems. The problems included simple varieties of factoring, reduction of fractions, solution of linear and quadratic equations, simplification of radicals and fractions containing irrational denominators and evaluation of quantities containing zero, negative and fractional exponents. Of the fifteen hundred papers from Kansas students which have been examined, there has been just one perfect paper.

3. Those present were given opportunity to write out the questions they desired to have answered on either of the subjects. The questions concerning college mathematics were handed to Professor Ashton who answered a number of them. He especially emphasized the fact that the student who does not take all the regular high school courses in mathematics is handicapped if he enters an engineering school. Professor Ashton called on Professor Mitchell and Professor Stouffer to answer one question each.

Miss Dougherty in answering the questions concerning high school mathematics brought out the fact that the drift away from mathematics in high school is not as rapid as sometimes thought.

4. Professor Stratton traced briefly the historical development of the idea of statistics from the beginning of organized nations down to the present time, pointing out a number of mathematical topics that are employed in the scientific

treatment of the subject, and some of the famous mathematicians who have contributed directly or indirectly to the development of its mathematical treatment. He showed by means of tables and graphs the methods usually employed in handling correlated data, such as the length and weight of ears of corn. He also developed mathematically two of the most commonly used units of correlation; viz., the correlation ratio and the coefficient of correlation as defined by Karl Pearson.

5. Professor Myers defined hyperbolic functions both analytically and geometrically. He derived the identities between hyperbolic functions, showed their period and obtained the formulæ for their integration. Some of the numerous uses for hyperbolic functions were pointed out.

6. The problem may be stated as follows: A large pulley of radius b turns by means of a belt a smaller pulley of radius a. At a certain instant a radius of the larger pulley makes an angle with a radius of the smaller pulley. As the pulleys turn, in what position will the outer ends of the spoke be closest together? Professor White showed that the distance may be a maximum or a minimum when the radii are parallel or when the radii form an isosceles triangle with the line of centers.

7. Professor Harshbarger discussed briefly the growth of the present courses with the division of time brought about by changing high school curricula. The plan of college algebra, three hours, trigonometry, two hours, analytics, five hours, and calculus, five hours, was compared to two other plans. The first suggested college algebra, five hours, trigonometry, two hours, and analytics, three hours, for the freshman year, with five hours of calculus for the first half of the sophomore year. The second plan suggested algebra, three hours, and trigonometry, two hours, for the first semester with analytics, four hours, and calculus, six hours, during the next two semesters.

Discussion on this subject was participated in by Professor Ashton, Professor Mitchell, Professor Remick, Professor Stouffer, Professor Stratton, and Professor White.

E. B. STOUFFER, *Secretary-Treasurer.*

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The eighth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Md., on Dec. 11, 1920. The meeting consisted of two sessions with Professor L. S. Hulburt presiding.

The attendance was forty-seven, including the following thirty-four members of the Association: O. S. Adams, J. J. Arnaud, R. N. Ashmun, Clara L. Bacon, Sarah Beall, A. A. Bennett, G. A. Bingley, C. C. Bramble, J. A. Bullard, P. Capron, G. R. Clements, A. Cohen, G. H. Cresse, F. W. Darling, L. S. Dederick,

A. Dillingham, H. English, J. B. Eppes, A. Hall, W. M. Hamilton, W. E. Heal, P. E. Hemke, L. S. Hulburt, L. S. Johnston, W. D. Lambert, L. N. Morscher, F. D. Murnaghan, J. R. Musselman, C. H. Rawlins, Jr., H. M. Robert, Jr., R. E. Root, J. B. Scarborough, W. F. Shenton, C. A. Shook.

A generous luncheon was served to the members and their guests by the mathematical staff of the Naval Academy. A preliminary report of the committee appointed to cooperate with the National Committee on Mathematical Requirements was presented by Professor A. Cohen. The next meeting will be held in Washington, probably early in May.

The following papers were presented:

(1) "Some mechanical curiosities connected with the earth's field of force" by Mr. W. D. Lambert, U. S. Coast and Geodetic Survey.

(2) "Parallel components in statical equilibrium" by Professor G. R. Clements, U. S. Naval Academy.

(3) "Singular curves of a plane pencil" by Professor C. C. Bramble, U. S. Naval Academy.

(4) "Remarks on a problem in geometry" by Dr. F. D. Murnaghan, Johns Hopkins University.

(5) "Some arithmetic operations with transfinite ordinals" by Professor A. A. Bennett, Technical Staff, Army Ordnance.

(6) "Multiple improper integrals" by Mr. C. A. Shook, U. S. Naval Academy.

(7) "On a certain statistical problem" by Dr. J. R. Musselman, Johns Hopkins University.

Abstracts of the papers numbered in accordance with the above list of titles are given below.

1. By the earth's field of force, in Mr. Lambert's paper, was meant the field due to mass-attraction according to the Newtonian law combined with the centrifugal force of the earth's rotation. These forces are adequate in problems of equilibrium relative to the earth. As approximations to the form of the equipotential or level surfaces of the earth's field we frequently take parallel planes or concentric spheres. The limiting level surface of the earth's field is, however, not even spheroidal, but has an edge around the equator. The level surfaces lying near the earth are ellipsoids, the ellipticity of which increases with the major axis. The fact that they are not concentric spheres gives rise to what seem like paradoxes. Some of these paradoxes were examined and a geological application suggested for one of them. The peculiarities of the earth's field may be examined experimentally by the Eötvös torsion balance. The paper gave a brief description of the instrument and some account of the work done with it in determining the curvature of the geoid and adjacent level surfaces.

2. Professor Clements suggested a more explicit formulation of the conditions under which one system of parallel forces may be replaced by another which shall be called equivalent to it, than is found in many texts on elementary mechanics. By examples he pointed out that in computing the reactions at the pins of a

framed structure, considerable simplification can be had by replacing the load (either concentrated or distributed) on each member by an equivalent system of parallel forces acting at its points of connection with the structure. In particular, any bar having only two such points of connection becomes, for the purpose of computing the reactions at the pins, a 'two force piece' and may be 'cut' in the usual manner of the method of sections.

3. The number of curves of a plane pencil with a double point is $3(n-1)^2$ provided that the base points of the pencil are ordinary points. If all of the curves have an r-fold point at a base point, Professor Bramble showed that the number of curves which have a further double point is $(r-1)(3r+1)$ less than $3(n-1)^2$. This is shown by considering the behavior of the curves $F_1\phi_3 - F_3\phi_1 = 0$ and $F_2\phi_3 - F_3\phi_2 = 0$ at the r-fold point when $F_i = \partial F/\partial x_i$, $\phi_i = \partial\phi/\partial x_i$, F and ϕ being the base curves of the pencil.

4. Dr. Murnaghan discussed Darboux's problem of the cyclic quadrilateral (cf. *Nouvelles Annales de Mathêmatiques,* 1868, p. 138.) The quartic equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ with real or complex coefficients has four roots in the complex plane which are either cyclic or anticyclic (i.e., two lie on an Apollonian circle of the other two) provided that $A \equiv g_2^3/g_3^2$ is *real*; g_2 and g_3 being the two invariants of the quartic. When the points are cyclic the double-ratio of the four points is real and $(g_2^3/g_3^2) \geqq 27$, the equality holding only when two of the points coincide. In the anticyclic case the double ratio has modulus unity and $(g_2^3/g_3^2) \leqq 27$. When the coefficients are real, the anticyclic case is that when two roots are real and the others conjugate imaginaries; the cyclic case is that when all roots are real or all imaginary. If three of the roots are regarded as fixed and the fourth variable, parts of the geometry of the triangle may be interestingly discussed (thus the Apollonian circle is the locus of points whose pedal triangle is isosceles, etc.); $A = \bar{A}$ is the equation of the circumcircle and the three Apollonian circles. The center of the circumcircle is tied up with the other three centres.

5. In this paper, Professor Bennett noted that as a consequence of the uniqueness of left hand subtraction among Cantor ordinals, the highest common left-hand divisor of two ordinals may always be found by Euclid's algorithm. The system of ordinals is then extended by right-hand subtraction so that among the formal pairs thus resulting, addition, subtraction and multiplication are always uniquely possible.

6. In his paper Mr. Shook first defined uniform convergence of improper definite integrals. He then discussed a double integral, in which the upper limits are infinite and the integrand does not remain finite, giving a set of sufficient conditions for changing the order of integration. Finally he gave an application to a certain double integral which arises in the theory of beta and gamma functions.

7. An urn contains n balls numbered successively from 1 to n; the balls are drawn out one by one and the order of their appearance noted. Dr. Musselman derived a formula for the average error resulting from the drawings varying from

some given assigned arrangement. This result can be used for determining statistically whether stock judging is more than a matter of chance.

O. S. ADAMS, *Secretary-Treasurer.*

ON CERTAIN PROPERTIES OF MAKEHAM'S LAWS OF MORTALITY WITH APPLICATIONS.[1]

By H. L. RIETZ, University of Iowa.

1. Introduction. Early in the development of the mathematics of life insurance, the calculations became laborious. This situation led to efforts to discover laws of mortality from which monetary values for life insurance could be calculated. As early as 1725, De Moivre in his *Annuities upon Lives,* put forth his famous hypothesis that the decrements d_x of the number living l_x at age x in a mortality table is constant for all values of x. Very little data were available to verify this hypothesis, but even the data of that time should have made it appear that this assumption could at most be a rough approximation. The De Moivre hypothesis is, of course, equivalent to a statement that the number living l_x at age x can be given by a linear function of x.

Just about one hundred years later in the *Philosophical Transactions* of 1825,[2] Gompertz published the so called law of mortality which bears his name. Before discussing the hypothesis of Gompertz, it is desirable to explain the meaning of the expression "force of mortality." This expression is in common use by actuaries at the present time in connection with the doctrines of Gompertz and in other connections. It should perhaps be stated that Gompertz did not use the expression "force of mortality" in his papers, but used instead the expression "intensity of mortality" which has no technical meaning at present.

The "force of mortality" μ_x at any age x may be defined as the rate of change of the average number living l_x of an indefinitely large class of persons of age in the neighborhood of x, per individual of the group l_x. That is to say, if the function l_x has a derivative with regard to x,

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx},$$

the negative sign being used so that μ_x is positive.

In other words, we conceive a continuous function l_x which is proportional to the average number living at age x, out of an indefinitely large group of lives of age in the neighborhood of x. Then the force of mortality is defined as the derivative of this function with regard to age or time divided by l_x. Obviously

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[1] Read under slightly different title before the Mathematical Association of America, September 6, 1920.

[2] "On the nature of the function expressive of the law of human mortality and a new mode of determining the value of life contingencies," pp. 513–583.

l_x is a decreasing function. The negative sign is placed before the derivative dl_x/dx so that μ_x is positive. Gompertz gave a rather wordy discussion of his hypothesis showing its reasonableness.

Briefly, his hypothesis may be stated by saying that the rate of change of the force of mortality at any time is proportional to the force of mortality. That is,

$$\frac{d\mu_x}{dx} = c'\mu_x,$$

or

$$\mu_x = Bc^x, \tag{1}$$

where B and c are parameters. Then from (1) and the definition of force of mortality, we have

$$-\frac{1}{l_x}\frac{dl_x}{dx} = Bc^x. \tag{2}$$

Integrating, we have in the usual form of Gompertz's law,

$$l_x = kg^{c^x}, \tag{3}$$

where k, g, and c are the parameters.

It is of some interest to know the view of Gompertz in regard to the real character of his derivation after he gave such a lengthy discussion of the reasonableness of his hypotheses. He remarked that "this equation between the number living and the age becomes deserving of attention not in consequence of its hypothetical deduction, which in fact is congruous with many natural effects, as for instance the exhaustion of the receiver of an air pump by strokes repeated at equal intervals of time, but it is deserving of attention because it appears corroborated during a long portion of life by experience." In the paper of 1825, Gompertz applied his function to portions of the Northampton Tables, Carlisle Tables, Desparcieux Tables, and certain Swedish Tables.

In the *Philosophical Magazine* for 1839,[1] De Morgan showed that if the law of Gompertz applies to the whole of life, then Simpson's Rule, which has often been used as an approximation, to simplify joint life annuity calculations, would be rigidly correct.

In 1860, Makeham suggested[2] his first modification of the law of Gompertz and in 1865 published a valuable paper setting forth the development of his suggestion. The hypothesis back of Makeham's first modification is that the force of mortality is given by

$$\mu_x = A + Bc^x. \tag{4}$$

That is, the force of mortality consists of two parts—the one part increases in geometrical progression with age and the other is a constant. The addition of the constant seems reasonable when we recall that for certain diseases and accidents, the tendency to death seems practically independent of age.

[1] "On the rule for finding the value of an annuity on three lives," vol. 15, p. 337.

[2] *Journal of the Institute of Actuaries*, vol. 8, 1860, pp. 301–310; vol. 13, 1867, pp. 325–358.

Then from (4) and the definition of force of mortality, we have

$$-\frac{1}{l_x}\frac{dl_x}{dx} = A + Bc^x, \tag{5}$$

and by integration we have for Makeham's first modification

$$l_x = ks^x g^{c^x}, \tag{6}$$

where k, s, g, and c are the parameters.

The determination of these parameters to fit a given table of mortality under some criterion such as that of least squares is a problem of interest in attempts at accuracy and economy of time. This problem has received considerable attention in recent literature. It is perhaps sufficient for our purpose to state that the main methods used are:

(*a*) Method of averaging results obtained from using four values of l_x to correspond to four assigned values of x.

(*b*) Method of moments.

(*c*) Method of least squares.

It may be of interest to write down the values of the parameters for some well-known table. For the American Experience Table, Arthur Hunter found the following:

$$\log_{10} g = -\ .00013205$$

$$\log_{10} c = \quad .04579609$$

$$\log_{10} s = -\ .003296862$$

$$\log_{10} k = \quad 5.03370116$$

With respect to fitting tables well with the Makeham function $l_x = ks^x g^{c^x}$, there are many examples of tables that have been fitted well enough for practical purposes. As a striking example of closeness of fit, I wish to refer to a graduation of a rural life population table[1] for the registration area of the United States by C. H. Forsyth and the writer. The resulting curve crossed the curve of the given data eleven times in the range from age 19 to 85 which we treated.

2. The advantages of a Makehamized table for joint life and survivorship insurances. It is the main purpose of this paper to exhibit certain features of the application of Makeham's function in the treatment of joint life and survivorship insurances. It is fundamental in joint life and in survivorship insurances and annuities that we be able to find in the simplest way possible the value of joint life annuities to two or more persons of any assigned ages that may arise in the applications. We wish to show how it becomes possible with a Makehamized table to replace a cumbersome and laborious method of finding joint life annuities by a simple and elegant plan.

[1] *Record of the American Institute of Actuaries*, June, 1911, p. 19.

The laborious character of joint life calculations, when a table is not Makehamized, should perhaps be indicated first. This can be shown with sufficient clearness by an illustration with only two lives of ages x and y. The joint life annuity for (x)[1] and (y) is given by

$$a_{xy} = \frac{vl_{x+1}l_{y+1} + v^2 l_{x+2}l_{y+2} + v^3 l_{x+3}l_{y+3} + \cdots}{l_x l_y}, \tag{7}$$

where a symbol l with a subscript means the number living in the mortality table at the age indicated by the subscript, and v is the discount factor. That is,

$$v = \frac{1}{1+i},$$

where i is the rate of interest.

The formula (7) can be extended in an obvious manner to include three or more lives. For only two lives, in De Morgan's forms, we introduce commutation symbols as follows:

$$\begin{aligned} D_{xy} &= l_x l_y v^{(x+y)/2}, \\ C_{xy} &= (l_x l_y - l_{x+1}l_{y+1})v^{(x+y+2)/2}, \\ &= vD_{xy} - D_{x+1:y+1}. \\ N_{xy} &= D_{xy} + D_{x+1:y+1} + D_{x+2:y+2} + \cdots, \\ M_{xy} &= C_{xy} + C_{x+1:y+1} + C_{x+2:y+2} + \cdots, \\ S_{xy} &= N_{xy} + N_{x+1:y+1} + N_{x+2:y+2} + \cdots, \\ R_{xy} &= M_{xy} + M_{x+1:y+1} + M_{x+2:y+2} + \cdots. \end{aligned}$$

To exhibit the tabular values for any one of this set of symbols, say for D_{xy}, for all combinations of ages from 10 to 95 inclusive would require a double entry table with $(86)^2$ tabular entries if the rates of mortality of the lives of (x) and (y) are given by different mortality tables, as in the case of a husband and wife when we use McClintock's Annuitants' Tables[2] (Male and female). If the rates of mortality of x and y are given by the same mortality table, the $(86)^2$ values would involve duplicates since $D_{xy} = D_{yx}$, and the number of necessary tabular entries would be ${}_{87}C_2$.

For three lives of (x), (y), and (z) with rates of mortality from different mortality tables, there would be required $(86)^3$ tabular entries to give D_{xyz}. If the mortality rates for (x), (y), and (z) were given by a single mortality table, the number of necessary tabular entries would be ${}_{88}C_3$.

It becomes clear, without carrying the illustration to a larger number of lives, that, unless a very extensive joint life business were to be done, it would be impractical to prepare such tables.

[1] A symbol (x) is an abbreviation for "a person aged x."

[2] M. M. Dawson, *Practical Lessons in Actuarial Science*, New York, vol. 2, 1905, pp. 315–325.

We should ordinarily find it more practical to treat each separate case by a method of approximation which consists in finding the approximate sum of the series of terms in (7). The method of approximation very often adopted consists in applying formulas derived from the Euler-Maclaurin formula of finite differences. Even with these methods of approximation, this plan is generally somewhat laborious.

3. Lives of unequal ages can be replaced by lives of equal ages. We shall now attempt to give a brief exposition of the method that can be substituted for the rather laborious method of finding joint life annuities, when the table of mortality is a Makehamized table.

In the usual notation, let

$$_tp_{x_1x_2\cdots x_m}$$

be the probability that each of m persons (x_1), (x_2), (x_m) will live t years. Under Makeham's law, $l_x = ks^xg^{c^x}$ for all values of x in a certain interval. Then we have

$$\begin{aligned} _tp_{x_1} &= s^tg^{c^{x_1}(c^t-1)}, \\ _tp_{x_2} &= s^tg^{c^{x_2}(c^t-1)}, \\ &\vdots \\ _tp_{x_m} &= s^tg^{c^{x_m}(c^t-1)}. \end{aligned}$$

Then the value of the joint life annuity to (x_1), (x_2), $\cdots$, (x_m) is

$$a_{x_1x_2\cdots x_m} = vp_{x_1x_2\cdots x_m} + v^2\,{}_2p_{x_1x_2\cdots x_m} + v^3\,{}_3p_{x_1x_2\cdots x_m} + \cdots + v^t\,{}_tp_{x_1x_2\cdots x_m} + \cdots, \tag{8}$$

$$= \sum_{t=1}^{t=\omega} v^t_tp_{x_1x_2\cdots x_m}\cdots, \tag{9}$$

$$= \sum_{t=1}^{t=\omega} s^{mt}v^tg^{(c^{x_1}+c^{x_2}+\cdots+c^{x_m})(c^t-1)}, \tag{10}$$

where ω is the highest age in the mortality table.

Let ξ be the equal age of m persons that may, for the purpose of finding the value of the annuity replace ages x_1, x_2, $\cdots$, x_m.

Then $_tp_\xi = s^tg^{c^\xi(c^t-1)}$ and the probability that m persons each of age ξ survive t years is

$$(_tp_\xi)^m = s^{mt}g^{mc^\xi(c^t-1)}. \tag{11}$$

If we determine ξ so that

$$mc^\xi = c^{x_1} + c^{x_2} + \cdots + c^{x_m}, \tag{12}$$

we may clearly write in place of (10),

$$a_{x_1x_2\cdots x_m} = \sum_{t=1}^{t=\omega} v^t(_tp_\xi)^m = s^{mt}g^{mc^\xi(c^t-1)}. \tag{13}$$

The solution of (12) for ξ is generally obtained by the use of a table of forces of mortality. This plan is easily shown as follows:

From $l_x = ks^x g^{c^x}$, we have

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\log s - c^x \log g \log c, \tag{14}$$

From (12) and (14), we have that

$$m\mu_\xi = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m}. \tag{15}$$

Then the value of ξ that satisfies (15) also satisfies (12). As tables of forces of mortality usually accompany a Makehamized mortality table, we solve for ξ by interpolation from such tables. With m equal lives substituted for the m unequal lives, there would be required for tabular entries of a commutation symbol, say for a $D_{\xi\xi}\cdots$, to cover all ages from some age such as 10 to an age such as 95 inclusive, simply 86 values instead of the impractically large number of values mentioned above.

4. Law of uniform seniority. For simplicity, let us consider first two lives (x_1) and (x_2). The equivalent equal age ξ is given by

$$2c^\xi = c^{x_1} + c^{x_2}. \tag{16}$$

If (x_1) is the younger of the lives, then let $x_2 = x_1 + a$, and we have

$$2c^\xi = c^{x_1}(1 + c^a). \tag{17}$$

From (17) we have

$$\xi = x_1 + k,$$

where k depends only upon the difference of the ages a and not on the actual ages x_1 and x_2. Hence, for a given difference between ages, we add the same number to one age to get the equivalent equal ages no matter what the ages x_1 and x_2 are,

To look at this in the general case of m lives we may note that if

$$mc^\xi = c^{x_1} + c^{x_2} + \cdots + c^{x_m}, \tag{18}$$

then

$$mc^{\xi+t} = c^{x_1+t} + c^{x_2+t} + \cdots + c^{x_m+t} \tag{19}$$

for t years later, where t is any number. Thus, if we have obtained the equivalent equal age ξ at any date, then $\xi + t$ is the equivalent equal age for a date t years later. That is, the addition of a given number of years to each of the ages $x_1, x_2, \cdots, x_m$ will add the same number to ξ. This property of Gompertz's and Makeham's functions is known as uniform seniority. The applications of Makeham's formula are much simplified because of this property.

5. A single life may replace m lives by an appropriate change of the interest rate. For certain purposes, it is desirable to work with a single life ξ for which the cost of an annuity would be the same as for joint lives $(x_1), (x_2), \cdots, (x_m)$. When Gompertz's law holds, the replacement by a single life is possible without involving a change in the interest rate. But with Makeham's law, we proceed as follows.

We confine our attention to two lives of (x) and (y) since this appears simpler than m lives, although the plan can at once be extended to any number of lives.

Recalling that

$$ {}_tp_x = s^t g^{c^x(c^t-1)}, \tag{20} $$

and

$$ {}_tp_y = s^t g^{c^y(c^t-1)}, \tag{21} $$

let ξ be the single life to replace (x) and (y) in joint life probabilities. Then

$$ {}_tp_\xi = s^t g^{c^\xi(c^t-1)}. \tag{22} $$

From (20) and (21),

$$ {}_tp_{xy} = s^{2t} g^{(c^x+c^y)(c^t-1)} = s^t\,{}_tp_\xi, \tag{23} $$

where ξ is given by the equation

$$ c^\xi = c^x + c^y. \tag{24} $$

But the value of a life annuity is given by

$$ a_{xy} = vp_{xy} + v^2\,{}_2p_{xy} + v^3\,{}_3p_{xy} + \cdots + v^t\,{}_tp_{xy} + \cdots \tag{25} $$

Let $sv = v'$ be the modified discount factor. This indicates the change in interest rate to which we refer above. Then from (23),

$$ v^t\,{}_tp_{xy} = v'^t\,{}_tp_\xi. \tag{26} $$

Hence

$$ a_{xy} = v'p_\xi + v'^2\,{}_2p_\xi + v'^3\,{}_3p_\xi + \cdots, \tag{27} $$

and we obtain the joint life annuities from the single age ξ.

For special cases, and unusual rates of interest, the actuary is likely to find the substitution of a single life more convenient than the substitution of lives of equal age in number equal to the given number of lives. However, when complete tables are available, the plan that does not require a change of interest rate is likely to be preferred.

6. Makeham's second modification of the law of Gompertz. A second modification of the law of Gompertz was developed by Makeham in 1890.[1]

It may be recalled that the first modification assumed that the force of mortality

$$ \mu_x = A + Bc^x, $$

where A is a constant.

It is merely another step in the direction of generality to assume that μ_x is the sum of the term Bc^x used in the hypothesis of Gompertz, and a linear function $A + Dx$. We thus start from the assumption

$$ \mu_x = A + Dx + Bc^x, \tag{28} $$

or

$$ -\frac{1}{l_x}\frac{dl_x}{dx} = A + Dx + Bc^x. $$

[1] *Journal of the Institute of Actuaries*, vol. 28, pp. 152–159, pp. 185–192, pp. 316–331.

Then

$$l_x = ke^{-Ax-(Dx^2/2)-B'c^x} = ks^x r^{x^2} g^{c^x}. \tag{29}$$

$$l_{x+t} = ks^{x+t} r^{(x+t)^2} g^{c^{x+t}},$$

$$_tp_x = s^t r^{2xt+t^2} g^{c^x(c^t-1)},$$

$$_tp_y = s^t r^{2yt+t^2} g^{c^y(c^t-1)},$$

$$_tp_{xy} = s^{2t} r^{2t(x+y)+2t^2} g^{(c^x+c^y)(c^t-1)}. \tag{30}$$

Let ξ be the age of equivalent equal lives, where ξ is defined by

$$2c^\xi = c^x + c^y,$$

in case such a replacement by equal ages is possible.
Then

$$_tp_{\xi\xi} = s^{2t} r^{4t\xi+2t^2} g^{2c^\xi(c^t-1)}. \tag{31}$$

In the calculation of a joint life annuity,

$$a_{xy} = \sum_{t=1}^{t=\omega} v^t {}_tp_{xy}. \tag{32}$$

To determine a discount factor v' such that

$$v^t {}_tp_{xy} = v'^t {}_tp_{\xi\xi}, \tag{33}$$

we simply substitute in (33) from (30) and (31). This gives after some simplification

$$vr^{2(x+y)} = v'r^{4\xi}$$

or

$$v' = vr^{2(x+y)-4\xi}. \tag{34}$$

That is to say, using a discount factor v' given by (34) instead of v, enables us to replace lives of unequal ages by the same number of lives of equal age. Moreover, the law of uniform seniority holds for the same reasons given in other cases.

This second modification has been used to graduate the new American-Canadian Mortality Table for Men, recently prepared at much labor and expense, under the auspices of the Actuarial Society of America and the American Institute of Actuaries.

It seems that further extensions of Makeham's functions expressive of human mortality could be made by using in place of $A + Dx$ in (28) a polynomial of higher degree, but there would soon come a practical limit to the usefulness of such extensions.

In conclusion, let me say that the main purpose in presenting this paper to the Mathematical Association is accomplished if I have succeeded in giving a general notion of those properties of Makeham's functions that make these functions of much practical value in the treatment of joint life and survivorship insurances and annuities.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

4. Monge and the American Colonies.

A mathematician hearing the name of Gaspard Monge (1746–1818) would naturally and properly connect it with the science of descriptive geometry to which he was the first noteworthy contributor. The early genius that he showed at Beaune, where he occupied the chair of physics at the age of sixteen, his career as naval minister in the first and second years of the republic, his intimate relations with Napoleon, his noteworthy work in Egypt, his part in founding the École Polytechnique, his continued rise to high estate, and his complete downfall in honors, in health, and in mentality are hardly remembered by the profession in which he attained his greatest renown.

A considerable number of his letters, now in my possession, tell more or less of this story, but there are two official documents in the collection, signed by him, which show some little interest in the new world. Each was written in January, 1793, at which time he was naval minister under the Revolution. The first is dated on the seventeenth and is as follows:

State of the employés appointed by order of the Minister plenipotentiary of the French Republic to serve in the United States of America.

Memorandum.

One assistant in the works.

2 master carpenters, constructors on ordinary appointment. . . .

Supplement to the appointment because of the distance and of the extraordinary service which they have to perform, at the rate of two thousand four hundred livres for each, making 4,800 livres.

By the decision of January 17, 1793, the Minister has approved of the annual payment, to the two master carpenters who have been chosen by the Ordonnateur (director) of Rochefort, of two thousand four hundred livres apiece, in addition to the ordinary salary which they receive when in port, and which will serve to provide subsistance for their families.

Pour extrait.
Monge.

The other document, dated on the twenty-ninth of the month, refers to the shipments to the French colonies, including the "Colonies-amérique." It is an official order, beginning "Le Minister of the Navy to the Commissioners of the National Treasury," and is signed by Monge in his ministerial capacity.

The two orders show Monge in quite a different rôle from the one in which he is ordinarily imagined, and each testifies to the relations of France and America in the general period of the great revolutions in the two continents.

5. Descartes's Appreciation of Huygens the Elder.

After Descartes had served in the wars, part of the time in the army of Maurice, Prince of Orange, and part of the time under Maximilian, Duke of Bavaria, he decided to devote his life to scholarly pursuits. While with the

troops on the Danube; in 1619, when he was only twenty-three, he was already working on his *Discours de la Mêthode,* in an appendix to which his *Gêomêtrie* appeared, and in 1621 he definitely abandoned his military career and started out to see the intellectual world. He visited North Germany, Pomerania, Holstein, and Holland, tried to content himself for a time in his old home in Brittany, and then gave up the attempt and tried Switzerland and Italy. After this he settled for a while in Paris, found it too distracting in spite of the scientific group of which he was a member, and finally, at the age of thirty-two, decided that Holland was the country best suited to the intellectual work that he contemplated. He was thirty-three when, in March 1629, he took up his residence in Amsterdam,[1] and a year later he felt it worth while to enroll as a student at Leyden, probably because his friend Reneri (Renier) had just received an appointment there in philosophy. In 1632 he is known to have been at work on his *Gêomêtrie,* and in April he wrote to Mersenne about it, stating that it was nearly finished although it needed some revision and that he was bothered with the figures.

It is at this period that he wrote to his friend "Monsieur de Willhelme, Counseiller de Mon.r le Prince d Orange," at the Hague, a letter which is now in my collection and which is, indeed, one of the rarest I have. The letter was written a month after the one to Mersenne, above mentioned, and just before going to Deventer where he expected to finish his *Dioptrique.*

The letter is as follows:

Monsieur,

I have received the contents of the bill of exchange which you were so good as to send to me, and I thank you for it. I should have kept it a little longer so as to attempt to remit it to you with some profit, but I do not doubt that it will profit more by being in your hands than it will by remaining in mine, and I am now on the point of leaving here. I do not know how to respond to the courtesy of Monsieur Huguens, except that I cherish the honor of his acquaintance as one of the greatest pieces of fortune that has come to me, and I shall never be where I have the opportunity of seeing him without seeking to do so. Just as I shall ever seek the occasion to prove this fact, I am,

Monsieur, Your very humble and very affectionate servant,

DESCARTES.

AMSTERDAM, May 23, 1632.

This Huygens was Constantin, of Zuylichem, the father of Christian Huygens the celebrated physicist. Constantin was the brother-in-law of Willhelme (to take Descartes's spelling), and Descartes had met him through Jacob Golius, professor of mathematics at Leyden, and had formed a warm friendship with him and his talented wife Constance. He seems to have read the proofs of the first edition of the *Discours de la Mêthode,* and for many years

[1] On October 16, 1920, a tablet was placed on the house in Amsterdam where Descartes lived in 1634. It bears the inscription: "Dans cette maison, habita le célèbre philosophe français, René Descartes. A sa glorieuse mémoire cette plaque a été consacrée par l'Alliance française des Pays Bas," and also the following phrase extracted from the correspondence of the philosopher: "Quel autre pays où l'on puisse jouir d'une liberté si entière?" See also, elsewhere in this issue of the MONTHLY, page 179, in connection with the contents of *Revue générale des Sciences.*—EDITOR.

he and Descartes carried on an intimate correspondence, each being a great admirer of the other. Little, however, could either have guessed that Christian, then only a child of three, would one day far outrank his father and would, in point of mathematical ability, rival his father's distinguished friend.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

To a question raised by the editor in connection with Professor Light's "Note on curves whose evolutes are similar curves" [*1920*, 303], as to the existence of other curves possessing the stated property, an answer is given below by Mr. Franklin, who derives in a simple way the result, stated by Puiseux, that an infinite set of such curves exists. A rather interesting incidental feature is found at the end of the paper, in the construction of a sort of geometrical "sum" of two curves as the locus of the sum (as determined by vectors starting at the origin) of points on the two curves at which tangents are parallel. The especial interest lies in the fact that exactly this process of composition of curves has been mentioned recently in an entirely different connection by Professor W. B. Carver[1]; the curves to which he applies the process are algebraic, while those of Mr. Franklin are transcendental.

Proofs of the law of tangents in plane trigonometry have been given recently by Cheney [*1920*, 53], Lovitt [*1920*, 465] and Epperson [*1921*, 71]. In a letter to the editor, from which an extract is printed, Professor Mathews calls attention to various other proofs of the law. It is clear, as Professor Mathews states, that other new proofs can be devised in considerable number from a suitable figure; it therefore seems desirable to bar consideration of further proofs, unless they involve new principles.

As the third discussion, we print a note on the nature of expository papers for presentation to the Association.

I. ON CURVES WHOSE EVOLUTES ARE SIMILAR CURVES.

BY PHILIP FRANKLIN, Princeton University.

In the July–September number of the MONTHLY [*1920*, 303] there appeared a discussion of curves whose evolutes are similar to themselves, by Professor Light. He found that the only curves having this property and having their intrinsic equations of the special type $AR^n + BS^m + C = 0$ were the logarithmic spiral and the cycloidal curves. The editor inquired whether any other curves possessed this property.

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[1] "The failure of the Clifford chain," *American Journal of Mathematics*, vol. 42 (1920), p. 167.

That there is an infinite number of other classes of curves having the property in question was shown by Puiseux.[1] His method was similar to that used by Binet[2] to solve the special case of the problem where the evolute is equal to the original curve, although his work was independent of Binet's. The treatment here given follows closely that of Salmon.[3]

The simplicity of the derivation depends on the use of the relation between R (the radius of curvature) and t (the angle which the tangent at any point makes with a fixed tangent), instead of the ordinary intrinsic equation of the curve. Setting then

$$R = f(t) \tag{1}$$

as the equation of the original curve, we have

$$R_1 = af(t_1) \tag{2}$$

as the equation of its evolute, a being the ratio of similitude.

We may evidently set

$$t_1 = t + \alpha \tag{3}$$

and combining the last two equations with the relation:

$$\frac{dR}{dt} = R_1 \tag{4}$$

we obtain as the functional-differential equation for the determination of the function $f(t)$:

$$f'(t) = af(t + \alpha). \tag{5}$$

If the right member of this equation were capable of being expanded in a Taylor's series, the equation would have the *form* of a linear differential equation (of infinite order); hence we are led to seek a solution of the form $R = e^{mt}$ We find that this is a solution if the relation

$$m = ae^{m\alpha} \tag{6}$$

is satisfied. It is then evident that the equation

$$R = \Sigma C_i e^{m_i t}, \tag{7}$$

where the C_i are arbitrary, while the m_i are all roots of an equation of the form (6), defines a solution of (5). Whether (7) is the most general solution of (5) will not be discussed; it is probably the most general solution with no singularities except at infinity. Certain properties of the solutions of an equation similar to (5) have been obtained by Professor Fite.[4]

[1] M. Puiseux, Liouville's *Journal*, vol. 9, 1844, p. 377.

[2] J. Binet, Liouville's *Journal*, vol. 6, 1841, p. 61.

[3] G. Salmon, *Higher Plane Curves*, 1852, p. 280. The problem is not discussed in the later editions of the work.

[4] W. B. Fite, "Properties of the solutions of certain functional differential equations." *Bulletin of the American Mathematical Society*, vol. 26, pp. 245, 254.

As a particular case of (7) we have

$$R = Ae^{m_1 t} + Be^{m_2 t}, \tag{8}$$

where A and B are arbitrary constants, since a pair of values of a and α can be found such that m_1 and m_2 satisfy (6). If (8) is specialized by setting A and B equal, and taking a pair of conjugate complex numbers for m_1 and m_2, we obtain:

$$R = Ce^{mt} \cos nt, \tag{9}$$

which defines a logarithmic spiral when $n = 0$, and a cycloidal curve when $m = 0$, the cases obtained by Professor Light.

It is interesting to note that (6) has at most two real roots (since the equation obtained from it by differentiating both sides with respect to m evidently has at most one real root), but an infinity of complex roots.

Since the Cartesian equation of a curve given in the form (1) would be obtained by integrating the equations:

$$\begin{aligned} dx &= ds \cos t = R \cos t\, dt = f(t) \cos t\, dt, \\ dy &= ds \sin t = R \sin t\, dt = f(t) \sin t\, dt, \end{aligned} \tag{10}$$

and eliminating the parameter t, we see that the curve corresponding to (7) could be obtained from the curves corresponding to the separate terms of the sum by locating the points whose abscissas are the sums of the abscissas and whose ordinates are the sums of the ordinates of points on the component curves whose tangents are parallel, *i.e.*, points corresponding to the same value of t.

The problem admits of several generalizations. Puiseux[1] extended his solution to the case where it is merely required to find curves whose nth evolutes are similar to themselves. This extension presents no new difficulties. Binet[1] studied surfaces such that the locus of one of the two centers of curvature at each point was a surface equal to the original surface. We might also inquire whether there are any twisted curves such that one of their evolutes is similar; or such that the locus of centers of osculating spheres or of centers of curvature gives similar curves. This question is more difficult than that for the plane, owing to the greater number of constants determining a displacement, and the writer knows no successful method of attacking it, nor whether any particular solutions besides certain circular helices are known.

II. Geometric Proofs of the Law of Tangents.[2]

By R. M. Mathews, Wesleyan University.

Professor Lovitt's sixth proof, which uses the circumscribed circle, is given by: Killing und Hovestadt, *Handbuch des Mathematischen Unterrichts*, vol. 2, p. 27.

In *School Science and Mathematics*, vol. 15, pp. 798–801, I published an article, "Proofs of the Law of Tangents," in which I gave five different proofs.

[1] L. c.

[2] Extract from a letter to the editor.

The proofs of Hobson, Wilczynski, Hall and Frink, Paterson (*Elementary Trigonometry*), and Killing and Hovestadt were reduced to a common notation. This article called forth two notes, one from Professor E. R. Hedrick, *School Science and Mathematics*, vol. 16, pp. 347–348, containing a proof of his own which is used by his permission in Kenyon and Ingold, *Plane and Spherical Trigonometry*; and one from Professor C. N. Mills (*loc. cit.*, p. 607), who gives a proof found in "an old text-book of trigonometry."

Professor Lovitt begins by constructing segments for $a + b$ and $a - b$ and then discovering angles equal to $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$. We can also begin by constructing these angles and then discovering the segments. In this connection I gave in the article cited a proof of my own that I have not seen anywhere in print.

Take $a > b$; let XCY be the bisector of the exterior angle at C, so that $\angle ACY = \frac{1}{2}(A + B)$. Draw $AG \parallel XY$; $AF \perp XY$; $BGD \perp XY$. Then $\angle BAG = \frac{1}{2}(A - B)$.

$$\tan \tfrac{1}{2}(A - B) = \frac{BG}{GA} = \frac{BD - GD}{DC + CF} = \frac{(a - b) \sin \frac{1}{2}(A + B)}{(a + b) \cos \frac{1}{2}(A + B)} = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B).$$

By adding a few lines to Professor Lovitt's figure I have been able to reduce all the proofs at hand to one figure. From this figure so many other possible proofs appear and the temptation to add still other lines is so great that it seems the prudent thing to rest content with what we have.

III. Expository Papers for the Association.[1]

By E. J. Wilczynski, University of Chicago.

The Mathematical Association of America has reached the conclusion that it can assist very effectively in enlarging the mental horizon of its members by presenting, from time to time, properly conceived papers of an expository character. But the question immediately arises: what is meant by an expository paper, and what are the most desirable characteristics of such a paper? It is my purpose to answer this question very briefly and in a preliminary fashion. Later, when as we hope, a large number of successful expository papers will be available for analysis and comparison, it may become possible to answer this question far more fully.

1. *Choice of Subject.* We all observe, from time to time, in connection with our work of teaching and research, that certain subjects are either omitted entirely from our textbooks, or are treated in inadequate fashion. If it is a subject of general interest and importance which has been thus slighted, it clearly offers a desirable field for an expository paper. Or else we may be interested in some advanced work, and the idea may come to us to explain this work to a non-technical audience. We should follow such impulses, especially when they are

[1] An address delivered before the Mathematical Association of America, and Section L of the American Association for the Advancement of Science at the University of Chicago, December 28, 1920.

strong and spontaneous. But no one should attempt any work of this kind unless at least *he himself* feels strongly that his exposition of the subject fills a real need and constitutes, in some essential way, an improvement on other treatments which are already available.

2. *Prerequisites Required of the Reader.*—The form of the paper will, of course, be determined very largely by the question: for what class of readers are we writing? We may assume that the reader has the usual knowledge of elementary algebra, geometry, and trigonometry; we shall assume further that he has had a first course in analytic geometry and calculus. We must not, however, presume too much on these prerequisites but use our judgment in every case as to whether a certain matter of detail may be passed over without discussion, or whether it requires further explanation. There are plenty of propositions in elementary geometry which even a well trained mathematician may not be able to remember and reproduce on the spur of the moment. In such cases, a reference to a generally accessible book should be given. From analytic geometry and the calculus, we should assume only those propositions which may properly be regarded as being in the possession of all.

3. *Two Kinds of Expository Papers.*—Of course, there exists no absolute line of demarcation between expository papers and other papers. In fact, every mathematical paper is, or should be, expository. But in a research article the emphasis is placed on results and methods, rather than upon any attempt to explain the subject to a large audience. There are at least two kinds of expository papers, namely, those which are purely *descriptive* or *popular*, and those which are also *demonstrative*. That popular papers on mathematical subjects are both possible and desirable is a fact which has not been recognized sufficiently, much to the disadvantage of our science in popular estimation. To reach the maximum of their usefulness, popular papers should contain very full and exact references to those books in which actual proofs may be found for the propositions under discussion. The expository papers presented to the Association will ordinarily be both descriptive and demonstrative. Naturally they will tend to become descriptive, rather than demonstrative, as the subjects with which they deal become more advanced. For it may be quite impossible to discuss a subject demonstratively, within the prerequisites mentioned under section 2, while a descriptive treatment of it would be perfectly feasible.

4. *Definition.*—Every mathematical discipline has certain fundamental concepts which are, in most cases, expressible in the form of definitions. It is vital that these definitions be formulated with great distinctness and clearness. If the reader can only be made to understand these fundamental concepts, the rest is easy.

5. *Rigor and Clearness.*—All proofs given should be mathematically sound and clear. They cannot be clear unless they are sound. It is a fallacy to think that a dishonest proof can help a student. The principal source of obscurity in mathematics, next to logical unsoundness, is excessive brevity. Space is valuable; but excess of brevity, especially in an expository paper, does not save space but

wastes it since it defeats completely the pupose for which the paper was written and, therefore, makes it useless.

6. *Desirability of a Heuristic Treatment.*—The human mind does not care about isolated facts, but likes to gain a *point of view* which will enable it to include many facts at once. Therefore it is essential for the success of an expository paper that the general connection between the problems discussed be made apparent. For this purpose a free, informal style is desirable, and much to be preferred to the formal and dogmatic style of Euclid.

7. *Restriction of Scope.*—If an expository paper is primarily demonstrative, it will necessarily be concerned with a limited field and, in accordance with section 2, this field will have to be of a rather elementary character. Most of such papers, however, will be primarily descriptive. In such descriptive papers no proofs should be attempted except when such proofs can actually be furnished without using other prerequisites than those mentioned in section 2, and when this does not use up too much of the available space, which should properly be reserved for a full and clear exposition of the more important features of the theory. The object of these papers is to *arouse interest* and to convey *some* information, but *not* to carry the reader to the very confines of knowledge. The attempt to accomplish this latter purpose would defeat the former since it would necessarily lead to excessive brevity, thus offending against the principle laid down in section 5.

8. *Character of References Given.*—A reference may be accurate and satisfactory from the point of view of a scholar, and may, nevertheless, prove to be perfectly useless from the point of view of the reader, whom we are attempting to benefit, because the source referred to may be far beyond his comprehension. Therefore, the references should be classified, and special emphasis placed upon those books and articles which are not beyond the reader's scope. This should be supplemented by a short graduated list of standard books, study of which may have the effect of equipping the reader to approach the higher literature of the subject.

RECENT PUBLICATIONS.

REVIEWS.

Gli Scienziati Italiani dall'Inizio del Medio Evo ai Nostri Giorni. Diretto da ALDO MIELI. Volume I, parte 1. Roma, A. Nardecchia, 1921. Royal 8vo. 10 + 234 pages. Price 45 lire.

This is the first part of an important and a sumptuous repertory of which it is expected that one volume will be published each year for a series of years. It is a bio-bibliographical work of Italian scientists from the middle ages to our own day. The term scientists is to be interpreted as including: philosophers, mathematicians, astronomers, physicists, chemists, mineralogists, geologists, biologists, psychologists, geographers, etc. The sketches are all signed and in many of them

are facsimiles of manuscripts and reproductions of portraits. They are not arranged according to any special plan, except, possibly, the order in which they may have reached the editor.

Among the 34 sketches by 17 authors, in the part under review, are the following seven sketches which have interest for us: Redento Baranzano (1590–1622), philosopher and astronomer, by G. Boffito, 208–212; Ulisse Dini (1845–1918), mathematician, by G. Loria, 137–150; Leonardo Fibonacci (sec. XII–XIII), mathematician, by G. Loria, 4–12; Giovanni Inghirami (1779–1851), astronomer and geodesist, by G. Giovannozzi, 188–196; Giovanni Antonio Magini (1555–1617), astronomer and geographer, by A. Favaro, 101–111; Giuseppe Moletti (1531–1588), astronomer and cosmographist, by A. Favaro, 36–39; Giovanni Virginio Schiaparelli (1835–1910), astronomer and historian of science, by E. Millosevich, 45–67.

The sketch of Dini is divided up as follows: "Vita" (containing portrait and a two-page facsimile of a manuscript), pages 137–141; "Opera," 142–146; "Bibliografia" with a list of 62 scientific papers in addition to parliamentary papers, 146–152; "Letteratura" which contains simply the references to the sketches of Dini by L. Bianchi and W. B. FORD. [Compare *1919*, 205, 455; *1920*, 271.]

So also for Leonardo: "Vita," 4–5; "Opera," 6–10; "Bibliografia," 11; "Letteratura," 12, the last title in which is R. B. McClenon, "Leonardo of Pisa and his *Liber Quadratorum*," published in this MONTHLY, *1919*, 1–8.

Everyone interested in the history of science will wish to have in his library a copy of Mieli's valuable "Repertorio."

R. C. ARCHIBALD.

January 19, 1921.

Das Fermatproblem in seiner bisherigen Entwicklung. By PAUL BACHMANN. Berlin and Leipzig. Walter de Gruyter & Co., 1919. 8 + 160 pages. Price 12 marks.

This little book, written in the terrible days of 1918, is in honor of the fiftieth anniversary of the doctorate of Felix Klein. It is avowedly a glorification of German scientific effort in the theory of numbers, but the author is too great a scientist to allow any unworthy motive to color the presentation of the subject. He treats the work of German and non-German with scholarly impartiality and thoroughness.

The book gives in very convenient form the chief results of 284 years of struggle with the problem of proving the possibility or impossibility in integers of the equation

$$x^n + y^n = z^n$$

for values of n greater than 2. This problem, raised by certain comments of Fermat, was the object of a prize by the French Academy in 1823, and later in 1853. In 1908 a prize of 100,000 marks was offered by the Royal Academy of Science of Göttingen for a solution of the problem. Apart from the stimulus

added by these prizes the problem itself seems one that is particularly attractive to mathematicians, both amateur and professional. It is also one that has tripped up so many eminent thinkers that one is tempted to fancy that the great Fermat himself was deceived in thinking that he had a "truly remarkable proof."

If the value of a problem be measured by the number of different theories developed in trying to prove it, one would be puzzled to find a more valuable one than Fermat's last theorem. In some respects it would have been a calamity if Fermat had taken a little more paper and jotted down the method he employed to establish it, if he did establish it. The development of the theory of Algebraic Numbers might have been delayed many years.

D. N. LEHMER.

The Theory of Relativity. By R. D. CARMICHAEL. Second edition. (Mathematical Monographs No. 12). New York, J. Wiley & Sons, 1920. 112 pages. Price $1.50.

Preface—"The theory of relativity has now reached its furthest conceivable generalization in the direction of the covariance of the laws of nature under transformations of coordinates. The older theory of relativity remains valid as a special case of the general theory and may well serve as an introduction to its more far-reaching aspects. Accordingly, in the present (second) edition of this monograph I have retained the older theory in precisely the same form as in the first edition, the matter covering Chapters I to VI of the present treatment, and have added the longer Chapter VII to give a compact account of the generalized theory. The tendency now is to call the latter the theory of relativity and to distinguish the older from it by giving to the older theory the name of the restricted theory of relativity.

In the opening section (§ 37) of the new chapter, I give a brief summary of results from the restricted theory. Anyone who is acquainted with these, whether derived as in this book or otherwise, may proceed at once to the reading of Chapter VII. It is believed that he will find in it about as brief an account of the new theory as can be given so as to be easily intelligible and at the same time to reach the general theory of gravitation, to make clear the nature of the three famous crucial phenomena, to associate the theory with Maxwell's electromagnetic equations, and to place the whole in its proper setting with respect to the general body of scientific truth.

The new as well as the older matter in the booklet has been written from the point of view of the usefulness of the theory of relativity in the development of physical science. No applications are given other than those which are directly and immediately associated either with the fundamental ideas or with certain crucial phenomena for testing the validity of the theory. In this way only may the central elements of novelty most readily be brought to light.

No attempt has been made to give a complete account of the theory. The purpose of the monograph is best served by presenting only those fundamental developments which are needed for and contribute directly to making clear the main characteristics of the theory. The more detailed statements are to be found elsewhere, especially in the memoirs which have now reached a considerable number.

Every exposition of the general theory of relativity must be deeply indebted to the basic memoir of Einstein, published in 1916 in *Annalen der Physik*, volume 49. Very useful to me also, as every reader will observe, has been the report of A. S. Eddington to the Physical Society of London on "The Relativity Theory of Gravitation," a booklet to which one may be referred who wishes to go further into the theory than the exposition of the present monograph will carry him."

Contents—

A Course of Modern Analysis. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions. By E. T. WHITTAKER and G. N. WATSON. Third edition. Cambridge at the University Press, 1920. Royal 8vo. 4 + 608 pages. Price 40 shillings.

The first edition of this work was by Whittaker alone in 1902 (16 + 378 pp.). The second edition in collaboration with Watson appeared in 1915 (560 pp.). "Advantage has been taken of the preparation of the third edition . . . to add a chapter on Ellipsoidal Harmonics and Lamé's Equation, and to rearrange the chapter on Trigonometrical Series so that the parts which are used in applied mathematics come at the beginning of the chapter. A number of minor errors have been corrected and we have endeavored to make the references more complete" (Preface).

Contents.

Principles and Methods of Teaching Arithmetic. By J. R. OVERMAN. Chicago, Lyons and Carnahan, 1920. 6 + 350 pages. Price $1.60.

Contents:

NOTES.

Plane Geometry by H. E. HAWKES, W. A. LUBY, and F. C. TOUTON (Boston, Ginn. 8 + 305 pages; price $1.32) appeared in December, 1920.

The Cambridge University Press has published the concluding volume of Rayleigh's *Scientific Papers,* volume 6, 1911–1919.—From the Clarendon Press has come a fourth volume by HAROLD HILTON; it is entitled: *Plane Algebraic*

Curves (1920, 16 + 388 pages; price 28 shillings).—Another volume by Sir Thomas Heath (cf. *1921*, 133) appeared in the "Pioneers of Progress: Men of Science" series (London, Society for the Promotion of Christian Knowledge, 1921, 3 + 59 pages; price 2 shillings and 6 pence). It is entitled: *The Copernicus of Antiquity* (*Aristarchus of Samos*).

About a year ago a semi-monthly publication *Princeton Lectures* was founded by Princeton University for distribution to its alumni. Each issue contains a lecture by some member of the Princeton faculty, an introductory note on the lecture, and a brief biographical sketch of the lecturer. In number two, published May 1, 1920, the lecture was "Modifying our ideas of nature" by Professor H. N. RUSSELL. The introductory note was on "The Einstein theory of relativity."

"Contributions from the Mathematical and Physical Departments" is the subtitle of *Bryn Mawr College Monographs*, volume 4 (1904) and volume 8 (1909). In 1915 the mathematics department of the University of Edinburgh published a series of eleven "Research Papers" which were reprints from various journals. This plan has now been followed in connection with *Publications of the Massachusetts Institute of Technology* with its "Contribution from the Department of Mathematics," serial II, nos. 1–16, May, 1920–February, 1921. These articles are reprinted from *Annals of Mathematics, Bulletin of the American Mathematical Society, Proceedings of the American Academy of Arts and Sciences, Proceedings of the London Mathematical Society, Proceedings of the National Academy of Sciences, Proceedings of the Royal Irish Academy, Proceedings of the Royal Society of Edinburgh,* and *Transactions of the American Mathematical Society.* Five of the articles are by Norbert Wiener, three by F. L. Hitchcock, two by Joseph Lipka, two by S. D. Zeldin, one by J. S. Taylor, two by C. L. E. Moore and H. B. Phillips together, and one by C. L. E. Moore alone. Although these "contributions" are labelled "serial II" there was no serial I. It was feared that the labelling of this series as "serial I" would imply absence of antecedent activity.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 42, no. 4, October (published in December), 1920: "Geometrical significance of isothermal conjugacy of a net of curves" by E. J. Wilczynski, 211–221; "Observations weighted according to order" by P. J. Daniell, 222–236; "Some determinant expansions" by L. H. Rice, 237–242; "A general implicit function theorem with an application to problems of relative minima" by K. W. Lamson, 243–256; "On the Laplace-Poisson mixed equation" by R. F. Borden, 257–277; "Characteristic subgroups of an abelian prime power group" by G. A. Miller, 267–286.

ANNALS OF MATHEMATICS, second series, volume 22, no. 2, December, 1920: "The mean of a functional of arbitrary elements" by N. Wiener, 66–72; "On certain determinants associated with transformations employed in thermodynamics" by J. E. Trevor, 73–85; "The permanent gravitational field in the Einstein theory" by L. P. Eisenhart, 86–94; "On the structure of finite continuous groups with a finite number of exceptional infinitesimal transformations" by S. D. Zeldin, 95–100; "Conformal mapping of a family of real conics upon another" by T. H. Gronwall, 101–127; "On the location of the roots of the derivative of a polynomial" by J. L. Walsh, 128–144 [First sentence: "This paper contains some geometric results concerning the relative positions of the roots of a polynomial and those of its derivative. Although not entirely restricted to real polynomials, and although the cubic is especially treated in detail, most of the results here presented are naturally connected with the following theorem of Jensen's:

"*If circles are described whose diameters are the segments joining pairs of conjugate imaginary roots of a real polynomial $f(z)$, then every non-real root of the derivative $f'(z)$ lies on or within those circles.*" In this connection, Dr. Walsh remarks "No proof of Jensen's theorem has previously been published." This remark is inaccurate since the proof of the theorem was published in the AMERICAN MATHEMATICAL MONTHLY, *1920*, 299–300.]

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 3, December, 1920: "The October meeting of the American Mathematical Society" by F. N. Cole, 97–103; "The mathematical congress at Strasbourg" by D. E. Smith, 104–108; "Note on a method of proof in the theory of Fourier's series" by D. Jackson, 108–110; "On implicit functions" by F. H. Murray, 111–113; "On a pencil of nodal cubics. Second paper" by N. Altshiller-Court, 114–115; "Hausdorff's *Grundzüge der Mengenlehre*" by H. Blumberg, 116–129 [Review of Hausdorff's book published at Leipzig, 1914]; Review by D. E. Smith of F. P. Barnard's *The Casting-Counter and the Counting Board*, (Oxford, Clarendon Press, 1916), 129–131; Review by A. Emch of S. Ganguli's *Lectures on the Theory of Plane Curves* (Calcutta, 1919), 132–135; "Notes," 137–145; "New Publications," 145–148.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY, volume 11, nos. 1–2, September, 1920: "Potent divisors of the characteristic matrix of a minimum simple square ante-slope" by N. Ghosh, 1–6; "Liquid motion inside certain rotating curvilinear rectangles" by N. Sen, 7–20; "On the horpolhode" by S. Basu, 21–22; "On a generalization of Neumann's expansion in a series of Bessel functions" by A. Datta, 23–34; "A note on the theory of the magnetometer" by L. Srivastava, 35–42; "On the motion of an elongated spheroid in viscous fluid media" by B. Pal, 43–50; Review by A. C. Bose of Cullis's *Matrices and Determinoids* (volumes 1–2, Cambridge, 1913–1918), 51–82; "Notes on spherical waves of finite amplitude" by S. Banerji, 83–90; "On the theory of continued fractions" (third paper) by H. Datta, 91–100; "Notes and News," 101–103.

MATHEMATICAL GAZETTE, volume 10, December, 1920: "The Durham summer course in mathematics for teachers in secondary schools" by G. J. B. Westcott, 161–169; "Vector analysis in a university course" by C. E. Weatherburn, 170–172; "The rule of signs for a product: the completed multiplication table" by S. Lister, 173–174; "Conical projection of a conic" by H. E. Girdlestone, 174–176; "Convergence of series" by P. J. Heawood, 176–177; "Note on the integration of the difference between two Fagnano arcs of an ellipse" by E. M. Langley, 177–178; Reviews by H. G. F. of J. B. Shaw's *Lectures on the Philosophy of Mathematics* (Chicago, 1918) and C. J. Keyser's *The Human Worth of Rigorous Thinking* (New York, 1916), 181–183; Review of H. E. Licks's *Recreations in Mathematics* (New York, 1917), 186–187; Review (continued) of A. Macfarlane's *Lectures on Ten British Physicists of the Nineteenth Century* (New York, 1919), 187–190.

THE MATHEMATICS TEACHER, volume 13, No. 2, December, 1920: "Educational opportunity in the army of occupation" by J. T. Rorer, 45–52; "The outline method in mathematics" by R. R. Goff, 53–56; "Greek philosophers on the disciplinary value of mathematics" by F. Cajori, 57–62; New Books, 63–63; Notes and News, 65–68.

NATURE, volume 106, November 25, 1920: "Archimedes" [Review of T. L. Heath's *Archimedes* (London and New York, 1920)] by G. B. M[athews], 401–402. [First paragraph: "By the general consent of all competent judges Archimedes is one of the greatest mathematicians the world has ever seen. It is not easy to justify this opinion to a popular audience, most members of which know little and care less about mathematics; but Sir Thomas Heath's book ought to succeed in making the ordinary reader understand to some extent the nature of Archimedes' discoveries, and in arousing interest in the achievements of Greek mathematicians."]

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 79, November, 1920: "Étude des surfaces de translation de Sophus Lie" by B. Gambier, 401–423; "Sur la cubique à point double" by N. Altshiller-Court, 424–434; "Agrégation des Sciences Mathématiques (concours de 1920). Sujets de composition," 435–448.

PHILOSOPHICAL MAGAZINE, sixth series, volume 40, November, 1920: "The torsion of closed and open tubes" by J. Prescott, 521–541; "The modification of the parabolic trajectory on the theory of relativity" by W. B. Morton, 674–677.

PHYSICAL REVIEW, second series, volume 16, no. 5, November, 1920: "On the free oscillations of spheroids" by R. N. Ghosh, 477–480.

POPULAR ASTRONOMY, volume 28, December, 1920: "Historical notice of John Nelson Stockwell of Cleveland" by T. J. J. See, 565–584 (frontispiece portrait) [see *1920*, 383].

Proceedings of the Benares Mathematical Society, volume 1, 1919: "Constitution of the Society," i–viii; "List of members," ix–xii; "Report of the secretary," xiii–xvi; "Records of proceedings at meetings," xvii–xxi; "Remarks and criticisms on some results of Mrs. A. G. Kerkhoven-Wythoff" by H. Datta, 1–8; "The effect of the double suspension mirror on the sensitiveness of the balance" by Gorakh Prasad, 9–14; "On a result in the expansion of an arbitrary function" by Lakshmi Narayan, 15–29; "On cosmic synthesis (part I)" by S. V. Ramamurty, 30–37; "On certain new tautochrones determinable by quadratures" by S. Pande, 38–42; "On the functions and needs of a mathematical society" by Ganesh Prasad, 43–51—Vol. 2, part 1, 1920: "On the application of Burgess's method for determining the uniform motion of an ellipsoid of revolution through a viscous liquid along its axis of revolution" by D. K. Sen, 1–11; "On the expansion of the product of two parabolic cylinder functions in a series of parabolic cylinder functions" by Gorakh Prasad, 12–22; "Notes on vortices in a compressible fluid" by B. Datta, 23–31; "On the potential of a double layer whose strength has a discontinuity of the second kind" by Ganesh Prasad, 32–41.

Revue de l'Enseignement des Sciences, volume 14, July–October, 1920: "Sur la représentation paramétrique d'une surface" by F. Meyer, 148–151; "Propriétés focales des quartiques bicirculaires" by R. Dontot, 151–168; "Sur certains trièdres" by C. Bioche, 168–172; "Problèmes de mathématiques et de physiques donnés au baccalaureat en octobre, 1919," 173–191.

Revue de Mathématiques Spéciales, volume 13, December, 1920: "Au sujet de la ligne de striction d'une surface réglée" by H. Girárd, 345–347; Questions and solutions, 347–368—Volume 13, January, 1921: "Sur les moments de vecteurs" by —. Rech, 369–372; Questions and solutions (analytic geometry, descriptive geometry, and calculus), 372–380, 383–390; "Agrégation des sciences mathématiques, sujets de concours, session normale de 1920," 380–383.

Revue Générale des Scences, volume 31, November 30, 1920: "Un anniversaire ignoré" by J. and C. Felix, 713 ["La date du 11 november 1920, en plus de son intérêt national, ramène à trois cents ans d'intervalle un anniversaire de la pensée scientifique. Descartes avait, en effet, mis en marge d'un de ses manuscrits la note suivante: *XI Novembris 1620 coepi intelligere fundamentum inventi mirabilis:* le 11 novembre 1620 j'ai commencé à comprendre le fondement d'une découverte digne d'admiration. Il avait eu déjà en 1619 et à la même date sa nuit d'enthousiasme durant sa retraite d'hiver passée dans un poêle. Il l'avait ainsi notée: *X Novembris 1619, cum plenus forem enthusiasmo et mirabilis scientiæ fundamenta reperirem.* . . . D'après le texte même du *Discourse de la Méthode* (1637), c'est en 1619, que Descartes 'empruntant tout le meilleur de l'analyse géométrique et de l'algèbre, corrigeant tous les défauts, de l'une par l'autre,' jeta les bases de sa 'mathématique universelle' qui devait faire dans son esprit de toutes les sciences une même chaine. La découverte remarquable qu'il fit et nota en 1620 est probablement (d'après Carnot) cette méthode des indéterminées 'qui est si admirable qu'elle touche à l'analyse infinitésimale, et que l'analyse infinitésimale n'en est qu'une heureuse application.'

"A vingt-quatre ans, Descartes était en possession des bases de tout son système scientifique. Pendant que cette grande pensée va fonder à partir du célèbre 'Je pense donc je suis' toute une philosophie nouvelle, sa méthode va guider par ses principes ou ses effets la plupart des sciences modernes. C'est cette influence prépondérante d'un génie francais que l'on célébrait dernièrement aux fêtes organisées à Amsterdam en l'honneur de Descartes. On y a rappelé les grandes étapes de la pensée cartésienne. Et puisque le philosophe en a noté lui-même les dates, on peut rappeler celle du 11 novembre 1620 si intéressante pour l'histoire générale des sciences."] Compare pages 166–167 of this issue of the Monthly.

AMERICAN DOCTORAL DISSERTATIONS.

R. F. Borden, *On the Laplace-Poisson mixed equation.* [Reprinted from *American Journal of Mathematics*, Vol. 42, 1920]. Pages 257–277 + "Vita." (Univ. of Illinois, 1918.)

J. M. Kinney, *The general theory of congruences without any preliminary integrations.* Lancaster, Pa., 1920. 32 pages. (Univ. of Chicago, 1917.)

L. J. Rouse, *A contribution to the question of linear dependence in linear integral equations.* [Reprinted from *Tôhoku Mathematical Journal*, Vol. 15, 1919, pp. 184–216]. 33 pages. (Univ. of Michigan, 1918.)

W. G. Simon, *On the solution of certain types of linear differential equations in infinitely many variables.* [Reprinted from *American Journal of Mathematics*, Vol. 42, 1920]. Pages 27–47. (Univ. of Chicago, 1918).

T. McN. Simpson, Jr., *Relations between the metric and projective theories of space curves.* Lancaster, Pa., 1920. 4to. 26 pages. (Univ. of Chicago, 1917.)

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICAL CLUB OF THE UNIVERSITY OF ALBERTA, Edmonton, Canada.

This club was organized on October 29, 1913. The following paragraphs from its constitution outline the form and object of the club:

(1) Membership shall be open to all members of the staff of the University, to graduate students, to honor students in Mathematics and Physics, to third and fourth year students in Applied Science and to fourth year students in Arts.

(2) Meetings shall be held fortnightly throughout the session.

(3) The purpose of the society is the study and discussion of pure and applied mathematics.

The officers for the year 1913–14 were: President, Professor Ernest W. Sheldon; secretary, Professor Samuel D. Killam.

The following papers were presented during the year 1913–14: "Graphical methods" by Professor Killam; "Velocity of the stars" by Ross S. Sheppard Gr.; "Non-euclidean geometry" by George Robinson '14; "Mathematics of investments" by Cecil E. Race, lecturer in mathematics and accounting; "Mathematical approximations" by Ibrahim F. Morrison, associate professor of civil engineering; "The burial of Euclid" by Professor Sheldon; "The theory of least work" by W. Maxwell Fife, lecturer in civil engineering; "Rotations" by R. N. Parsons Gr.

The officers for the year 1914–15 were: President, Professor Killam; secretary, Professor Morrison.

During the year 1914–15 the following papers were read: "Mathematical impossibilities" by John M. Stetson, lecturer in mathematics; "Fundamental units of mechanics" by Robert W. Boyle, professor of physics. "Ionization of gases by collision" by Stanley Smith, assistant professor of physics; "The flight of projectiles" by Hector J. McLeod, lecturer in electrical engineering; "The fourth dimension" by Professor Sheldon; "The gyroscope" by Professor Morrison; "Darwin's tidal theory" by George Robinson Gr.; "Short cuts in mathematics" by Professor Killam; "Some problems in aeronautics" by Charles A. Robb, Associate professor of mechanical engineering.

During the year 1915–16 Professor Morrison was president and Professor McLeod secretary.

The papers presented during 1915–16 were: "Life assurance problems" by Professor Killam; "The prismoidal formula" by Alex J. Cook '16; "Philosophy and mathematics" by Rupert C. Lodge, Instructor in mathematics; "The rotation period of Venus" by Ross S. Sheppard Gr.; "Finite and infinite numbers" by Professor Sheldon; "Electrical forces" by Russell E. Westberg '17; "Some

applications of mathematics in chemistry" by Alfred D. Cowper, assistant professor of chemistry; "Mechanics of mountain building" by John A. Allan, professor of geology; "The problems of the ether" by Professor Smith.

No meetings of the club were held during the years 1916 to 1919 on account of the fact that most of the members were in active war service. In the fall of 1919 it was decided to merge the Mathematical Club with the new Science Association which was then being formed and to carry on its activities as a section of the larger organization. Membership in the new organization is by election and is open only to members of the staff and to students doing graduate and research work.

The mathematical and physical sciences section meets every month for the presentation and discussion of papers. During the year 1919–20 Professor Killam was chairman and Professor Morrison secretary and during 1920–21 Professor Smith is chairman and Professor Cowper secretary.

THE MATHEMATICS CLUB OF FAIRMOUNT COLLEGE, Wichita, Kansas.

On October 26, 1920, a group of students of Fairmount College who were interested in mathematics organized, with the aid of the mathematics faculty, the Mathematics Club of Fairmount College. Active membership is limited to students who have taken or are taking differential calculus. At present the club has fifteen members. The officers are: President, Frank Isely '21; vice-president, Lucretia Switser '22; secretary-treasurer, Frances Brown '21; faculty adviser, Professor Arthur J. Hoare.

Meetings are held twice a month. Programs for the year are given below.

November 15, 1920: "My first impressions of calculus" by Donice Brees '22 and Harold Higgins '22; "What is mathematics and why do we study it?" by Castle Foard '21.
December 6: "Non-euclidean geometry" by Ellice Seelye '23 and Frances Brown '21.
January 3, 1921: "Logarithms and the slide rule" by Harold Higgins '22.
January 17: "Famous problems of antiquity" by Donice Brees '22.
February 7: "Theory of numbers" by Frances Brown '21.
February 21: "Hyperbolic functions" by Persis Lehman '21.
March 7: "Quaternions" by Castle Foard '21.
March 21: "Applications of mathematics to art" by Elizabeth Sprague, professor of fine arts.
April 4: "Celestial mechanics" by Lucretia Switser '22.
April 18: "Meaning of the symbol π" by Jesse Beams '21.
May 2: "Calculating machines" by Frank Isely '21.
May 16: Social and business meeting. Election of officers for 1921–22.

THE MATHEMATICAL CLUB OF THE JOHNS HOPKINS UNIVERSITY, Baltimore, Md. [*1920*, 478.]

We have already given some account of the record kept of the proceedings of the Johns Hopkins Mathematical Club. From that record the list of programs for the first half of the year 1920 are taken as given below.

March 24, 1920: "Calculation of amplifying and detecting properties of electron tubes" by Gregory Breit Gr.
April 7: "Formulae for the approximation of factorials" by Flora D. Sutton Gr.
April 13: "Prime factors in the quadratic domain" by Professor Abraham Cohen.
April 21: "An analytical treatment of the quadrilateral" by Frank V. Morley Gr.
April 28: This meeting was devoted to the solution of problems which had been proposed at previous meetings for solution and to the proposal of new problems.
May 5: "Quaternions" by Professor Frank D. Murnaghan; "Wilkinson's theorem"[1] by Frank V. Morley Gr.
May 12: "Induction motors" by Charles T. Zahn Gr.
May 19: "A self-dual Lüroth quartic" by Professor Frank Morley.
May 26: "The parabolic group of line-to-line transformations" by Dr. Tobias Dantzig, Instructor in mathematics.

A very noticeable feature of the work of this club, as the record of its proceedings shows, has been the proposal and solution of problems. Some part of the time at nearly every meeting has been devoted to this kind of activity.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel and Otto Dunkel.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

The Relation of Caustics to Certain Envelopes.

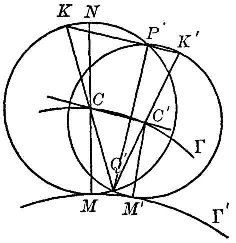
By Otto Dunkel, Washington University.

A number of problems have been proposed in the Monthly relating to the envelope of circles moving in a prescribed fashion.[2] Many such problems may be solved more simply by pure geometry than by the usual calculus procedure and it is desired to indicate here a method which is applicable in such cases and also to show how important facts may be learned by making use of caustic curves and their properties (see *1920*, 225). Suppose that a variable circle rolls on a fixed curve Γ' while its center C remains on a second fixed curve Γ and that the envelope of the moving circle is to be studied. Let C and C' be the centers of two such circles touching Γ' in M and M', respectively, and let $Q'P'$ be their common chord. Draw the diameters $Q'CK$, $Q'C'K'$. As C' approaches C the chord CC' approaches the tangent at C to Γ, M' and Q' approach M, and K and K' approach the extremity N of the diameter MCN. Since the angles $\angle KP'Q'$,

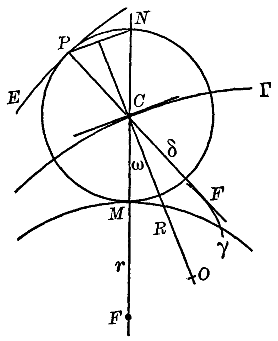
[1] Cf. Mackay, *Proc. Edinb. Math. Soc.*, Vol. 11, 1893, p. 24; Casey's *Sequel to Euclid*, 6th ed., p. 66, exs. 30, 31.

[2] For example problems 2819 (*1920*, 134); 2827 (*1920*, 186); 2861 (*1920*, 428); 2868 (*1920*, 482).

$\angle K'P'Q'$ are right, the points K, K', P' lie in a straight line parallel to the chord CC' of Γ. Hence the limiting position of P', say P, is found by drawing through N a straight line parallel to the tangent at C to Γ. This line cuts the circle in P, its point of tangency with its envelope.[1]



Suppose now that any curve Γ is given with a fixed point F on its concave side and that γ is its caustic with respect to F. With F as a center and any radius r describe a circle, and let us suppose that a variable circle with its center C on Γ touches this circle at M, and that N is the other extremity of the diameter of the variable circle. Then the point P in which the variable circle touches its envelope$E$ is found as indicated above. Produce PC and it will be seen from the figure that this prolongation and FC are equally inclined to the tangent at C. Hence this prolongation touches the caustic γ in F'. Since γ is the envelope of the lines PF' it must be the evolute of the curve E, for PCF' is a normal at P. The relation between $FC = \delta$, $CF' = \delta'$, R the radius of curvature of Γ at C has been given in the article cited above and by use of this relation the radius of curvature $\rho = MC + \delta'$ of E may be calculated. Also if the equation of the envelope E is desired, it may be obtained from the construction given for P. In examples of envelopes to which this applies it will be found that these principles are more fruitful in results which lead to ready interpretation than the process of setting up equations and differentiating according to rule.



An interesting application of this method is as follows. Let s denote the length of arc of γ suitably measured from a fixed point on it. It is well known that $PF' + s = k$ (a constant). But $PF' = PC + \delta' = FC - r + \delta' = \delta + \delta' - r$. Hence

$$\delta + \delta' + s = r + k = \text{a constant}, \tag{1}$$

a known property of caustics which has been derived by other methods not quite so simple as the above. If F is at infinity, the circle whose center is F becomes a fixed straight line perpendicular to the incident parallel rays. If this line be taken as x-axis, then $MC = y$ for the point C and the above relation becomes

$$y + \delta' + s = \text{a constant}. \tag{2}$$

If the curve Γ is such that γ reduces to a point and hence $s = 0$, the equation (1) shows that Γ must be a central conic with foci F and F'. In the case of parallel rays, if γ is a point, (2) shows that Γ must be a parabola with F' as its focus.

[1] In his solution of problem 2691 (*1919*, 131) R. A. Johnson gives a simpler derivation of an equivalent result.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2890. Proposed by B. F. FINKEL, Drury College.

Having given a triangle whose base is $2c$ and (a) the sum of whose other two sides is $2a$, (b) the difference of whose other two sides is $2a$, determine the envelope of the perpendicular bisectors of the variable sides.

2891. Proposed by D. F. BARROW, Philomath, Ga.

Let A', A'', A''', and P denote, respectively, the vertices of a triangle and any point in its plane; and let P', P'', P''', denote the feet of the perpendiculars from P upon the sides opposite A', A'', A'''. Now suppose each of the lines PP', PP'', PP''' to revolve about P through an angle α; and let P_α', P_α'', P_α''' denote the intersections of this new triad of lines with the corresponding sides of the triangle. As α varies, find the envelope of the variable circle through P_α', P_α'', P_α'''

2892. Proposed by R. T. McGREGOR, Bangor, Calif.

Two parabolas have parallel axes. Prove that their common chord bisects their common tangent.

2893. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two given skew lines in a variable plane turning about a fixed axis, not coplanar with either of the given lines.

2894. Proposed by PHILIP FRANKLIN AND E. L. POST, Princeton University.

Given the following set of assumptions concerning a set S and certain undefined sub-classes of S, called m-classes:

I. If A and B are distinct elements of S, there is at least one m-class containing both A and B.

II. If A and B are distinct elements of S, there is not more than one m-class containing both A and B.

Def. Two m-classes with no elements in common are called *conjugates.*

III. For every m-class there is at least one *conjugate* m-class.

IV. For every m-class there is not more than one conjugate m-class.

V. There exists at least one m-class.

VI. Every m-class contains at least one element of S.

VII. Every m-class contains not more than a finite number of elements.

Develop some of the propositions of the "mathematical science" (cf. Veblen and Young, *Projective Geometry*, Vol. I, pp. 1 f.) based on them and in particular develop a sufficient number of theorems to prove that the set of assumptions is categorical and give a concrete representation of the set S which satisfies them. Also prove that the assumptions are independent.

2895. Proposed by R. M. MATHEWS, Wesleyan University.

To construct an equilateral triangle with its vertices lying on: (a) any three coplanar lines; (b) three parallels in space; and (c) any three lines in space.

PROBLEMS—NOTES

10. A Curve of Pursuit. The extended discussion of a curve of pursuit in a recent issue of this MONTHLY (*1921*, 54–61, 91–97) suggests this note. In *Nouvelle Correspondance Mathématique,* volume 3, 1877, E. Lucas proposed the following problem in May (pages 175–176): "Three dogs are placed at the vertices of an equilateral triangle; they run one after the other. What is the curve described

by each of them?" In the issue for August, 1877, H. Brocard gave (page 280) the following result: Supposing the dogs start at the same time and with the same velocity, the curve of pursuit of each of the dogs is a logarithmic spiral, having for pole the center of the triangle, and tangent at a vertex of the triangle to one of its sides. ARC.

11. In *Revista Matemática Hispano-Americana,* September, 1920, the following problem is solved (pages 228–229): "Construct a square knowing the points of intersection of its sides with a line of its plane;" three solutions are found. The more general problem: To describe a square circumscribing a given quadrilateral, has been discussed many times since Diesterweg's solution in 1828.[1] There are six solutions, as Lehmus remarked in 1847,[2] and these are illustrated by a figure in I. Ghersi, *Matematica dilettevole e curiosa,* Milano, 1913, p. 587. The problem was also discussed by T. Clausen.[3] If the diagonals of the quadrilateral are equal and orthogonal there is an infinite number of solutions; this result is a particular case of a theorem given by J. Murènt in *Nouvelles Annales de Mathêmatiques,* 1855, p. 365: The necessary and sufficient condition that it is possible to circumscribe to a given quadrilateral an infinite number of rectangles, similar to a given rectangle, is that the diagonals of the quadrilateral shall be at right angles to one another and proportional to the sides of the given rectangle.[4] Hence, when diagonals so related are unequal, there must be a finite number of solutions of the square problem.

But Diesterweg's problem is only a particular case of a problem discussed by Lamê in his *Examen des différentes Mêthodes employées pour rêsoudre les Problèmes de Gêomêtrie,* Paris, 1818, pp. 16–17: About a given quadrilateral describe another similar to a third quadrilateral. He points out that there are, in general, eight solutions, closely allied to those of the inverse problem: To inscribe in a given quadrilateral another similar to a third quadrilateral. Numerous discussions of these problems are to be found in periodicals and books. The construction of a square inscribed in a quadrilatered was discussed analytically by Carnot[5] in 1803, and he stated that there were three solutions in general. T. Clausen in 1864 showed[6] that the problem had, in general, six solutions, and he commented on dual relations connecting it with the problem which we have traced to Diesterweg. ARC.

12. Milner's Lamp. In *The Journal of the Indian Mathematical Society,* June, 1920, the following problem is proposed for solution, on page 119, by A. Narasinga Rao: "Determine generally the form of a vessel whose contents are

[1] W. A. Diesterweg, *Geometrische Aufgaben nach der Methode der Griechen.* Andere Sammlung. Elberfeld, 1828, pp. 172–173.

[2] *Journal für die reine und augewandte Mathematik,* vol. 35, p. 281.

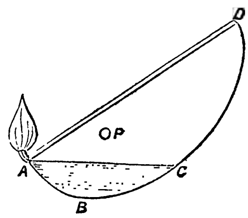
[3] *Archiv der Mathematik und Physik,* vol. 15, 1850, pp. 238–239.

[4] C. M. Herbert seemed to regard the special case of this theorem for the square as new, in his articles in *Annals of Mathematics,* second series, vol. 16, 1914, pp. 42 and 67.

[5] Carnot, *Géométrie de Position,* pp. 374–377.

[6] *Bulletin de l'acad. imp. d. sc. de St. Pétersbourg,* vol. 7, 1864, cols. 177–181.

just spilling over in the position of equilibrium, whatever the amount of liquid it contains (1) when it rests on a horizontal plane, (2) when it is suspended about a horizontal axis." This reminds one of problem number 2353 proposed by De Morgan in *The Educational Times* about fifty-five years ago: "The late Dr. Milner, President of Queen's College, Cambridge, constructed a lamp, which General Perronet Thompson remembers to have seen. It is a thin cylindrical bowl, revolving about an axis at P, and the curve $ABCD$ is such that, whatever quantity of oil ABC may be in the bowl, the position of equilibrium is such that the oil just wets the wick at A. What is the curve $ABCD$?" (Cf. *Mathematical Questions with their Solutions from the Educational Times*, volume 7, 1867, p. xvi. A few years later De Morgan referred to the problem in his *A Budget of Paradoxes*, London, 1872, p. 149; second edition by D. E. Smith, 1915, vol. 1, p. 252.) A solution by D. Biddle was published in *Mathematical Questions* . . ., volume 49, 1888, pp. 54–55. He found that a very near approach to the curve required was $r = \cos^{1/2}\theta$. This is one of a family of curves $r = a\cos^n m\theta$, arising in applications of descriptive geometry (cf. Gabriel Marie, *Exercices de Géomêtrie Descriptive,* 4e éd. Tours, 1909, pp. 835–842; indeed the special case $r = a\cos^{1/2}\theta$ is discussed on page 841).



The problem of the curve for Milner's lamp was considered by Tait, who refers to its formulation in De Morgan's *Budget,* in a paper read before the Edinburgh Mathematical Society in 1887.[1] He quoted De Morgan's statement that the lamp was "a hollow-semi-cylinder, but not with a circular curve," and arrived at a "direct contradiction" of this statement. As a question in connection with the differential equation caused trouble he applied to Cayley who in reply showed,[2] that starting with Tait's differential equation, the solution found was correct. Without any reference to Tait, Biddle discussed the circular form and the consequent lack of "bias to cause rotation." ARC.

PROBLEMS—SOLUTIONS

2799 [1920, 31]. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

A newspaper recently gave this problem: Cut a regular six-pointed star into the fewest number of pieces which will fit together and make a square. The newspaper gave a solution in seven pieces. First cut off two opposite points of the star. Divide each into two parts, and fit to the remaining portion of the star so as to make a rectangle. Find the mean proportional between the length and breadth of this rectangle (construction not shown); this is the side of the required square. Using this dimension on the two long sides of the rectangle, divide the latter into three pieces, which make the square. Total seven pieces.

How may the square be formed with not more than five pieces?

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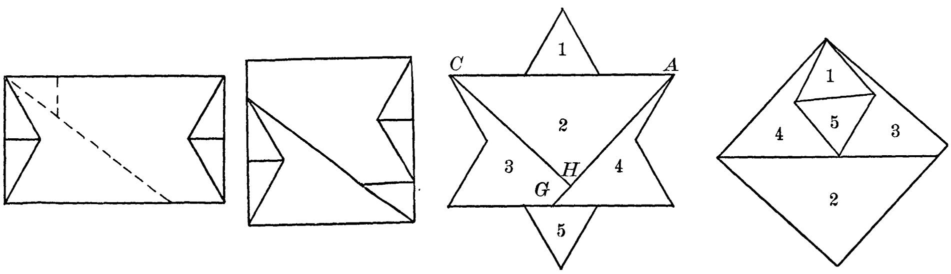
[1] P. G. Tait, "Note on Milner's lamp," *Proceedings of the Edinburgh Mathematical Society,* vol. 5, 1887, pp. 97–98; *Scientific Papers by Peter Guthrie Tait,* vol. 1, 1900, pp. 215–216.

[2] A. Cayley, "On a differential equation and the construction of Milner's lamp," *Proceedings of the Edinburgh Mathematical Society,* vol. 5, 1887, pp. 99–101; *Collected Mathematical Papers of Arthur Cayley,* vol. 13, 1897, pp. 3–5.

Solution by E. B. Escott, Chicago, Illinois.

Find the side of the equivalent square. Draw AG from one point of star equal to the side of the square. Draw CH perpendicular to AG. Then cut and arrange the pieces as in the figure.



The proposer gave a similar solution and the accompanying figures to illustrate the newspaper solution and his own.—Editors.

2819 [1920, 134]. Proposed by B. F. FINKEL, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle with a given base and a given vertical angle.

I. Solution and Remarks by H. S. Uhler, Yale University.

Let the rectangular coördinates of the ends of the base of the triangle be $(b, 0)$ and $(-b, 0)$, let 2ϕ be the constant vertical angle, and let γ denote the angle which the bisector of this angle makes with the positive direction of the x-axis.

Since the slope-angles of the bisectors of the base angles at $(b, 0)$ and $(-b, 0)$ are respectively $\frac{1}{2}(\pi + \gamma + \phi)$ and $\frac{1}{2}(\gamma - \phi)$, the equations of these lines may be written

$$y = -(x - b)\cot\tfrac{1}{2}(\gamma + \phi),$$
$$y = (x + b)\tan\tfrac{1}{2}(\gamma - \phi).$$

Solving for x and y, the coördinates of the center of the inscribed circles are found to be

$$\left.\begin{aligned} x_c &= b\cos\gamma/\cos\phi, \\ y_c &= b(\sin\gamma - \sin\phi)/\cos\phi. \end{aligned}\right\} \tag{1}$$

Since the radius of the inscribed circle equals y_c, the equation of this circle is

$$x^2 + y^2 - 2x_cx - 2y_cy + x_c^2 = 0. \tag{2}$$

Differentiating equation (2) with respect to γ, and substituting the values of $dx_c/d\gamma$ and $dy_c/d\gamma$ as obtained from equations (1), we obtain

$$\cos\phi\sin\gamma\cdot x - \cos\phi\cos\gamma\cdot y - b\sin\gamma\cos\gamma = 0. \tag{3}$$

Solving the equations (2) and (3) for x and y, with due regard to equations (1), the parametric equations of the envelope are found to be

$$x = x_c, \qquad y = 0. \tag{4}$$

$$\left.\begin{aligned} x &= b[2(\sin\gamma - \sin\phi)\sin\gamma + 1]\cos\gamma/\cos\phi, \\ y &= 2b(\sin\gamma - \sin\phi)\sin^2\gamma/\cos\phi. \end{aligned}\right\} \tag{5}$$

Equations (4) signify the base of the triangle. This branch of the locus is obviously generated by the lowest point of the inscribed circle as it rolls along the base of the triangle.

The rectangular equation of the other branch may be obtained as follows. Write equation (3) as

$$\cos\gamma = x\cos\phi\sin\gamma/(b\sin\gamma + y\cos\phi)$$

then square the two members and use $\cos^2\gamma = 1 - \sin^2\gamma$ to obtain

$$b^2s^4 + 2bcy\cdot s^3 + [c^2(x^2+y^2) - b^2]s^2 - 2bcy\cdot s - c^2y^2 = 0, \tag{6}$$

where $c \equiv \cos\phi$ and $s \equiv \sin\gamma$.

In the same notation, the second one of equations (5) may be written

$$2bs^3 - 2abs^2 - cy = 0, \tag{7}$$

where $a \equiv \sin\phi$.

Employing Sylvester's dialytic (or any other) method to eliminate s from equations (6) and (7) it will be found that

$$4(\xi^2 + 6av)(3\xi + 4a^2) - (2a\xi - 9v)^2 = 0, \tag{8}$$

where $u \equiv cx/b$, $v \equiv cy/b$, and $\xi \equiv u^2 + v^2 + 2av - 1$.

Since nothing is gained by expanding equation (8) into an explicit function of x and y, the solution may be considered as formally complete. Nevertheless it may not be superfluous to call attention to the fact that the act of squaring did not introduce any spurious factors into equation (8), that is, this equation is the simplest non-parametric rational rectangular form of which the upper branch of the locus is susceptible. For, this process amounts to multiplying equation (3), the left member of which is

$$csx - (bs + cy)\cos\gamma,$$

by the rationalizing factor

$$csx + (bs + cy)\cos\gamma,$$

obtained from (3) by changing the sign of $\cos\gamma$, which replaces γ by $\pi - \gamma$. Since s is not influenced by supplementary angles, and as the locus in question is symmetrical with respect to the y-axis, the introduction of the rationalizing factor should have no other influence than that of repeating or "double-laying" the points on the curve. In conclusion, the very simple but significant relation $dy/dx = \tan 2\gamma$ may be noted.

DISCUSSION: Writing $\omega \equiv 3\sin\gamma - 2a$, equations (5) lead to

$$\frac{dx}{d\gamma} = b\omega\cos 2\gamma/c, \qquad \frac{dy}{d\gamma} = b\omega\sin 2\gamma/c,$$

$$\frac{d^2x}{d\gamma^2} = \frac{b}{c}\left(\cos 2\gamma\cdot\frac{d\omega}{d\gamma} - 2\omega\sin 2\gamma\right), \qquad \frac{d^2y}{d\gamma^2} = \frac{b}{c}\left(\sin 2\gamma\cdot\frac{d\omega}{d\gamma} + 2\omega\cos 2\gamma\right).$$

γ	x	y	dy/dx	d^2y/dx^2	ρ
$\frac{\pi}{2}$	0	$2b(1-a)/c$	0	$\frac{-2c}{b(3-2a)}$	$\frac{b(3-2a)}{2c}$
$\frac{\pi}{4}$	$b(\sqrt{2}-a)/c$	$\frac{b(1-a\sqrt{2})}{c\sqrt{2}}$	∞	∞	$\frac{b(3\sqrt{2}-4a)}{4c}$
ϕ	b	0	$\tan 2\phi$	$\frac{2\cot\phi}{b\cos^3 2\phi}$	$ab/(2c)$
$\sin^{-1}(\frac{2}{3}a)$	$bR^3/(27c)$	$-8a^3b/(27c)$	$\frac{4aR}{9-8a^2}$	∞	0
0	b/c	0	0	$-c/(ab)$	$-ab/c$
$-\frac{\pi}{4}$	$b(\sqrt{2}+a)/c$	$\frac{-b(1+a\sqrt{2})}{c\sqrt{2}}$	∞	∞	$\frac{-b(3\sqrt{2}+4a)}{4c}$
$-\frac{\pi}{2}$	0	$-2b(1+a)/c$	0	$\frac{2c}{b(3+2a)}$	$\frac{-b(3+2a)}{2c}$

Direct substitution in the formula

$$\frac{d^2y}{dx^2} = \left(\frac{dx}{d\gamma}\cdot\frac{d^2y}{d\gamma^2} - \frac{dy}{d\gamma}\cdot\frac{d^2x}{d\gamma^2}\right)\bigg/\left(\frac{dx}{d\gamma}\right)^3$$

gives

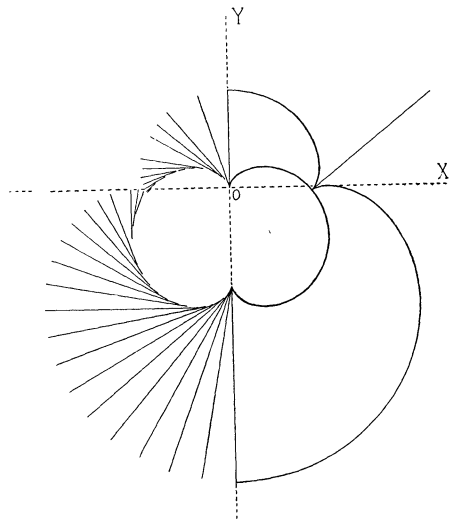
$$\frac{d^2y}{dx^2} = 2c/(b\omega \cos^3 2\gamma).$$

The cartesian formula for the radius of curvature is

$$\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2};$$

hence, for the envelope $\rho = b\omega/(2c)$.

The preceding table shows in condensed form the extreme limits and other characteristic properties of the envelope. $R \equiv (9 - 4a^2)^{1/2}$.



The accompanying diagram was constructed for the case where 2ϕ, the vertical angle of the triangle, equals 60°. Since the loci under consideration are symmetrical with respect to the y-axis, the curves have been drawn as continuous lines only on the right side of this axis. The outer curve represents the envelope and the inner one, the evolute of the envelope. The envelope always possesses two cusps the common ordinate of which is negative. A portion of the right cuspidal tangent is shown in the figure. The straight lines in the second and third quadrants represent normals to the envelope. As drawn, each normal extends from a point on the envelope to the corresponding point of tangency on the evolute. Hence, the length of each line gives the radius of curvature of the envelope. These lengths also signify the distances measured along the evolute from the point where it passes through the cusp of the envelope to the point of tangency of the chosen normal.

To show that the envelope always has a cusp at the point whose coördinates are given in the fourth horizontal line of the table, we may proceed as follows. Let $d\sigma$ denote an element of arc of the curve, so that

$$\left(\frac{d\sigma}{d\gamma}\right)^2 = \left(\frac{dx}{d\gamma}\right)^2 + \left(\frac{dy}{d\gamma}\right)^2 = \frac{b^2\omega^2}{c^2}, \qquad \text{or} \qquad \frac{d\sigma}{d\gamma} = \frac{b\omega}{c} = \frac{b}{c}(3 \sin \gamma - 2a).$$

Since this expression vanishes and changes sign for $\gamma = \sin^{-1}(\frac{2}{3}a)$, it is evident that the envelope has a simple cusp at this point.

The terms of the equation

$$d\sigma = \frac{3b}{c} \sin \gamma d\gamma - \frac{2ab}{c} d\gamma$$

can be integrated at once between definite limits and hence the lengths of chosen arcs of the envelope can be determined without difficulty. It will be left to the reader to elaborate this part of the discussion.

Attention will now be turned to the evolute of the envelope. As $dy/dx = \tan 2\gamma$ for the envelope, the equation of the normal at the point (x', y') is

$$y - y' = -(x - x') \cot 2\gamma$$

where x' and y' are given by equations (5). Substituting these values of x' and y', and reducing, we obtain

$$c \cos 2\gamma \cdot x + c \sin 2\gamma \cdot y - b(1 - 2a \sin \gamma) \cos \gamma = 0. \tag{9}$$

Differentiating equation (9) with respect to γ we find

$$2c \sin 2\gamma \cdot x - 2c \cos 2\gamma \cdot y - b(\sin \gamma + 2a \cos 2\gamma) = 0. \tag{10}$$

Solving equations (9) and (10) for x and y, the parametric equations of the evolute are found to be

$$\left.\begin{aligned} x &= b \cos^3 \gamma/c \\ y &= b(\omega - 2 \sin^3 \gamma)/(2c). \end{aligned}\right\} \tag{11}$$

Eliminating γ from equations (11), and introducing the abbreviations defined above, the cartesian formula for the evolute comes out as

$$27u^2 = [4u^2 + 4(v + a)^2 - 1]^3. \tag{12}$$

By direct substitution of the coördinates of the two cusps of the envelope,

$$u = \pm R^3/27, \qquad v = -8a^3/27,$$

it is found that equation (12) is satisfied identically. Hence,—as is true in general,—the evolute passes through the cusps of the original curve.

When $\gamma = \pi/2$ equations (11) give a point on the y-axis having the ordinate $b(1 - 2a)/(2c)$. The last expression shows that this point will lie above, or on, or below, the base of the given triangle according as ϕ is less than, or equal to, or greater than 30°, respectively. That this point is a simple cusp of the evolute follows at once from the equations

$$\frac{dx}{d\gamma} = -\frac{3b}{2c}\cos\gamma\sin 2\gamma, \qquad \frac{dy}{d\gamma} = \frac{3b}{2c}\cos\gamma\cos 2\gamma, \qquad \frac{d\sigma}{d\gamma} = \frac{3b}{2c}\cos\gamma,$$

since $\cos\gamma$ passes through zero and changes sign when $\gamma = \pi/2$.

It will not be necessary to investigate the nature of the lower point of the evolute on the y-axis because the occurrence of $(v + a)^2$ in equation (12) shows that this locus is symmetrical with respect to the horizontal line $v = -a$ or $y = -ab/c$.

II. Solution[1] by Otto Dunkel, Washington University.

The locus of the center of the inscribed circle is a circle passing through the extremities of the base of the triangle, with center at the lowest point of the circumscribed circle, and with radius equal to $b \sec \phi$.

Let $A'OA$ be the diameter parallel to the base of the triangle. C being the center of the variable circle, angle AOC will be equal to γ, and the radius of the variable circle will be

$$\frac{b}{\cos\phi}(\sin\gamma - \sin\phi).$$

Applying formula (1) [*1920*, 225],

$$\frac{1}{\delta} + \frac{1}{\delta'} = \frac{2}{R\cos\omega},$$

where $\delta = \infty$, $R = b\sec\phi$, and the angle there called ω is $90° - \gamma$, we have

$$CF' = \delta' = \frac{b\sin\gamma}{2\cos\phi}.$$

It is therefore easy to construct the point F' of the evolute. Hence the radius of curvature $PF' = \rho = (R\sin\gamma)/2 + R(\sin\gamma - \sin\phi) = R(3\sin\gamma - 2\sin\phi)/2$. Hence the envelope has a cusp where $\sin\gamma = \frac{2}{3}\sin\phi$. There are then two cusps lying upon the evolute, corresponding to the angles $\gamma < \phi$ and $180° - \gamma$. If C is taken at A, P will lie on the base produced which forms the tangent at P. Hence this line cuts the envelope in two distinct points and in a pair of points of tangency, in all six points. This serves to illustrate how to determine synthetically the important properties of the envelope and to construct it point by point.

Also solved by William Hoover, Joseph Rosenbaum, and F. L. Wilmer.

[1] This is an application of the article on "The Relation of Caustics to Certain Envelopes" printed above. The notation of the previous solution is here employed and the presentation is somewhat condensed.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

We have already referred to the American Field Service Fellowships for French Universities (*1921*, 44). Among the twenty fellows appointed for 1921–1922 there is one mathematician, Mr. F. H. MURRAY, now a Frederick Sheldon Fellow from Harvard University, studying in Paris.

Professor ALBERT EINSTEIN of Berlin University accepted the invitation of the head of the Zionist World Movement to accompany the Zionist delegation from Europe to the United States last March. It is reported that Professor Einstein is appealing to the Jews in this country for support of the Hebrew University to be erected on the Mount of Olives in Jerusalem.

The Benares Mathematical Society was founded in August, 1918, "for the encouragement and promotion of research in various branches of pure and applied mathematics, and in the history of mathematics." Dr. GANESH PRASAD was elected president for three years and Professor Lakshmi Narayan, secretary. At the end of 1919 there were 51 ordinary members, and two honorary members: Professors E. T. WHITTAKER of Edinburgh University and E. B. WILSON of Massachusetts Institute of Technology. The contents of the *Proceedings* are referred to elsewhere in this issue, [*1921*, 179; see also page 31]. Volume 1, 1919, contained 22 + 51 pages; volume 2, part 1, 1920, 3 + 42 pages.

In July, 1920, the editors of *Scientific American* announced the offer, by one of its friends, Mr. Eugene Higgins, an American resident of Paris, of a prize of five thousand dollars ($5,000.00) for "the best essay on the Einstein postulates and their consequences, written so that a person with no special mathematical training may read it profitably." The contest was left in the hands of the *Scientific American.* No essay was to contain more than 3,000 words and all "must be in English, and written as simply, lucidly and non-technically as possible." The essays had to be in the office of the *Scientific American* by November 1, 1920. Professors LEIGH PAGE of Yale and E. P. ADAMS, of Princeton, were the judges. In January, 1921, the prize was awarded to Mr. LYNDON BOLTON, a senior examiner in the Patent Office, London. His "two most immediate rivals" were also Englishmen. Among the competitors were: Professor H. H. TURNER, of Oxford University; Professor A. G. WEBSTER, of Clark University; and Professor G. D. BIRKHOFF, of Harvard University, who "heads the list of mathematicians pure and simple who competed."

Nearly 300 essays were received. They came in greater quantity from Germany than from any other foreign country. England stood next on the list and one or more essays were received from Austria, Canada, Chili, Cuba, Czecho-

slovakia, India, Jamaica, Jugoslavia, France, Switzerland, the Netherlands, Denmark, the Fiji Islands, Italy, Mexico and South Africa.

Mr. Bolton's essay was published in the *Scientific American,* February 5, and in the *Westminster Gazette,* London, February 14.

It is now announced that it was the Mr. Higgins mentioned above who was the anonymous donor of the prize of five hundred dollars ($500.00) offered by the *Scientific American* in 1909 for the "best popular explanation of the fourth dimension, the object being to set forth in an essay not longer than twenty-five hundred words the meaning of the term so that the ordinary lay reader could understand it." Professors H. P. MANNING, of Brown University, and S. A. MITCHELL, of Columbia University, were the judges.

The members of the committee on arrangements for the Wellesley meeting of the American Mathematical Society, September, 1921, are as follows: Professors E. V. HUNTINGTON, HELEN A. MERRILL, A. D. PITCHER, R. G. D. RICHARDSON, and CLARA E. SMITH.

The council of the American Mathematical Society has voted to affiliate itself with the American Association for the Advancement of Science. As a result the secretary of the Society will be a member of the council of the Association.

Professor W. F. OSGOOD has withdrawn from service on the committee of the Bôcher Memorial Fund (*1921*, 151). Professor DUNHAM JACKSON has been added to the committee and Professor E. B. VAN VLECK is its chairman.

We have already recorded (*1921*, 150) the presentation to Professor F. N. COLE of an address accompanied by a purse containing about four hundred and seventy-five dollars ($475.00). Professor Cole has presented this sum to the American Mathematical Society. The Council of the Society named it The Cole Fund and appointed a committee to report on the most desirable method of expending the income from the Fund.

In February, 1921, the secretary of the American Mathematical Society sent to MAGNUS GÖSTA MITTAG-LEFFLER, professor emeritus of the University of Stockholm, the following letter:

"As March sixteenth approaches I have been directed by the Council of the American Mathematical Society to extent to you heartiest greetings and felicitations on your seventy-fifth birthday.

"Your thirty-five years of service in universities has been recalled, also your authorship of many contributions to research, your founding and direction of a leading mathematical journal of which forty volumes have already been published, your establishment of a Mathematical Institute, and your leadership in varied movements. The Council feels that few have more widely, consistently, and ably fostered the maintenance of high ideals in connection with the development of mathematics in the world."

At the meeting of the American Mathematical Society in New York City, February 26, 1921, the following papers were presented: "The equations of interior ballistics" by A. A. BENNETT; "A geometrical characterization of the paths of particles in the gravitational field of a mass at rest" by L. P. EISENHART; "On the polar equation of algebraic curves" by ARNOLD EMCH; "Generalization of the concept of invariancy derived from a type of correspondence between

functional domains. Second proof of the finiteness of formal binary concomitants modulo p" by O. E. GLENN; "Some empirical formulas in ballistics" and "Summation of a double series" by T. H. GRONWALL; "The mathematical theory of proportional representation with a substitute for least squares" by E. V. HUNTINGTON; "A property of the Pellian equation with some results derived from it" by JOHN McDONNELL; "Concerning the sum of a countable number of closed point-sets" by R. L. MOORE; "A necessary and sufficient condition that the sum of two bounded, closed, and connected point-sets should disconnect a plane" by ANNA M. MULLIKIN; "On the apportionment of representatives" by F. W. OWENS; "Coefficient of the general term in the expansion of a product of polynomials" by L. H. RICE; "Periodic functions with a multiplication theorem" and "Note on equal continuity" by J. F. RITT; "Expressions for the Bernoulli function of order p," "The expansion of a continued product," "Method for the summation of a family of series" and "Note on the evaluation of a definite integral" by I. J. SCHWATT; "On the simplification of the structure of finite continuous groups with more than one two-parameter invariant sub-group" by S. D. ZELDIN.

At the twenty-seventh annual meeting of the American Mathematical Society, held in New York, December 28, 29, 1920, the following papers were presented: "An extension of Poincaré's geometric theorem" by G. D. BIRKHOFF; "On certain simple skew frequency curves" by R. W. BURGESS; "Systems of linear inequalities" by W. B. CARVER; "Differential geometry of the complex plane" by J. L. COOLIDGE; "The value of a bond to be redeemed ultimately, both principal and interest, in equal installments" and "Valuation of bonds bought to realize a specified rate of interest assuming the amortizations to accumulate at a savings bank rate" by C. H. FORSYTH; "On a new treatment of theorems of finiteness (second paper)" (preliminary report) by O. E. GLENN; "Parallel maps of surfaces" by W. C. GRAUSTEIN; "Zeros of Legendre functions" by EINAR HILLE; "A mathematical theory of proportional representation" by E. V. HUNTINGTON; "Some properties of methods of evaluation of divergent sequences" by W. A. HURWITZ; "Properties of orbits in the general theory of relativity" and "The solar gravitational field in finite form" by EDWARD KASNER; "Conformal transformations of period n and groups generated by them" by HARRY LANGMAN; "The Hilbert integral and Mayer fields for the problem of Mayer in the calculus of variations" by GILLIE A. LAREW; "Some special cases of the flecnode transformation of ruled surfaces" by J. W. LASLEY, Jr.; "Transformations of trajectories on a surface" by JOSEPH LIPKA; "Generalizations of the classical construction of the strophoid" by R. M. MATHEWS; "Note on minimal varieties in hyperspace" by C. L. E. MOORE; "Pleasant questions and wonderful effects" (presidential address) by FRANK MORLEY; "Recurrent motions of the discontinuous type" by H. M. MORSE; "Certain theorems concerning connected point sets" by ANNA M. MULLIKIN; "On the projectivity assumption in projective geometry" by F. W. OWENS; "The theory of relative maxima and minima of quadratic and Hermitian forms and its application to a

new foundation for the theory of bilinear forms. First paper: Equivalence of pairs of bilinear forms" by R. G. D. RICHARDSON; "Divergent double series and sequences" by G. M. ROBISON; "The efficiency of projectile and gun" by J. E. ROWE; "Independent expressions for the Bernoulli numbers," "Relations involving the numbers of Bernoulli and Euler," "Independent expressions for Euler numbers," "Independent expressions for the Euler numbers of higher order" and "Summation of a type of Fourier's series" by I. J. SCHWATT; "On homogeneous functions as generators of an abstract field" and "The concept of an iterative compositional algebra" by A. R SCHWEITZER; "The analytic geometry of complex variables with some applications to function theory" by J. S. TAYLOR; "On the convergence of the Sturm-Liouville series" and "On the location of the roots of polynomials" by J. L. WALSH; "On the maximum value of a determinant" and "On the automorphic transformation of a bilinear form" by J. H. M. WEDDERBURN; "The average of an analytic functional," "The average of a functional" and "Further properties of the average of a functional" by NORBERT WIENER; "Einstein's four-dimensional space is not contained in a five-dimensional linear space" by C. E. WILDER; "On quadratic congruences and the factorization of integers" by H. S. VANDIVER; "On the structure of finite continuous groups with one two-parameter subgroup" and "On the structure of finite continuous groups with a finite number of exceptional infinitesimal transformations" by S. D. ZELDIN.

The following reports of courses in mathematics offered at Summer Sessions in 1921 have been received.

Columbia University, July 5–August 13. Undergraduate courses are given as follows: Elementary and intermediate algebra, 4 points (that is, the equivalent of 4 semester hours), by Professor W. W. RANKIN; Plane geometry, 4 points, by Professor RANKIN; Logarithms and trigonometry, 2 points, by Professor G. W. MULLINS, Dr. J. K. RITT and Dr. K. W. LAMSON; Solid geometry, 2 points, by Dr. JESSE DOUGLAS and Professor RANKIN; Algebra, 2 points, by Professors W. B. FITE and MULLINS; Analytical geometry, 2 points, by Professor L. P. SICELOFF, Dr. LAMSON and DR. DOUGLAS; Calculus (first part), 2 points, by Professor SICELOFF; Calculus (second part), 2 points, by Dr. G. A. PFEIFFER. Graduate courses are offered as follows: By Professor EDWARD KASNER: General survey of modern mathematics, 3 points, and Mathematical introduction to Einstein's theory of relativity, 2 points. By Dr. RITT: Theory of numbers, 3 points. By Professor FITE: Differential equations, 3 points. By Dr. PFEIFFER: Theory of functions of a real variable, 3 points.

Harvard University, July 5–August 13. By Professor G. D. BIRHKOFF: Trigonometry, Analytic geometry, and Differential and integral calculus. By Professor O. D. KELLOGG: Differential and integral calculus. These courses will be accepted as regular semester courses towards the degrees of A.B., A.A., and S.B.

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ON AMICABLE NUMBERS AND THEIR GENERALIZATIONS.[1]

By THOMAS E. MASON, Purdue University.

Introduction. Amicable numbers m and n are defined by the equations

$$S(m) = S(n) = m + n,$$

where $S(m)$ means the sum of all the divisors of m, including m and unity. The great interest in amicable numbers is shown by the number of papers concerning them reviewed in Dickson's *History of the Theory of Numbers*. There are nearly as many papers as there are known amicable number pairs. This interest began in very early times when the first pair—220 and 284—was discovered. The belief in their power to bring friendship lasted through a long period. Euler made the most extensive contributions to the methods of discovering such number pairs. The above pair is referred to by Iamblichus[2] (283–330 A.D.), the second known pair was discovered by Fermat, the third by Descartes. Euler with new methods added fifty-nine others, Legendre one, Paganini one, Seelhoff two, Dickson two. This paper adds fourteen.

Dickson[3] defined an amicable k-tuple as k numbers satisfying the equations

$$S(n_1) = S(n_2) = \cdots = S(n_k) = n_1 + n_2 + \cdots + n_k.$$

He searched for amicable triples only. This paper gives amicable k-tuples for $k = 3, 4, 5, 6$.

Carmichael[4] defined a multiply amicable number pair as numbers m and n satisfying the equations

$$S(m) = S(n) = t(m + n),$$

where t is an integer. The same generalization applies to the k-tuples. A large number of such multiply amicable number pairs for $t = 2, 3$ and amicable triples for $t = 2$ are readily found by methods indicated in § 2.

No attempt has been made here to find k-tuples in which two or more of the numbers are alike. That such exist is shown in some of the sets of triples found by Dickson. Any multiply perfect number of multiplicity k (equals 2 for ordinary perfect numbers) is, of course, a multiply amicable k-tuple where all the k numbers are alike.

1. **Amicable Number Sets.** One of Euler's methods is to search for amicable numbers of the form apq and arf, where p, q, r are primes and a and f are composite, but prime to each other and to p, q, and r. Let $f = (g - 1)(h - 1)$, where $g - 1$ and $h - 1$ are primes. Then $S(f) = gh$. Put $p + 1 = hx$,

[1] Read before the American Mathematical Society, Chicago, December, 1920.

[2] See Dickson, *History of the Theory of Numbers*, vol. 1, 1919, pp. 38–50.

[3] In this MONTHLY, *1913*, 84–92.

[4] In this MONTHLY, *1919*, 399.

$r + 1 = xy$ and $q + 1 = gy$. We shall have

$$ghxy\, S(a) = S(afr) = a(pq + fr) = a\{(hx - 1)(gy - 1) + f(xy - 1)\}.$$

In the equation formed by setting the first member above equal to the fourth, replace $a/(2a - S(a))$ by b/c and $bf - bgh + cgh$ by e and the result can be written

$$(E) \qquad (ex - bg)(ey - bh) = b^2gh + be(f - 1).$$

In applying this method a and f are assumed. $S(f)$ gives g and h. It remains to solve for x and y, and hence p, q and r. The restriction that g and h of formula (E) be the g and h of $f = (g - 1)(h - 1)$ is not necessary. It is sufficient to take g and h of formula (E) any factorization of $S(f)$. This more general g and h will occasionally give amicable number pairs when the restricted factorization of Euler fails to do so. The methods used to find the amicable number pairs of this paper were those of Euler with the modification just mentioned and the change in the fifth method of Euler made by Dickson as noted below.

If we seek for an amicable k-tuple an_1, an_2, $\cdots$, an_k where

$$S(n_1) = \cdots = S(n_k)$$

we have

$$S(a)S(n_1) = a(n_1 + n_2 + \cdots + n_k) \quad \text{or} \quad S(a)/a = (n_1 + n_2 + \cdots + n_k)/S(n_1).$$

In applying this method n_1, n_2, $\cdots$, n_k are chosen and a value of a is sought. This method is given by Dickson in his search for amicable triples. It has been used in this paper more extensively than any other method. It is a modification of Euler's fifth method.

The use of the methods mentioned has produced the following:

Amicable pairs.

$2\cdot5^2\cdot31\left\{\begin{matrix}47\cdot149\\19\cdot359\end{matrix}\right.$	$2^3\left\{\begin{matrix}13\cdot23\cdot251\\97\cdot863\end{matrix}\right.$	$3^2\cdot7^2\cdot13\cdot19\left\{\begin{matrix}11\cdot10499\\89\cdot1399\end{matrix}\right.$
$2^3\left\{\begin{matrix}11\cdot41\cdot173\\71\cdot1217\end{matrix}\right.$	$2^3\left\{\begin{matrix}13\cdot23\cdot1109\\71\cdot5179\end{matrix}\right.$	$3^2\cdot7^2\cdot13\cdot19\left\{\begin{matrix}17\cdot23\cdot1335949\\3079\cdot187379\end{matrix}\right.$
$2^3\left\{\begin{matrix}11\cdot31\cdot233\\127\cdot701\end{matrix}\right.$	$2^3\left\{\begin{matrix}17\cdot19\cdot281\\53\cdot1879\end{matrix}\right.$	$3^3\cdot5^3\left\{\begin{matrix}29\cdot41\cdot43\cdot59\\19\cdot131\cdot1259\end{matrix}\right.$
$2^3\left\{\begin{matrix}11\cdot31\cdot2099\\79\cdot10079\end{matrix}\right.$	$3^2\cdot5^2\cdot31\left\{\begin{matrix}29\cdot41\cdot43\cdot59\\19\cdot131\cdot1259\end{matrix}\right.$	$3^4\cdot5\cdot11\left\{\begin{matrix}41\cdot599\\59\cdot419\end{matrix}\right.$
$2^3\left\{\begin{matrix}13\cdot23\cdot149\\199\cdot251\end{matrix}\right.$		$3^4\cdot5\cdot11\cdot59\left\{\begin{matrix}89\cdot5309\\477899.\end{matrix}\right.$

Amicable triples.

$3\cdot89\cdot a$, $11\cdot29\cdot a$, $359\cdot a$ $(a = 2^8\cdot5\cdot19\cdot37\cdot73)$ and $(a = 2^{20}\cdot7\cdot13^2\cdot31\cdot61\cdot127\cdot337)$
$17\cdot79\cdot b$, $19\cdot71\cdot b$, $1439\cdot b$ $(b = 2^{15}\cdot5^2\cdot11\cdot31\cdot43\cdot257)$
$3\cdot17\cdot109\cdot c$, $29\cdot263\cdot c$, $71\cdot109\cdot c$ $(c = 2^9\cdot7\cdot11^2\cdot19)$ and $(c = 2^{14}\cdot7\cdot11\cdot19\cdot151)$.

Amicable quadruples.

$7\cdot 107\cdot a$, $11\cdot 71\cdot a$, $17\cdot 47\cdot a$, $863\cdot a$ $(a = 2^8\cdot 3^3\cdot 5\cdot 37\cdot 73)$
$17\cdot 79\cdot b$, $19\cdot 71\cdot b$, $23\cdot 59\cdot b$, $1439\cdot b$ $(b = 2^5\cdot 3^5\cdot 5\cdot 13)$
$19\cdot 71\cdot c$, $23\cdot 59\cdot c$, $29\cdot 47\cdot c$, $1439\cdot c$ $(c = 2^{15}\cdot 3\cdot 5^2\cdot 11\cdot 31\cdot 43\cdot 257)$
$5\cdot 139\cdot d$, $13\cdot 59\cdot d$, $19\cdot 41\cdot d$, $839\cdot d$, $(d = 2^{33}\cdot 3^6\cdot 7\cdot 23\cdot 83\cdot 137\cdot 331\cdot 547\cdot 1093\cdot 43691\cdot 131071)$,
$(d = 2^{33}\cdot 3^{10}\cdot 7\cdot 23\cdot 83\cdot 107\cdot 331\cdot 3851\cdot 43691\cdot 131071)$,
$(d = 2^{37}\cdot 3^6\cdot 7\cdot 23\cdot 83\cdot 137\cdot 331\cdot 547\cdot 1093\cdot 43691\cdot 174763\cdot 524287)$ and
$(d = 2^{37}\cdot 3^{10}\cdot 7\cdot 23\cdot 83\cdot 107\cdot 331\cdot 3851\cdot 43691\cdot 174763\cdot 524287)$.
$17\cdot 1999\cdot e$, $23\cdot 1499\cdot e$, $71\cdot 499\cdot e$, $149\cdot 239\cdot e$, $(e = 2^9\cdot 3^2\cdot 7\cdot 11\cdot 13)$, $(e = 2^9\cdot 3^3\cdot 5\cdot 11)$,
$(e = 2^{14}\cdot 3\cdot 5\cdot 7\cdot 19\cdot 151)$, $(e = 2^{14}\cdot 3^2\cdot 7^2\cdot 13\cdot 19^2\cdot 127\cdot 151)$,
$(e = 2^{25}\cdot 3^3\cdot 5^2\cdot 19\cdot 683\cdot 2731\cdot 8191)$, $(e = 2^{33}\cdot 3^4\cdot 7\cdot 11^3\cdot 61\cdot 83\cdot 331\cdot 43691\cdot 131071)$
and $(e = 2^{37}\cdot 3^4\cdot 7\cdot 11^3\cdot 61\cdot 83\cdot 331\cdot 43691\cdot 174763\cdot 524287)$.

Amicable quintuples.

$17\cdot 79\cdot a$, $19\cdot 71\cdot a$, $23\cdot 59\cdot a$, $29\cdot 47\cdot a$, $1439\cdot a$ $(a = 2^{15}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 43\cdot 257)$.
$17\cdot 439\cdot b$, $29\cdot 263\cdot b$, $43\cdot 179\cdot b$, $59\cdot 131\cdot b$, $71\cdot 109\cdot b$,
$(b = 2^{33}\cdot 3^6\cdot 5\cdot 7\cdot 11^2\cdot 19\cdot 83\cdot 331\cdot 43691\cdot 13107)$ and
$(b = 2^{37}\cdot 3^6\cdot 5\cdot 7\cdot 11^2\cdot 19\cdot 83\cdot 331\cdot 43691\cdot 174763\cdot 524287)$.

Amicable sextuples.

$17\cdot 1999\cdot a$, $23\cdot 1499\cdot a$, $59\cdot 599\cdot a$, $71\cdot 499\cdot a$, $79\cdot 449\cdot a$, $35999\cdot a$
$(a = 2^{36}\cdot 3^9\cdot 5^2\cdot 7^5\cdot 11^2\cdot 13\cdot 19^2\cdot 31^2\cdot 61\cdot 83\cdot 127\cdot 223\cdot 331\cdot 7019\cdot 112303\cdot 898423\cdot 616318177)$.

$23\cdot 1499\cdot b$, $59\cdot 599\cdot b$, $71\cdot 499\cdot b$, $79\cdot 449\cdot b$, $149\cdot 239\cdot b$, $179\cdot 199\cdot b$, where b is to be chosen as follows: take any multiply perfect number of multiplicity 6 which contains the factors $79\cdot 157$ and which does not contain any of the other factors multiplying b in the numbers above, from this number omit $79\cdot 157$ and what remains will give a value for b. The known multiply perfect numbers furnish at least six such values of b.

2. **Multiply Amicable Number Sets.** If we seek for multiply amicable number pairs of the form apq and arf, where p, q and r are primes and a and f are composite, but prime to each other and to p, q and r, we have the equations

$$S(a)S(pq) = S(a)S(rf) = ta(pq + rf).$$

By the same process as that used in § 1 we arrive at equation (E), using the same substitutions, except that we replace $ta/(2ta - S(a))$ by b/c. The numbers p, q, and r depend only on the ratio b/c. It is evident, therefore, that the same numbers p, q, r and f that will form an amicable number pair with a given a will serve to form a multiply amicable number pair if we can find a new a' such that $ta'/(2ta' - S(a')) = a/(2a - S(a)) = b/c$. For example, the numbers $a\cdot 11\cdot 41\cdot 173$

and $a \cdot 71 \cdot 1217$ form

an amicable number pair for . . . . $a = 2^3$,
a multiply amicable pair ($t = 2$) for . . . . $a = 2^{10} \cdot 3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 89$,
a multiply amicable pair ($t = 3$) for

$$a = 2^{21} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 13^3 \cdot 17 \cdot 19^3 \cdot 23 \cdot 89 \cdot 181 \cdot 379 \cdot 683 \cdot 757.$$

A table of multiply perfect numbers can be used in connection with formula (E) to find multiply amicable numbers. In case $f = 1$, formula (E) reduces to[1] $(cx - b)(cy - b) = b^2$. An illustration with this simpler formula will be given. Take any multiply perfect number of multiplicity 4 containing the factor 31, for example, $2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31$, and choose for a all of this except the factor 31, $a = 2^9 \cdot 3^3 \cdot 5 \cdot 11$. For $t = 2$ this will give $b = 16$, $c = 1$. Using these values of b and c we get $p = 23$, $q = 47$, $r = 1151$. This gives the multiply amicable pair $23 \cdot 47 \cdot a$ and $1151 \cdot a$ of multiplicity 2. It is necessary that the number a do not already contain any of the factors p, q and r. Since the values of b and c depend only on the fact that the multiply perfect number is of multiplicity 4 and contains the factor 31, any number having these properties will give a value for a. It can readily be seen that if we had taken a multiply perfect number of multiplicity 6 in the above illustration and had used $t = 3$ we should have found $b = 16, c = 1$. This would have given the same values $p = 23, q = 47, r = 1151$, and hence, apq and ar would be a multiply amicable pair of multiplicity 3.

If we omit from any multiply perfect number of multiplicity $2t$ a factor found in column numbered 1, in the table below, and use the remainder of the number for a value of a, then a taken with the corresponding set in column 2 will give a multiply amicable number pair of multiplicity t, provided that a does not contain the factors in column 2.

1	2		1	2	
5	3·11	47	31	17·137·2990783	10103·735263
7	5·11	71	31	17·167·13679	809·51071
11	7·23	191	31	17·137·262079	12959·50231
19	11·59	719	59	31·479	15359
23	19·29	599	71	59·89	5399
29	17·89	1619	79	41·839	35279
31	19·1439	149·191	79	47·239	11519
31	23·47	1151	89	59·179	10799
31	23·47·9767	1583·7103	103	59·389	23399
31	47·89	53·79	139	83·419	35279
31	23·467	103·107	149	89·449	40499
31	23·479	89·127	191	97·4703	460991
31	23·1367	53·607	307	167·1847	310463
31	17·5119	239·383	463	239·6959	1670399
31	17·10303	167·1103			

[1] This is not a particular case of equation (E) but is obtained in the same way, putting $p + 1 = x$ and $q + 1 = y$; that is, setting $g = h = 1$, as well as f.—EDITOR.

By use of this table in connection with a table[1] of the known multiply perfect numbers about six hundred multiply amicable number pairs can be obtained of multiplicity $t = 2, 3$.

Amicable number sets of the form am and an can be found by the use of the equations $S(m) = S(n)$ and $S(a)S(m) = ta(m + n)$. From the latter equation we have $S(a)/ta = (m + n)/S(m)$. Select m and n so that $S(m) = S(n)$ and then find the ratio $(m + n)/S(m)$ and seek a value for a. This method will serve for amicable k-tuples.

For the case of multiplicity 2 we have the following[2]: If m and n are amicable numbers and a is a perfect number, prime to m and n, then am and an are multiply amicable numbers of multiplicity 2. By hypothesis $S(a) = 2a$ and $S(m) = S(n) = m + n$, and hence, $S(a)S(m) = S(a)S(n) = 2a(m + n)$.

A few multiply amicable number sets are given here that were not found by means of the table above.

For $t = 2$:

$53 \cdot 431 \cdot a$, $23327 \cdot a$ ($a = 2^8 \cdot 3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 37 \cdot 73$)

$13 \cdot 23 \cdot 1109 \cdot b$, $71 \cdot 5179 \cdot b$ ($b = 2^{15} \cdot 3^4 \cdot 11^3 \cdot 17 \cdot 31 \cdot 43 \cdot 61 \cdot 257$) and for the same value of b

$13 \cdot 23 \cdot 251 \cdot b$, $97 \cdot 863 \cdot b$ and $13 \cdot 23 \cdot 149 \cdot b$, $199 \cdot 251 \cdot b$

$7 \cdot 59 \cdot c$, $19 \cdot 23 \cdot c$ ($c = 2^7 \cdot 3^3 \cdot 5$)

$7 \cdot 53 \cdot d$, $17 \cdot 23 \cdot d$ ($d = 2^6 \cdot 3^3 \cdot 5$)

$43 \cdot 167 \cdot 449 \cdot e$, $59 \cdot 131 \cdot 419 \cdot e$ ($e = 2^7 \cdot 3 \cdot 5^2 \cdot 7 \cdot 31$)

$7 \cdot 89 \cdot f$, $23 \cdot 29 \cdot f$ ($f = 2^{13} \cdot 3^3 \cdot 5 \cdot 127$)

$59 \cdot 599 \cdot g$, $71 \cdot 499 \cdot g$ ($g = 2^{20} \cdot 3 \cdot 5^2 \cdot 7 \cdot 31 \cdot 127$).

A multiply amicable triple with $t = 2$ is given by $59 \cdot 331 \cdot a$, $23 \cdot 829 \cdot a$, $19919 \cdot a$, where a is found by omitting the factor 47 from a multiply perfect number of multiplicity 6 and using the remainder of that number as a, provided that the multiply perfect number does not contain any of the factors 23, 59, 331, 829, 19919.

Conclusion. The methods of finding amicable number sets are very largely those of trial. Experience in working with such numbers will suggest the likely numbers to try, but there is no sure guide yet known. The number of cases that need to be tried out becomes very large for some of the larger numbers that we might choose for $n_1, n_2, \cdots, n_k$ in seeking k-tuples of the form $an_1, an_2, \cdots, an_k$. For example, the number 1108800 is the sum of the divisors of at least twenty different sets of prime factors no one of which is less than 19. If we are using the method suggested by Dickson and are seeking amicable sextuples we shall have as many possibilities as there are combinations of twenty, six at a time. The same set offers ${}_{20}C_2$ possibilities for amicable pairs, ${}_{20}C_3$ for triples, etc. Any systematic search for amicable numbers among the large numbers will furnish a vast amount of work.

The author of this paper was attracted to the subject on reading the section

[1] Carmichael and Mason, *Proceedings of the Indiana Academy of Science*, 1911, pp. 257–270.

[2] Indicated but not explicitly stated by Carmichael, in this MONTHLY, *1919*, 399.

on amicable numbers in Dickson's *History of the Theory of Numbers* and seeing how large a number of papers had been written on the subject and how relatively few numbers had been found. After some time spent in searching he is convinced that any one with a little skill in manipulation of numbers, considerable patience, and access to the Lehmer's *List of Prime Numbers,* can add to the known list of amicable number pairs, amicable k-tuples or multiply amicable number sets.

AN ELEMENTARY TREATMENT OF FOURIER'S SERIES.

By GEORGE D. BIRKHOFF,[1] Harvard University.

The aim of this note is to treat the "remainder" after $n + 1$ terms of the Fourier's series for a given function $f(x)$:

$$\tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \cos kx dx, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \sin kx dx.$$

It will be assumed that the function $f(x)$ is periodic of period 2π and continuous together with its first three derivatives.

Let $T_n(x)$ stand for the sum of the first $n + 1$ terms of the above series; its derivative $T_n'(x)$ is seen at once not to exceed in numerical value

$$(|a_1| + |b_1|) + 2(|a_2| + |b_2|) + \cdots + n(|a_n| + |b_n|).$$

Furthermore, by integration by parts and use of the periodicity of $f(x)$, we find

$$a_k = -\frac{1}{\pi k}\int_{-\pi}^{+\pi} f'(x) \sin kx dx = -\frac{1}{\pi k^2}\int_{-\pi}^{+\pi} f''(x) \cos kx dx$$

$$= \frac{1}{\pi k^3}\int_{-\pi}^{+\pi} f'''(x) \sin kx dx.$$

Consequently $|a_k|$, and similarly $|b_k|$, is less than $2F_3/k^3$ where F_3 is the maximum of $|f'''(x)|$. We conclude that $|T_n'(x)|$ is not greater than

$$4F_3\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) < 4F_3\left(1+\left(1-\frac{1}{2}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)\right),$$

[1] Professor Birkhoff's earlier paper in this MONTHLY, "Note on certain quadratic number systems for which factorization is unique," appeared almost exactly fifteen years ago. His first published paper, in collaboration with H. S. Vandiver, "On the integral divisors of $a^n - b^n$," was published in *Annals of Mathematics*, 1904. His second paper appeared also in *Annals . . .*, 1905. His third paper was a thirty page memoir in *Transactions of the American Mathematical Society*, 1906; and his fourth paper is the one referred to above. A complete collection of his mathematical papers 1904–1919, 37 in number, is preserved in a bound volume at the mathematical seminary of Brown University.—EDITOR.

so that

$$|T_n'(x)| < 8F_3. \tag{1}$$

The expression $T_n(x)$ has also the properties

$$\left\{\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{+\pi} T_n(x)\cos kx dx, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{+\pi} T_n(x)\sin kx dx, \end{aligned}\right. \tag{2}$$

for $k \leqq n$, as follows by direct integration. Using the equations of definition of a_k, b_k in combination with (2) we deduce at once

$$\int_{-\pi}^{+\pi} R_n(x)\cos kx dx = \int_{-\pi}^{+\pi} R_n(x)\sin kx dx = 0, \qquad (k \leqq n), \tag{3}$$

where $R_n(x)$ denotes the "remainder" $f(x) - T_n(x)$.

Now by means of the familiar product formulas of trigonometry we may express successively $\cos^2 x$, $\cos x \sin x$, $\sin^2 x$, $\cos^3 x$, $\cdots$, $\sin^n x$ as linear combinations of 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$, $\sin nx$. Therefore we find from (3)

$$\int_{-\pi}^{+\pi} R_n(x)\cos^\alpha x \sin^\beta x dx = 0, \qquad (\alpha + \beta \leqq n), \tag{3'}$$

and, noting the equation

$$\cos^2\frac{x - x_0}{2} = \tfrac{1}{2}(1 + \cos x \cos x_0 + \sin x \sin x_0),$$

we infer from (3′)

$$\int_{-\pi}^{+\pi} R_n(x)\cos^{2n}\frac{x - x_0}{2} dx = 0 \tag{4}$$

for any x_0. For convenience we make the change of variables $x = x_0 + t$ and obtain

$$\int_{-\pi}^{+\pi} R_n(x_0 + t)\cos^{2n}\frac{t}{2} dt = 0. \tag{4'}$$

The limits can be made $\pm\,\pi$ since the integrand is periodic of period 2π in t.

By the aid of (4′) we propose to obtain an upper limit for the "remainder" $R_n(x_0)$. We have

$$\begin{aligned} |R_n(x_0 + t) - R_n(x_0)| &\leqq |f(x_0 + t) - f(x_0)| + |T_n(x_0 + t) - T_n(x_0)| \\ &\leqq (F_1 + 8F_3)|t|, \end{aligned}$$

where F_1 stands for the maximum of $|f'(x)|$ and $8F_3$ is at least as great as

$|T_n'(x)|$ by (1). Now the ratio

$$\frac{y}{\sin y} \qquad \left(|y| \leqq \frac{\pi}{2}\right)$$

is positive and may be seen geometrically or otherwise to have its maximum value $\pi/2$ for $y = \pm\,\pi/2$. Therefore it is clear that

$$|t| \leqq \pi\left|\sin\frac{t}{2}\right|, \qquad (|t| \leqq \pi),$$

whence we find

$$(5) \qquad |R_n(x_0 + t) - R_n(x_0)| \leqq N|\sin\tfrac{1}{2}t|, \quad (N = \pi(F_1 + 8F_3)).$$

Thus we may substitute in (4′)

$$(5') \qquad R_n(x_0 + t) = R_n(x_0) + \theta N \sin\tfrac{1}{2}t$$

where $|\theta| \leqq 1$. When we do so there results

$$(6) \qquad R_n(x_0) = -\frac{\displaystyle\int_{-\pi}^{+\pi} \theta N \cos^{2n}\frac{t}{2}\sin\frac{t}{2}\,dt}{\displaystyle\int_{-\pi}^{+\pi}\cos^{2n}\frac{t}{2}\,dt}.$$

The numerator on the right in (6) is not greater numerically than

$$N\int_{-\pi}^{+\pi}\cos^{2n}\frac{t}{2}\left|\sin\frac{t}{2}\right|dt = 2N\int_0^{\pi}\cos^{2n}\frac{t}{2}\sin\frac{t}{2}\,dt = \frac{4N}{2n+1}.$$

Also the familiar formulas

$$\int_0^{\pi/2}\cos^n x\,dx = \begin{cases} \dfrac{1\cdot 3\cdots n-1}{2\cdot 4\cdots n}\cdot\dfrac{\pi}{2} & (n \text{ even}),\\[2ex] \dfrac{2\cdot 4\cdots n-1}{3\cdot 5\cdots n} & (n \text{ odd}),\end{cases}$$

show that

$$\int_0^{\pi/2}\cos^n x\,dx\cdot\int_0^{\pi/2}\cos^{n+1}x\,dx = \frac{\pi}{2(n+1)}.$$

Of the two integrals on the left the first is of course the greater, and it follows that

$$\int_0^{\pi/2}\cos^n x\,dx > \sqrt{\frac{\pi}{2(n+1)}}, \qquad (n = 0, 1, \cdots),$$

and thus that

$$\int_{-\pi}^{+\pi}\cos^{2n}\frac{t}{2}\,dt = 4\int_0^{\pi/2}\cos^{2n}x\,dx > 2\sqrt{\frac{2\pi}{2n+1}}.$$

If we replace the numerator and denominator in (6) by the greater and lesser values respectively which have now been found, the fraction is increased numer-

ically; whence finally

$$|R_n(x)| < N\sqrt{\frac{2}{\pi(2n+1)}} \tag{7}$$

for every value x_0 of x and for every $n \geqq 0$. Thus the remainder approaches 0 as n increases indefinitely, and the Fourier's series converges to $f(x)$.

The method outlined above may easily be modified so as to deal with a more general type of function $f(x)$, for instance any periodic function with two derivatives. Essentially the only novel feature of the method is the direct use of an equation like (4) to obtain an upper limit for the remainder.[1]

A CUBIC SPACE CURVE CONNECTED WITH THE TETRAHEDRON.

By FRANCIS D. MURNAGHAN, Johns Hopkins University.

In connection with the geometry of the triangle there is a well known circumconic, known as Kiepert's hyperbola,[2] which passes through the orthocenter and centroid of the triangle. In the following note an analogue of this hyperbola for the tetrahedron is given, and it is hoped that interest may thereby be aroused in the still comparatively unexplored field of the "geometry of the tetrahedron." [3]

It will be convenient to recall an interesting property of Kiepert's hyperbola—a property which may be used to define the curve and at the same time to give a simple parametric representation of the points on it. Let A_1, A_2, A_3 be the triangle and on each of the segments A_2A_3, A_3A_1, A_1A_2 describe a circle containing an angle θ; θ being in each case the angle subtended by the segment at points of that arc of the corresponding circle which is on the same side of the segment as the opposite vertex, so that θ can vary from 0° to 180°. For any given value of θ the three circles have a radical center and Kiepert's hyperbola is the locus of these radical centers. In order to find the coördinates of the radical center for a given value of θ, it is probably simplest to introduce, momentarily, rectangular axes with origin at O_1, the mid-point of A_2A_3, and with the X axis along A_2A_3. If $2a_1$ is the length of A_2A_3, the coördinates of C_1, the center of the circle through A_2A_3, are $(0, a_1 \cot\theta)$ and the radius is $a_1 \operatorname{cosec}\theta$, so that the equation is

[1] See Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1906, pp. 37–38, and de la Vallée Poussin, *Bulletins de l'Académie royale de Belgique*, classe des sciences, 1908, pp. 193–254, in particular p. 230 *et seq.*

[2] Kiepert, *Nouvelles Annales de Mathématiques*, 1869, p. 42. Other discussions of this hyperbola are: by Brocard, in *Journal de Mathématiques Spéciales*, 1884, pp. 197–209 and 1885, pp. 12–15, 30–33, 58–64, 76–80, 104–112, 123–131; by de Longchamps, *idem*, 1886, pp. 77–79 231–235; by M'Cay, in *Mathesis*, 1887, pp. 208–220; by Laisant, in *Compte rendu . . . Association Française pour l'Avancement des Sciences*, seconde partie, 1887, pp. 113–114; by J. Casey, in his *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections*, second edition, 1893, pp. 431, 442–445, 449, 452, 453; and by I. J. Schwatt, in his *A geometrical Treatment of Curves which are isogonal conjugate to a straight line with respect to a triangle.* Part 1, Boston, New York, and Chicago, [c. 1895], pp. 3–28.—EDITOR.

[3] Those interested may be referred to two papers by J. Neuberg in *Archiv der Mathematik und Physik*, 3. Reihe, vols. 16 and 18, 1910–1911.

his illustrious brother and survived him by nine years. He was well and favorably known both as a mathematician and as a hydrographer. He was attached to the ill-fated voyage of discovery led by La Pérouse (La Peyrouse), but fortunately was compelled to abandon the trip owing to illness which developed at Teneriffe. He became professor of mathematics in the Ecole militaire, and later became professor and examiner in hydrography in the navy. Thanks to his brother's influence, the latter having a position as minister in the revolutionary period, he passed safely through the Days of Terror, and in 1797, when the worst was over, he occupied the position which the letter indicates.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

The discussion by Professor Moritz on geometric illustrations of indeterminate forms should prove interesting and valuable. It is not superfluous to take this occasion to point out the extraordinary misunderstanding and logical inaccuracy often associated with the subject of indeterminate forms. Errors are common in our textbooks, and some teachers have never become acquainted with the correct point of view. It will be necessary first to touch upon the ideas of zero and infinity, which are scarcely less hazy.

In the first place, it should be clearly understood that the number system which is agreed upon by mathematicians as best representing the practical needs of analysis—irrespective of the obvious possibility of other self-consistent number systems, whose properties may agree to a greater or less extent with those of the usual system—does contain a number *zero*, as real, with as definite a meaning as any other number, and having, with one exception, properties like those of any other number. This ordinary number system, on the other hand, does *not* contain a number *infinity*. If any one prefers a system which shall not contain zero or which shall contain infinity, then, so long as he formulates self-consistent laws of operation, he is entirely free to use his system; but he cannot expect mathematicians generally to follow his lead unless he can show that his system involves fewer exceptions or inconveniences than that to which they are accustomed.

If we are to use the ordinary system, we shall have no difficulty provided we take account of the fact that its laws are uniform and general, except in one single particular: the number zero cannot be used as a divisor. The expression 2/0 does not represent some mysterious unattainable large number; it represents no number whatever. The expression 0/0 does not represent several, many, or all numbers[1]; it represents no number whatever.

[1] A logical system might be set up in which 0/0 would have this multiple-valued interpretation; such a system would apparently be about as convenient as the usual one; but it is not the usual one.

In what way, then, is the statement

$$\frac{2}{0} = \infty$$

to be interpreted? Strictly, in an arithmetical sense, it signifies merely that the operation $2/0$ is meaningless;—that the "result" of the operation is of one kind with the unicorn and the sea serpent. It is also used in a sense not exclusively arithmetical, generally with some sacrifice in clearness, to mean that "as x approaches zero, $2/x$ becomes infinite," or, in other words, "$2/x$ can be made as large as we please by taking x sufficiently small." It should be noted that even this interpretation implies no division by zero, but does imply division by many other numbers,—numbers near zero.

When we consider the fraction $(x^2 - 1)/(x - 1)$ for the value $x = 1$, we find that both numerator and denominator are entirely respectable numbers,—both are in fact zero; the fraction however is meaningless, since division by zero is a meaningless operation. When on the other hand we consider the fraction $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ for $x = 1$, we find that even the numerator and denominator individually are meaningless.

But in both cases, when we speak of "evaluating the indeterminate form," we mean (or ought to mean) something which is unfortunately quite different from the literal implication of the words. We do not mean that we are to discover, by astute detective work, a "true value" which unkind fate is endeavoring to conceal from us. It is in no sense true that $(x^2 - 1)/(x - 1)$, which for $x = 1$ assumes the form $0/0$, really takes on for $x = 1$ the "true value" or "determinate value" 2. The value of $(x^2 - 1)/(x - 1)$ for $x = 1$ is not 2, does not exist, and cannot be brought into existence unless some other than the usual number system be used. What we can do, and what is often useful for us to do, is to find the limit approached by $(x^2 - 1)/(x - 1)$ as x approaches 1; this limit is indeed 2. If we write, as do some text-books:

$$\text{“(1)} \quad \frac{x^2 - 1}{x - 1} = x + 1;$$

hence

$$\text{(2)} \quad \frac{x^2 - 1}{x - 1} = 2 \qquad \text{when} \qquad x = 1,\text{”}$$

we are merely confusing the issue by a bald untruth. The following statement is true, and is the most that can be said without departing from the truth:

$$\text{“(1)} \quad \frac{x^2 - 1}{x - 1} = x + 1 \qquad \text{when} \qquad x \neq 1;$$

hence

$$\text{(2)} \quad \frac{x^2 - 1}{x - 1} \text{ approaches 2 as } x \text{ approaches 1.”}$$

As indicated, (1) is true for $x \neq 1$. It is untrue for $x = 1$; the algebraic

operation by which it was obtained—formal division of the numerator by the denominator—cannot be performed when $x = 1$, although it can be performed for every other value of x.

Similarly, when we set out to "evaluate the indeterminate form $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ for $x = 1$," we must not think that we shall by sharpness of vision pierce an algebraic mist which hides from us the "true value" 1 of this expression for $x = 1$. We should mean that we are seeking the limit (if any exists) approached by $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ as x approaches 1; and we should mean nothing further than this.

For the value $x = 1$, $(x^2 - 1)/(x - 1)$ does assume the form $0/0$, and $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ may be said to assume the form ∞/∞; but neither fraction has any *value* for the value of x in question, nor can we by any amount of ambiguous argument or inaccurate analogy force it to have such a value.

On The Geometrical Representation of Indeterminate Forms.

By R. E. Moritz, University of Washington.

Every teacher of the calculus will have witnessed the difficulty beginners have in properly interpreting the so-called indeterminate forms $0/0$, ∞/∞, $0\cdot\infty$, $\infty - \infty$, etc. This should cause no surprise; for the meaning of the symbols 0 and ∞, even when taken by themselves, is not too clear to the average student and the difficulty is, of course, accentuated when these symbols are combined as in the indeterminate forms.

It should be remembered in this connection that the indeterminate forms have not always been correctly interpreted even by mathematicians and that some of them remained subjects of controversy until comparatively recent times. The first complete evaluation of the form $0/0$ dates from John Bernoulli[1] (1704) and it was not until 1755 that Euler[2] gave the first published treatment of the forms ∞/∞, $0\cdot\infty$, and $\infty - \infty$. But the true meaning of 0^0 eluded even the great Euler, for he arrived at the conclusion[3] that 0^0 must always equal 1 by a process of reasoning which would show equally well that $0/0$ must always equal 1. Euler's erroneous conclusion was shared by many subsequent mathematicians for three quarters of a century. New proofs (!) of his conclusion appeared in creditable journals by eminent mathematicians as late as the middle third of the nineteenth century,[4] and this notwithstanding the great weight of the authority of Cauchy, who, in his *Cours d'Analyse*,[5] had given the interpretation of the exponential indeterminate forms which is universally accepted today.

In view of the fact that modern textbooks on the calculus generally avail themselves of geometrical illustrations to introduce new or difficult concepts, it

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[1] *Opera*, vol. 1, pp. 401–405.

[2] *Opera Omnia*, series I, vol. 10, 1913, p. 576.

[3] Euler, *Opera Omnia*, series I, vol. 1, 1911, p. 65.

[4] Libri, G., Crelle's *Journal*, vol. 10, 1833, p. 303; Moebius, A. F., Crelle's *Journal*, vol. 12, 1834, p. 134.

[5] *Oeuvres*, 2d series, vol. 3, p. 70.

seems unfortunate that such aids are not used in introducing the concept of indeterminate forms. In this connection geometrical illustrations are so conspicuously absent that the cynically inclined reader might suspect writers of a conspiracy to mystify rather than to clarify the subject under discussion. With one notable exception[1] in the case of a text long out of use (and now out of print) the writer does not know of a single textbook in either English, French, or German, which so much as hints at the existence of geometrical illustrations in treating the subject of indeterminate forms, although every modern text, so far as the writer is aware, freely uses geometrical aids in introducing the concepts of the derivative and the definite integral.

Lovitt,[2] in a short paper on the subject, gave five illustrations of indeterminate forms chosen from the fields of solid geometry, trigonometry, the geometry of the triangle, the geometry of the circle, and hydraulics respectively. These illustrations lack, however, the desired perspicuity and, in the majority of cases, are faulty in two respects: first, in that the variables actually reach their limits, thus rendering the expressions which give rise to the indeterminate forms meaningless; and second, in that the ratio, product, or difference which assumes the indeterminate form is constant and thus fails to illustrate the really important case of a function of x which approaches a limiting value but an indeterminate form as the variable approaches some definite limit.

It is the purpose of the present paper to exhibit some simple plane-geometric illustrations of the elementary indeterminate forms. Care has been exercised that in each case the function which in the limit assumes an indeterminate form is really a function of the variable assuming different values and not a constant in disguise. Moreover, the constructions are so conditioned that the question of dividing zero by zero, or infinity by infinity, cannot arise since the quantities considered are never zero or infinite. All our ratios are those of finite quantities which may be increased or decreased indefinitely but which, from the nature of their generation, can never be actually zero or infinite. The point here is that the problem of indeterminate forms may be so conceived that the question of dividing zero by zero, or infinity by infinity, which is the real source of confusion, need not arise.

Each of the figures that follow is so chosen that it illustrates at the same time each of the possible values zero, finite, infinite, which the indeterminate form under consideration may take. The last figure illustrates all the elementary indeterminate forms at once for the whole range of values which any of these forms may have. This figure is remarkable in that it interrelates one each of the four possible types of triangles, the ultimate values of whose sides are respectively

$$0, 0, 0; \qquad 0, a, b; \qquad a, \infty, \infty; \qquad \infty, \infty, \infty.$$

It would be interesting to know whether it is possible to construct a simpler figure that will serve the same purpose equally well or better.

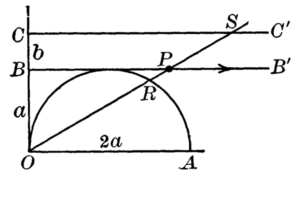
[1] DeMorgan, A., *Differential and Integral Calculus*, Introductory, p. 19.

[2] AMERICAN MATHEMATICAL MONTHLY, *1916*, 41.

Definitions. A variable distance is said to approach 0 as a limit, if it ultimately becomes, and remains, less than any preassigned distance however small.

A variable distance is said to approach a constant distance a as a limit if the difference between the constant distance and the variable distance ultimately becomes, and remains, less than any preassigned distance however small.

A variable distance is said to become infinite if it ultimately becomes, and remains, greater than any preassigned distance however large.



1. $\infty - \infty$. Consider the semicircle having $OA = 2a$ as a diameter. BB' is a tangent to the semicircle drawn parallel to OA, and CC' a line drawn parallel to BB' at a distance b from it on the side opposite from 0.

Let P be a point on BB' which moves indefinitely in the direction BB' at a uniform velocity. Join O to P and produce OP to meet CC' at S.

Now it is obvious that each of the variable distances OP, BP, RP, OS, ultimately becomes, and remains, larger than any preassigned distance however large, so that ultimately

$$OP - BP = \infty - \infty, \qquad OP - RP = \infty - \infty, \qquad OS - OP = \infty - \infty.$$

But

$$OP - BP = \frac{a^2}{OP + BP}, \text{ which approaches 0 as a limit,}$$

$$OP - RP = OR, \text{ which approaches } OA = 2a \text{ as a limit,}$$

$$OS - OP = PS, \text{ which becomes infinite;}$$

hence

$$OP \to \infty, \qquad BP \to \infty, \qquad RP \to \infty, \qquad OS \to \infty;$$

$$OP - BP \to 0, \qquad OP - RP \to 2a, \qquad OS - OP \to \infty.$$

If we take O as origin of coördinates, OA and OB as the directions of coördinate axes, and denote the coördinates of the point P by (x, a), we find

$$OP = \sqrt{x^2 + a^2}, \quad BP = x, \quad RP = (x - a)^2/\sqrt{x^2 + a^2}, \quad OS = \frac{a + b}{a}\sqrt{x^2 + a^2},$$

and as x becomes infinite we have

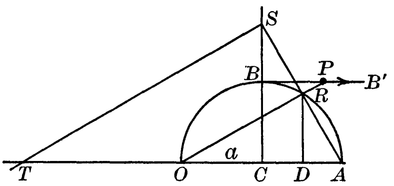
$$OP - BP = \sqrt{x^2 + a^2} - x = a^2/[\sqrt{x^2 + a^2} + x] \to 0,$$

$$OP - RP = \sqrt{x^2 + a^2} - (x - a)^2/\sqrt{x^2 + a^2} = 2ax/\sqrt{x^2 + a^2} \to 2a,$$

$$OS - OP = \frac{a + b}{a}\sqrt{x^2 + a^2} - \sqrt{x^2 + a^2} = \frac{b}{a}\sqrt{x^2 + a^2} \to \infty,$$

which verifies the preceding results analytically.

2. 0/0 and ∞/∞. Consider the semicircle having $OA = 2a$ as a diameter. BB' is the tangent to the semicircle drawn parallel to OA, P a point on it which moves indefinitely in the direction BB' at a uniform velocity.



Join P to O and denote the point in which OP intersects the semicircle by R. Join A and R and denote the point in which AR produced intersects the perpendicular CB to OA through the center C of the circle by S. Through S draw a perpendicular to AS and let this perpendicular intersect AO produced in T. Through R draw RD perpendicular to OA.

Now it is obvious that as P moves indefinitely along BB' in the direction of BB' each of the variable distances DA, AR, and RD ultimately becomes, and remains, less than any preassigned distance however small, and each of the variable distances CS, ST, and TC ultimately becomes, and remains, greater than any preassigned distance, however large, so that ultimately

$$\frac{DA}{RD}=\frac{0}{0},\qquad \frac{AR}{RD}=\frac{0}{0},\qquad \frac{RD}{DA}=\frac{0}{0};\qquad \frac{CS}{TC}=\frac{\infty}{\infty},\qquad \frac{ST}{TC}=\frac{\infty}{\infty},\qquad \frac{TC}{CS}=\frac{\infty}{\infty}.$$

But the triangles RDA, SCA, ODR, and TCS are similar, so that

$$\frac{DA}{RD}=\frac{CS}{TC}=\frac{DR}{OD},\ \text{which approaches 0 as a limit,}$$

$$\frac{AR}{RD}=\frac{ST}{TC}=\frac{RO}{OD},\ \text{which approaches 1 as a limit,}$$

$$\frac{RD}{DA}=\frac{TC}{CS}=\frac{SC}{CA},\ \text{which becomes infinite;}$$

hence

$$DA\rightarrow 0,\qquad RD\rightarrow 0,\qquad AR\rightarrow 0,\qquad CS\rightarrow\infty,\qquad TC\rightarrow\infty,\qquad ST\rightarrow\infty;$$

$$\frac{DA}{RD}\rightarrow 0,\qquad \frac{CS}{TC}\rightarrow 0,\tag{1}$$

$$\frac{AR}{RD}\rightarrow 1,\qquad \frac{ST}{TC}\rightarrow 1,\tag{2}$$

$$\frac{RD}{DA}\rightarrow\infty,\qquad \frac{TC}{CS}\rightarrow\infty.\tag{3}$$

We may obtain the same results analytically as follows. Take O as origin and OA as the direction of the x-axis of coördinates. Denote the coördinates of the point P by (x, a), then we find

$$DA = 2a^3/(x^2+a^2),\qquad AR = 2a^2/(\sqrt{x^2+a^2}),\qquad RD = 2a^2x/(x^2+a^2),$$

$$CS = x,\qquad ST = x\sqrt{x^2+a^2}/a,\qquad TC = x^2/a.$$

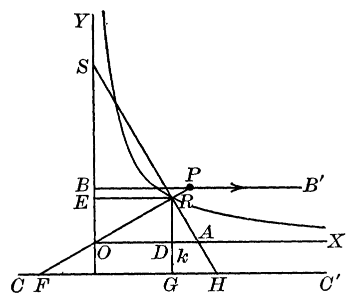
Substituting these values in the left members of (1), (2), (3), we have as x becomes infinite

$$\frac{DA}{RD} = \frac{2a^3/(x^2+a^2)}{2a^2x/(x^2+a^2)} = \frac{a}{x} \to 0, \qquad \frac{CS}{TC} = \frac{x}{x^2/a} = \frac{a}{x} \to 0;$$

$$\frac{AR}{RD} = \frac{2a^2/(\sqrt{x^2+a^2})}{2a^2x/(x^2+a^2)} = \frac{\sqrt{x^2+a^2}}{x} \to 1, \qquad \frac{ST}{TC} = \frac{x\sqrt{x^2+a^2}/a}{x^2/a} = \frac{\sqrt{x^2+a^2}}{x} \to 1;$$

$$\frac{RD}{DA} = \frac{2a^2x/(x^2+a^2)}{2a^3/(x^2+a^2)} = \frac{x}{a} \to \infty, \qquad \frac{TC}{CS} = \frac{x^2/a}{x} = \frac{x}{a} \to \infty.$$

3. $\infty \cdot 0$. Consider a branch of the equilateral hyperbola with $\sqrt{2}$ for the semi-major axis. Let O represent the center and OX, OY, the asymptotes to the hyperbola. Draw two parallels, BB' and CC', to one of the asymptotes on opposite sides of it and at distances a and k respectively.



Let P be a point on BB' which moves indefinitely in the direction of BB' at a uniform velocity. Join OP and denote its intersection with the hyperbola by R and its intersection with CC' by F. Through R draw a perpendicular to OR and denote its intersection points with OY, OX, and CC', by S, A, and H respectively. From R draw a perpendicular to OX, intersecting OX and CC' in D and G respectively. Through R draw a parallel to OX intersecting OY at a point E.

As the point P moves indefinitely in the direction BB' it is obvious that each of the variable distances FG, OD, ER, ES, ultimately becomes, and remains, greater than any preassigned distance however great, and each of the variable distances OE, DR, DA, GH, ultimately becomes, and remains, less than any preassigned distance however small, while GR approaches $GD = k$ as a limit; hence ultimately

$$OD \cdot DA = \infty \cdot 0, \qquad FG \cdot GH = \infty \cdot 0, \qquad ES \cdot OE = \infty \cdot 0.$$

But from the pairs of similar triangles ODR,RDA, FGR,RGH, SER,REO, we have

$$OD \cdot DA = (DR)^2, \text{ which approaches 0 as a limit,}$$

$$FG \cdot GH = (GR)^2, \text{ which approaches } k^2 \text{ as a limit,}$$

$$ES \cdot OE = (ER)^2, \text{ which becomes infinite;}$$

hence

$$OD \to \infty, \quad FG \to \infty, \quad ES \to \infty; \quad DA \to 0, \quad GH \to 0, \quad OE \to 0;$$

$$OD \cdot DA \to 0, \tag{4}$$

$$FG \cdot GH \to k^2, \tag{5}$$

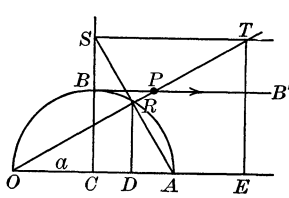
$$ES \cdot OE \to \infty. \tag{6}$$

To verify these results analytically, take O as the origin, OX as the x-axis of coördinates. Then, if we denote the coördinates of the point P by (x, a), we find

$$OD = \sqrt{(x/a)}, \qquad FG = (x/a)[k + \sqrt{(a/x)}], \qquad ES = \sqrt{(x/a)^3},$$
$$DA = \sqrt{(a/x)^3}, \qquad GH = (a/x)[k + \sqrt{(a/x)}], \qquad OE = \sqrt{(a/x)}.$$

On substituting these values in the left members of (4), (5), (6), we have as x becomes infinite,

$$OD \cdot DA = \sqrt{(x/a)} \cdot \sqrt{(a/x)^3} = a/x \to 0,$$
$$FG \cdot GH = (a/x)[k + \sqrt{(a/x)}] \cdot (x/a)[k + \sqrt{(a/x)}] = [k + \sqrt{(a/x)}]^2 \to k^2,$$
$$ES \cdot OE = \sqrt{(x/a)^3} \cdot \sqrt{(a/x)} = x/a \to \infty.$$



4. The figure illustrates all the preceding cases of indeterminate forms at the same time. The figure is like the one in § 2 with two additional lines, a line through S drawn parallel to OA and a perpendicular to OA through T, the point in which this line intersects OP produced.

As the point P moves indefinitely with a uniform velocity in the direction BB' the variable sides of the similar right triangles

$$RAD, \qquad OAR, \qquad SAC, \qquad TSR,$$

ultimately become or approach

$$0, 0, 0; \qquad 0, 2a, 2a; \qquad a, \infty, \infty; \qquad \infty, \infty, \infty.$$

We have then

$$AS - CS = \infty - \infty, \qquad OT - RT = \infty - \infty, \qquad OT - PT = \infty - \infty;$$
$$RS/ST = \infty/\infty, \qquad ST/RT = \infty/\infty, \qquad RT/RS = \infty/\infty;$$
$$CS \cdot DA = \infty \cdot 0, \qquad CS \cdot DR = \infty \cdot 0, \qquad RT \cdot AR = \infty \cdot 0;$$
$$DA/DR = 0/0, \qquad AR/DR = 0/0, \qquad DR/DA = 0/0.$$

Again it appears from the obvious geometrical relations that

$$AS - CS = (CA)^2/(AS + CS), \qquad RS/ST = AR/OA, \qquad CS \cdot DA = DR \cdot CA,$$
$$DA/DR = AR/OR,$$

the right member of each of which approaches 0 as a limit;

$$OT - RT = OR, \quad ST/RT = OA/OR, \quad CS \cdot DR = CA \cdot OD, \quad AR/DR = OA/OR,$$

and the right members of these approach the limits $2a$, 1, $2a^2$, 1, respectively;

finally

$$OT - PT = OP, \quad RT/RS = CS/CA, \quad RT \cdot AR = RS \cdot OR, \quad DR/DA = CS/CA,$$

the right member of each of which becomes infinite.

We have therefore as the limits of the ratios involved

$$\begin{array}{rrr} AS - CS \to 0, & OT - RT \to 2a, & OT - PT \to \infty; \\ RS/ST \to 0, & ST/RT \to 1, & RT/RS \to \infty; \\ CS \cdot DA \to 0, & CS \cdot DR \to 2a^2, & RT \cdot AR \to \infty; \\ DA/DR \to 0, & AR/DR \to 1, & DR/DA \to \infty. \end{array}$$

RECENT PUBLICATIONS.

REVIEWS.

Elementary Calculus. By W. F. Osgood. New York, The Macmillan Company, 1921. 12mo. 10 + 224 pages. Price $2.40.

Preface: "The object of this book is to present the elements of the Differential Calculus in a form easily accessible for the undergraduate. It is possible, from the very beginning, to illustrate the ideas and methods of the Calculus by means of applications to physics and geometry, which the student can readily grasp, and which will seem to him of interest and value. To do this, the stress in the illustrative examples worked in the text must be laid first of all on the thought which underlies the method of solution, in distinction from the exposition of a process, reduced in the worst teaching to rules, whereby the answer can be obtained. The treatment of maxima and minima, Chapter III, §§ 2, 3, and curve tracing, Chapter III, § 5, and Chapter VII, § 10, will serve to show what is here meant.

"It is, however, also essential that the student receive thorough training in the formal processes and the technique of the Calculus, and this side has been treated with care and completeness. Note, for example, the differentiation of composite functions in Chapter II, § 8, and the exposition of the use of differentials in differentiating in Chapter IV, §§ 4, 5.

"An important application of the graphical methods, with which the Calculus is so intimately associated, is that of solving approximately numerical equations which do not come under the standard rules of algebra and trigonometry. Hitherto, however, little attempt has been made to present this subject, simple as it is, in any systematic and elementary manner. In Chapter VII the common methods in use by physicists and others who apply the Calculus are set forth and illustrated by simple examples.

"The book might have included a brief treatment of curvature and evolutes, and the cycloid. But probably most teachers of the Calculus will prefer to take up integration next, and so the closing chapter is devoted to the last of the elementary functions, the inverse trigonometric functions, with special reference to their one great application in the elements of mathematics, namely, their application to integration.

"The book is so written that it can be adapted, if desired, to an abridged course, in which, after the fundamentals of the first three chapters have been covered, any of the remaining topics can be treated briefly, and thus a wide scope in subject matter is possible, even when the time is short."

Contents—Chapter I: Introduction, 1–12; II: Differentiation of algebraic functions. General theorems, 13–45; III: Applications, 46–80; IV: Infinitesimals and differentials, 81–104; V: Trigonometric functions, 105–145; VI: Logarithms and exponentials, 146–165; VII: Applications, 166–205; VIII: The inverse trigonometric functions, 206–224.

The Dynamics of the Airplane. (Mathematical Monographs, No. 21). By KENNETH P. WILLIAMS, Associate professor of mathematics, Indiana University. New York, John Wiley & Sons, 1921. 8vo. 8 + 138 pp. Price $2.50.

Preface: "It was the good fortune of the author to attend the University of Paris during the spring semester of 1919. One of the special courses which the French authorities, with their characteristic hospitality, arranged for the large number of students from the American army, was a course in aërodynamics, given by Professor Marchis. The comprehensive knowledge that Professor Marchis possessed of all branches of the new science of aëronautics, the inestimable value of his advice to the French Republic during the war, the interest he took in his rather unusual class, could not fail to be an inspiration.

"This book is an outgrowth of those parts of Professor Marchis' lectures that were of particular interest to the author. It is in no sense a complete treatise on aviation. Questions of design and construction are passed over with bare mention. The book is intended for students of mathematics and physics who are attracted by the dynamical aspect of aviation. The problems presented by the motion of an airplane are novel and fascinating. They vary from the most pleasing simplicity to the most stimulating difficulty. The question of stability, particularly, exhibits at the same time the elegance and the power of analysis, and shows the adaptability of some of the general developments in dynamics. The field is assuredly a fruitful one of study, and increasing demands will be put upon the mathematician as the science of aviation continues its rapid development. The mathematician can well own a sense of pride that he had already at hand, in the developments inaugurated by Euler and Routh, a means of dealing accurately with the question of stability, that plays so fundamental a rôle in the science of flying.

"The treatment in the text is for the most part elementary. The last chapter alone demands of the student familiarity with more advanced dynamical methods. In the treatment of descent a slight digression is made to consider in part the nature of the solution of a system of two differential equations. This was done in order not to completely evade what seems a problem of considerable difficulty. It might seem that a treatment of the propeller should not find a place in a book with the purpose of this one. No student of mathematics, however, could fail to own a curiosity as to a propeller's action, and it is hoped the discussion, while not complete, will at least serve as a sufficient introduction."

Contents—Chapter I: The plane and cambered surface, 1–17; II: Straight horizontal flight, 18–30; III: Descent and ascent, 31–51; IV: Circular flight, 52–68; V: The propeller, 69–85; VI: Performance, 86–93; VII: Stability and controllability, 94–106; VIII: Stability (continued), 107–129; Appendix, 131–136; Index, 137–138.

NOTES.

It is announced that John Wallis's *Arithmetica Infinitorum,* translated and edited by J. M. Child, is soon to be published by The Open Court Company, London—The second edition of Hobson's *The Theory of Functions of a Real Variable* is to be in two large volumes. The first volume appeared early in this year (Cambridge University Press; 16 + 671 pages; price 45 shillings).

At the meeting of the Academy of Sciences of the Institute of France last December, several sums of money from the Loutreuil Foundation were voted to assist scientific publications. Among them was 3000 francs to T. Lemoyne and H. Brocard "for the publication of the second and third volumes of their work *Courbes géométriques remarquables (courbes spéciales) planes et gauches*" (*1920,* 130).

The MONTHLY draws the attention of mathematicians to the excessive charges for publications of the Cambridge University Press made by the present American agent, The Macmillan Company. A single illustration will suffice.

The third edition of Whittaker and Watson's *A Course of Modern Analysis* was published at 40 shillings (*1921*, 31); the American agent's price is \$12.50. Hence by ordering from a London bookseller, and paying the duty, a saving of at least \$3.00 on the purchase of this single volume could be effected.—There is no duty for books ordered for college and public libraries. The American Branch of the Oxford University Press appears to count more definitely on the ignorance of purchasers in the United States. To illustrate: H. Hilton's *Plane Algebraic Curves*, 1920, was published at 28 shillings (about \$5.60 at the present rate of exchange); the price of the American Branch is \$12.60!

The second and concluding number of *Mathematische Annalen*, volume 80, is to be a Generalregister of volumes 51–80. The Generalregister, volumes 1–50 (11 + 202 pp.), was published in 1898.

Julius Springer, of Berlin, published early in 1921, the first volume (price 186 marks, "Zuschlagfrei," 12 + 612 pages + portrait) of Felix Klein's *Gesammelte Mathematische Abhandlungen*, "herausgegeben von R. FRICKE und A. OSTROWSKI (von F. Klein mit ergänzenden Zusätzen versehen)." The volume contains: "Liniengeometrie, Grundlegung der Geometrie zum Erlanger Programm."

In *Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 29, nos. 1–6, September, 1920, we find (pp. 28–40) "Ein Beitrag zur Lebensbeschreibung von L. Fuchs" by L. SCHLESINGER, who has for years been collecting material for a life of FUCHS. Schlesinger refers to LEO KÖNIGSBERGER'S book, *Mein Leben* (Heidelberg, 1919), where there are various "assertions and judgments concerning L. Fuchs which do not in any wise correspond to the picture which I bear in my heart of my unforgettable teacher, and fatherly friend." In support of his contentions Schlesinger publishes several letters written by Weierstrass in 1875.

We have already referred (*1921*, 32) to the first of a series of memoirs by G. H. HARDY and J. E. LITTLEWOOD, on "Some problems of 'partitio numerorum'" published in the Göttingen *Nachrichten*, 1920, pp. 33–54. The second memoir was published January 31, 1921, in *Mathematische Zeitschrift*, volume 9, nos. 1–2, pp. 14–27, and deals with the "Proof that every large number is the sum of at most 21 biquadrates." The same issue of the *Zeitschrift* contains an article on "Congruence properties of partitions" by S. Ramanujan (see *1920*, 338). This is extracted by G. H. Hardy from one of the author's manuscripts which "contains a large number of further results. It is very incomplete, and will require very careful editing before it can be published in full. I have taken from it," Hardy writes, "the three simplest and most striking results, as a short but characteristic example of the work of a man who was beyond question one of the most remarkable mathematicians of his time."

Nature for February 17, 1921, was a special number on relativity, and contained the following articles, pages 782–811: "A brief outline of the development

of the theory of relativity" by A. EINSTEIN; "Relativity: the growth of an idea" by E. CUNNINGHAM; "Relativity and the eclipse observations of May, 1919" by F. DYSON; "Relativity and the motion of Mercury's perihelion" by A. C. D. CROMMELIN; "The displacement of solar lines" by C. E. ST. JOHN; "Non-Euclidean geometries" by G. B. MATHEWS; "The general physical theory of relativity" by J. H. JEANS; "The Michelson-Morley experiment and the dimensions of moving bodies" by H. A. LORENTZ; "The geometrisation of physics, and its supposed basis on the Michelson-Morley experiment" by O. LODGE; "Electricity and gravitation" by H. WEYL; "The relativity of time" by A. S. EDDINGTON; "Theory and experiment in relativity" by N. CAMPBELL; "The relation between Geometry and Einstein's theory of gravitation" by DOROTHY WRINCH and H. JEFFREYS; "The metaphysical aspects of relativity" by H. W. CARR. The "Bibliography of relativity" (pages 811–813) lists (1) 83 books and pamphlets (among the names of authors are: F. S. WOODS, E. V. HUNTINGTON, R. D. CARMICHAEL, L. SILBERSTEIN, R. C. TOLMAN and E. E. SLOSSON); and (2) the articles on the subject, which have appeared in *Nature*. These were mainly selected from a bibliography of about 650 titles for the period 1886–1920, compiled by H. F. Morley, director of the *International Catalogue of Scientific Literature*.—This special number of *Nature* was in great demand and was entirely out of print on March 17.

A new part of the great *New English Dictionary on Historical Principles*, Visor-Vywer, was issued from the Clarendon Press in March, 1920 (see *1919*, 256–257, *1920*, 128–129). The words in the section of special interest to the mathematician are: viss, volume and vortex (and their derived words and compounds), vorty, vulcan, vulgar fractions, and vulpecula—Viss is a weight used in southern India and Burmah equal to about three and one half pounds; it has appeared in English publications since 1588—Of the article on volume, paragraphs 8–9 give illustrations of the use of the word since 1792 as "the bulk, size, or dimensions of a thing," "the mass or solid body of something," "the amount or quantity of something," and "without article: bulk, mass, dimensions." Volumenometer, volumometer, volumenometry and volumetry are illustrations of derived words—Vortex has been used since its introduction in older theories of the universe, especially that of Descartes. Derived words are vorticose, vorticular and vortiginous—Vorty, a "south-western dialect form for forty," occurs in seventeenth century literature—Vulcan a hypothetical planet supposed to have its orbit between the sun and mercury—The earliest quotation for the term "vulgar fractions" is in Jeake's *Arithmetic*, 1696. The obsolete expression "vulgar arithmetic" was employed in the seventeenth and eighteenth centuries —Vulpecula is a small northern constellation lying between Hercules and Pegasus.

Among universities of America which have recently published complete lists of those on whom the degree of doctor of philosophy, with mathematics as a major, has been conferred are: (*a*) The University of Chicago, (*b*) Harvard University, and (*c*) The Johns Hopkins University.

(*a*) This list is in *Circular of the Departments of Mathematics, Astronomy and Astrophysics, Physics, Chemistry, 1920.* There have been 85 mathematical doctors, 1896–1919, L. E. Dickson and J. I. Hutchinson being the first. With mathematical astronomy as a major, there were 11 more, 1900–1915. No indication is given as to whether or not the dissertations have been published, although it is known that this occurred in most cases.

(*b*) This list is to be found in *Official Register of Harvard University, Division of Mathematics, 1920–1921.* There were 47 doctors in mathematics, 1873–1919, the first being W. E. Byerly. Indications are given of the place of publication of 32 of the dissertations.

(*c*) *Doctors' Dissertations 1878–1919* is the title of *The Johns Hopkins University Circular,* new series, 1920, no. 1. In mathematics there were 69 doctors, 1878–1918, the first being Thomas Craig; in astronomy there were 5, 1891–1898. Indications are given regarding the publication of all of the dissertations except for 14 in mathematics and 2 in astronomy. Of these, 24 were published in the *American Journal of Mathematics.*

Since 1896, according to the above mentioned lists for mathematics, Harvard has had 41 doctors and Johns Hopkins, 46; so that during the period 1896–1919 the University of Chicago has conferred almost as many of such doctorates as Harvard and Johns Hopkins together.

The following extracts are from John Dewey's *Reconstruction in Philosophy* (New York, Holt, 1920).

"Mathematics is often cited as an example of purely normative thinking dependent upon *a priori* canons and supra-empirical material. But it is hard to see how the student who approaches the matter historically can avoid the conclusion that the status of mathematics is as empirical as that of metallurgy. Men began with counting and measuring things just as they began with pounding and burning them. One thing, as common speech profoundly has it, led to another. Certain ways were successful—not merely in the immediately practical sense, but in the sense of being interesting, of arousing attention, of exciting attempts at improvement. The present-day mathematical logician may present the structure of mathematics as if it had sprung all at once from the brain of a Zeus whose anatomy is that of pure logic. But, nevertheless, this very structure is a product of long historic growth, in which all kinds of experiments have been tried, in which some men have struck out in this direction and some in that, and in which some exercises and operations have resulted in confusion and others in triumphant clarifications and fruitful growths; a history in which matter and methods have been constantly selected and worked over on the basis of empirical success and failure.

"The structure of alleged normative *a priori* mathematics is in truth the crowned result of ages of toilsome experience. The metallurgist who should write on the most highly developed method of dealing with ores would not, in truth, proceed any differently. He too selects, refines, and organizes the methods which in the past have been found to yield the maximum of achievement. Logic is a matter of profound human importance precisely because it is empirically founded and experimentally applied. So considered, the problem of logical theory is none other than the problem of the possibility of the development and employment of intelligent method in inquiries concerned with deliberate reconstruction of experience. And it is only saying again in more specific form what has been said in general form to add that while such a logic has been developed in respect to mathematics and physical science, intelligent method, logic, is still far to seek in moral and political affairs" (pages 137–138). "Such a deductive science as mathematics represents the perfecting of method. That a method to those concerned with it should present itself as an end on its own account is no more surprising than that there should be a distinct business for making any tool. Rarely are those who invent and perfect a tool those who employ it. There is,

indeed, one marked difference between the physical and the intellectual instrumentality. The development of the latter runs far beyond any immediately visible use. The artistic interest in perfecting the method by itself is strong—as the utensils of civilization may themselves become works of finest art. But from the practical standpoint this difference shows that the advantage as an instrumentality is on the side of the intellectual tool. Just because it is not formed with a special application in mind, because it is a highly generalized tool, it is the more flexible in adaptation to unforeseen uses. It can be employed in dealing with problems that were not anticipated. The mind is prepared in advance for all sorts of intellectual emergencies, and when the new problem occurs it does not have to wait till it can get a special instrument ready" (page 149).

ARTICLES IN CURRENT PERIODICALS.

American Oxonian, Concord, N. H., volume 8, no. 1, January, 1921: "The record of the American Rhodes Scholars" by R. W. Burgess, 1–36 [Of the 351 American Rhodes Scholars at Oxford, classes matriculating 1904–1914, 32.7 percent studied law; 17.1 percent modern history and economics; 16.8 percent humanities, including the classics, philosophy, and anthropology; and 6 percent mathematics, physics, chemistry and engineering].

Bulletin of the American Mathematical Society, volume 27, no. 4, January, 1921: "The October meeting of the San Francisco Section" by B. A. Bernstein, 149–153; "An image in four-dimensional lattice space of the theory of the elliptic theta functions" by E. T. Bell, 153–160; "Note on the median of a set of numbers" by D. Jackson, 160–164; "Note on closure of orthogonal sets" by O. D. Kellogg, 165–169; "The mathematical work of Thomas Jan Stieltjes" [review of *Oeuvres complètes de Thomas Jan Stieltjes* (2 volumes, Groningen, 1914–1918)] by R. D. Carmichael, 170–178; Reviews by D. E. Smith of *Opere di Evangelista Torricelli* (edited by G. Loria and G. Vassura, 2 volumes, Faenza, 1919) and of W. W. R. Ball's *An Introduction to String Figures* (Cambridge, 1920), 178–181; Review by C. N. Moore of A. R. Forsyth's *Solutions of the Examples in a Treatise on Differential Equations* (London, 1918), 181–182; Reviews by E. B. Wilson of A. S. Eddington's *Space, Time, and Gravitation; an Outline of the General Relativity Theory* (Cambridge, 1920) and of R. W. Wood's *Researches in Physical Optics* (New York, 1919), 182–186; Notes, 187–193; New publications, 194–196.

Educational Administration and Supervision, volume 6, no. 9, December, 1920: "The scoring of geometry test W" by J. H. Minnick, 509–511.

Hibbert Journal, volume 19, January, 1921: Review by C. D. Broad of A. N. Whitehead's *The Concept of Nature* (Cambridge, 1920), 360–366 [Last paragraph: "The thanks rendered in the preface by Professor Whitehead to the Cambridge University Press officials seem to me excessive. No doubt their hearts are in the right place, but they have passed at least six bad mistakes. On p. 51, l. 4, for *sight* read *touch;* p. 86, l. 8, for *external, eternal;* p. 148, l. 4, for *agree, argue;* p. 155, l. 17, for *sense-object, perceptual object;* p. 180, l. 4, for *universely* (a pleasant conceit!), *inversely;* and on p. 188, l. 9, for *by* read *from.* In conclusion, I must say that anyone who has read *Principles of Natural Knowledge* will find his understanding of that book much improved by reading *The Concept of Nature;* and that anyone who has read neither should go at once to his (or her) bookseller and order both."]

Journal of Philosophy, volume 28, no. 2, January 20, 1921: "Eddington on Einstein" by E. E. Slosson, 48–51 [Last paragraph: "Some mathematicians and physicists have manifested impatience at the impertinent curiosity of the public and declare that Einstein's theory concerns only themselves, and whatever they may decide to do with it can have no possible effect upon anybody's religion, philosophy or view of life. But the public knows better. And Professor Eddington agrees with the majority on this question. Galileo, Newton and Darwin were specialists, speculating in fields remote from common life, yet they have revolutionized the thought and altered the conduct of the world. Einstein's theory is even more fundamental and uncontentional and if it is verified by experiment or generally adopted as a working hypothesis it will be found in the course of time to have a profound influence upon the minds of men outside of the realm of science."]

Journal of the United States Artillery, volume 51, December, 1919: "Charts for the calculation of the effect of small changes in the elements of fire" by P. L. Alger, 585–603—Volume 53, August, 1920: "Two misconceptions" by R. S. Hoar, 179–181—October: "On weighting factor curves for flat fire" by J. F. Ritt, 404–410—December: "Wind weighting factors" by J. J. Johnson, 578–587; "Mirror and window position finders" by W. C. Graustein, 588–610.

Mathematical Gazette, volume 10, January, 1921: "Unicursal plane curves" by G. B. Mathews, 193–194; "The lighter side of mathematics" by C. A. Stewart, 195–200 ["In our profession we often come into contact with those who do not understand Pure Mathematics. Some of these are respectful as if entering a shrine; but others, of the baser sort, are contemptuous. My object in this paper is to consider the position of the mathematician in two of his possible moods—during his moments of leisure and during his moments of reflection]; "Gleanings far and near," 200, 219; "The tracing of conics" by E. H. Neville, 201–203; "An odd method for determining the year of birth" by G. A. Miller, 208–209; "Lagrange's tribute to Maclaurin" by C. Tweedie, 209; "Note on the differentiation of sin x, and on the limit of (sin x)/x as x tends to zero" by C. H. Hardingham, 212–215. [G. A. Miller's note[1]: "In the second edition of Cajori's *History of Mathematics*, 1919, page 330, there appears an interesting biographical sketch of Augustus De Morgan, the second sentence of which is as follows: 'For the determination of the year of his birth (assumed to be in the nineteenth century) he proposed the conundrum, "I was x years of age in the year x^2."' It may be of interest to note that after the year 1936 the conditions here given are insufficient to determine the year of birth of DeMorgan, since

$$1806 + 43 = (43)^2 \text{ and } 1892 + 44 = (44)^2.$$

"Perhaps the interest in the given remark, which is said[2] to have been made by DeMorgan, is enhanced by the observation that in every later century there is no more than one year, such that by adding an integer x to it there results a sum which is equal to x^2. For instance, if a man born in the twentieth century will be x years old in the year x^2, he must be born in 1980, since

$$1980 + 45 = (45)^2.$$

"For the twenty-first and twenty-second centuries the corresponding years are 2070 and 2162 respectively, since

$$2070 + 46 = (46)^2 \text{ and } 2162 + 47 = (47)^2.$$

"The thirty-third century is the first century which does not contain a year such that if you add x to it you obtain the year x^2, and the fifteenth century is the next to the nineteenth in which there are two such years; viz: 1406 and 1482, since

$$1406 + 38 = (38)^2 \text{ and } 1482 + 39 = (39)^2.$$

"These very elementary observations tend to show that if DeMorgan made the said remark it does not exhibit his usual thoughtfulness and accuracy."] [C. Tweedie's note: "In the recent work by Professor Cajori on the *History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, the author preludes his remarks upon Colin Maclaurin's *Treatise of Fluxions* with the following statement:

"'Maclaurin's book on fluxions has been considered the ablest and most rigorous text of the eighteenth century. It was pronounced by Lagrange, 'le chef d'oeuvre de géométrie qu'on peut comparer à tout ce qu'Archimède nous a laissé de plus beau et de plus ingénieux.''

"And there is a footnote '*Mém. de L'Acad. de Berlin*, 1773; quoted in the art. Maclaurin in Sidney Lee's *Dict. of National Biography*.'

"The statement by Lagrange occurs in his memoir, 'Sur l'attraction des Sphéroides Elliptiques' (*Mém. de L'Acad. de Berlin*, 1773, p. 121).

"Here is the passage: 'M. Maclaurin qui a le premier résolu ce problème dans son excellent Pièce sur le flux et reflux de la mer, couronnée par l'Académie des Sciences de Paris en 1740, a suivi une méthode purement géométrique, et fondée uniquement sur quelques propriétés de l'ellipse et des sphéroides elliptiques: et il faut avouer que cette partie de l'ouvrage de M. Maclaurin est un chef-d'oeuvre de Géométrie qu'on peut comparer à tout ce qu'Archimède nous a laissé de plus beau et de plus ingénieux.'

"Clearly this has nothing whatever to do with the *Treatise of Fluxions* of Maclaurin, and it is most unfortunate that such a sublime appreciation of the work of one great mathematician by another should be so cruelly misapplied by two such eminent authorities. There may have been confusion in the mind of the earlier writer for the *D.N.B.* between *flux* and *fluxion*, but such an excuse cannot be tendered for a scholar of Professor Cajori's standing.

[1] The gist of this note appeared in *School and Society*, August 7, 1920. Cf. this Monthly, *1920*, 476.

[2] "[v. *Budget of Paradoxes*, p. 332; 2nd ed., p. 124, vol. ii. All that DeMorgan says is, 'I was x years old in A.D. x^2'; not 4 in A.D. 16, nor 5 in A.D. 25, but still in one case under that law.]"

"The passage from Lagrange is correctly quoted in the *Life of Maclaurin* published in the *Mathematical Gazette*, October 1916, and is to be found also in Chasles' *Aperçu Historique*.

"Maclaurin was, of course, 'the creator of the theory of the attraction of ellipsoids.'"]

Mathematics Teacher, volume 14, no. 1, January, 1921: "The National Council of Teachers of Mathematics" by C. M. Austin, 1–4; "Progress of the National Committee on Mathematical Requirements" by J. W. Young, 5–15; "Junior high school mathematics: A discussion of the National Committee Report" by C. B. Walsh, E. R. Breslich, W. Betz, Marie Gugle, R. Schorling, 16–41; "First lessons in demonstrative geometry" by M. J. Newell and G. A. Harper, 42–45; Editorials, News and Notes, Questions and Answers, Book Reviews, 46–54.

Messenger of Mathematics, volume 50, nos. 2 and 3, June and July, 1920: "4-tic & 3-bic residuacity-tables" (continued) by A. Cunningham and T. Gosset, 17–30; "On plane curves of degree n with tangents of n-point contact (second paper)" by H. Hilton, 31–40; "On Laplace's theorem of simultaneous errors" by L. V. Meadowcroft, 40–48—No. 4, August: "Four-vector algebra and analysis (part II)" by C. E. Weatherburn, 49–61; "On a Diophantine problem (third paper)" by H. Holden, 62–64.

Nature, volume 106, December 23, 1920: Review of E. T. Whittaker and G. N. Watson's *A Course of Modern Analysis* (3d edition, Cambridge, 1920), 531—December 30: "Mathematics in secondary education" [in the United States], 583–584—January 6, 1921: "Nomography" [review of S. Brodetsky's *A First Course in Nomography* (London, 1920)], 593–594—January 13: "The Mathematical Association" by C. G., 644–645—January 20: "General dynamics" by A. Gray, 655–656 [Review of Lamb's *Higher Mechanics* (Cambridge, 1920)]; "The history of determinants" by G. B. M., 658 [Review of Muir's *The Theory of Determinants*, vol. 3 (London, 1920)]; "The mechanics of solidity" by J. Innes, 662–663; "Measurements of the angular diameters of stars" by A. C. D. C., 676–677; "The late Srinivasa Ramanujan" by E. H. Neville, 661–662 [Last two paragraphs: "Ramanujan walked stiffly, with head erect, and his arms, unless he was talking, held clear of his body, with hands open and palms downward. In conversation he became animated, and gesticulated vividly with his slender fingers. He had a fund of stories, and such was his enjoyment in telling a joke that often his words struggled incomprehensible through the laughter with which he anticipated the climax of a narrative. He had serious interests outside mathematics; he was always ready to discuss whatever in philosophy or politics had last caught his attention, and Indians speak with admiration of a mysticism of which his English friends understood little.

"Perfect in manners, simple in manner, resigned in trouble and unspoilt by renown, grateful to a fault and devoted beyond measure to his friends, Ramanujan was a lovable man as well as a great mathematician. By his death I have suffered a personal loss, but I do not feel that his coming to England is to be regretted even for his own sake. Prof. Hardy speaks of disaster because of the hopes he entertained. If he pictures Ramanujan as he might have been throughout a long life, tormented by a lonely genius, unable to establish effective contact with any mathematicians of his own class, wasted in the study of problems elsewhere solved, Prof. Hardy must agree that the tragedy averted was the greater. Shortly before he left England, at a time of great depression, Ramanujan told me that he never doubted that he did well to come, and I believe that he would have chosen as he did in Madras in 1914 even had he known that the choice was the choice of Achilles."]

Nouvelles Annales de Mathematiques, volume 79, December, 1920: "Charles-Ange Laisant (1841–1920)" by R. Bricard, 449–454; "Étude des surfaces de translation de Sophus Lie" (suite et fin) by B. Gambier, 454–479; "Sur certaines relations qui existent entre l'épicycloide et l'hypocycloide à trois rebroussements" by J. A. Moren, 479–484; "Considérations sur le frottement de glissement" by E. Delassus, 485–496; "Note géométrique sur une généralisation du théorème de composition des vitesses et le théorème de Coriolis" by L. Pomey, 496–501; "Table des matières par ordre méthodique," 503–507.

Philosophical Review, volume 30, January, 1921: "Dr. Whitehead's theory of events" by D. S. Robinson, 41–56 [First paragraph: "Dr. Whitehead has rightly said: 'It is a safe rule to apply that, when a mathematical or philosophical author writes with a misty profundity, he is talking nonsense.'[1] Now much of his own writing is assuredly impervious to this criticism, being crystal-clear as well as genuinely profound. But does not his singular and noteworthy theory of events, as recently expounded in his *Enquiry concerning the Principles of Natural Knowledge*,

[1] "*Introduction to Mathematics*, p. 227."

supply suitable material for the application of this 'safe rule'? If, as Dr. Whitehead claims, events are the ultimate facts of nature and the ultimate data of science, it is manifestly important that philosophers should have accurate and clear knowledge of what an event is. But a careful study of his account has convinced me that it is needlessly abstruse and nebulous, indeed, filled with what may well be called misty profundity. The attempt to substantiate this contention involves a somewhat minute examination of that part of his exposition setting forth his conception of an event."]

Proceedings of the National Academy of Sciences, volume 6, no. 10, October, 1920: "Motion on a surface for any positional field of force" by J. Lipka, 621–624.

Revue générale des sciences pures et appliquées, volume 31, December 15, 1920: Review by C. Maillard of Karpinski, Benedict and Calhoun's *Unified Mathematics* (Boston, 1920), 765.

Revue philosophique de la France et de l'Etranger, volume 89, March–April, 1920: "Les idées de temps, de durée et d'éternité dans Descartes" by J. Vigier, 196–223—May–June: "Les idées de temps etc." (suite) by J. Vigier, 312–348—Volume 90, November–December: "Un savant français: Henri Poincaré" by A. Denjoy, 321–350 [First paragraph: "Quand Poincaré mourut en 1912 à l'âge de cinquante-neuf ans, il laissait une oeuvre immense, dont le mérite recueillait le suffrage unanime des représentants de toutes les catégories du savoir qui empruntent aux mathématiques une aide indispensable. Le nombre de ses notes, articles, mémoires parus dans des publications périodiques réservées à la science, approchait 500. De grands ouvrages de mécanique céleste, quatorze volumes de Physique mathématique, et enfin,—originalité lui conférant une place unique parmi tous les grands mathématiciens des deux derniers siècles,—trois volumes philosophique s'ajoutaient, pour en accroître notablement l'importance, aux pièces fragmentaires déjà signalées. Et l'on ne doit point s'imaginer que cette énorme masse de text sortie de la plume de notre savant, se fût artificiellement enflée par la prolixité de son auteur. Les lectures de Poincaré qui porteraient de ce point de vue un jugement sur son oeuvre, ne songeraient pas précisement à lui adresser ce reproche-là. L'expression est tout juste suffisamment étendue pour déterminer sans aucune équivoque le sens de la pensée. Selon la bonne règle française, il ne demeure pas un seul membre de phrase dont la disparition trouverait suppléance dans le texte restant. Cette production formidable d'écrits représente uniquement une somme incompressible d'idées dont l'éclusion ne pouvait se dérouler que dans l'un des cerveaux les plus fertiles de tous les temps."]; "Descartes et Harve" (à suivre) by E. Gilson, 432–458.

School Science and Mathematics, volume 21, no. 1, January, 1921: "Minimum high school mathematics" by F. Cajori, 25–28 [First paragraph: "A pupil permitted to graduate from a high school without any mathematics is in danger of remaining the unenvied occupant of a blind alley. The attempt to reduce the minimum high school course in mathematics to one year is unwise, because thus far experience shows that one year is too short a time for a pupil to acquire a knowledge of elementary mathematics that can be applied in practical life. Two years would seem to be a better minimum."]; "The first month of geometry" by J. A. Nyberg, 29–36; "High school and college mathematics" by T. E. Mason, 37–44; Problem department, 83–86—February: "The teaching of graphs" by J. A. Nyberg, 144–149; "Tests of mathematical ability and their prognostic values. A discussion of the Rogers tests" by L. E. Mensenkamp, 150–162; Problems and solutions, 173–177.

Science, new series, volume 53, January 14, 1921: "Romancing in science" by D. W. Horn, 44 [With reference to F. Cajori's communication listed in this Monthly, *1921*, 138 (see also contents of *Science, 1921*, 35)]—January 28: "Musical notation" by T. P. Hall, 91–92 [First paragraph: "To the Editor of *Science*: In the September number of *The Scientific Monthly* Professor E. V. Huntington describes a new way of writing music [cf. this Monthly, *1921*, 35], which for simplicity and clearness can hardly be surpassed. It consists in using the ordinary staff for the twelve notes of the tempered chromatic scale, instead of (as now done) for the seven notes of the diatonic scale. This new 'normalized' notation does away with all sharps and flats. Since there are just twelve lines and spaces (including the added line below) in each staff, each letter will have always the same position on the staff, whether soprano, alto, tenor or bass. It is hoped that teachers will take advantage of the normalized notation to smooth out the road for beginners, particularly in the grade schools."]; "Star-time observations with an engineer's Y-level" by W. J. Fisher, 94–95—February 4: "The biographical directory of American men of science" by J. McK. Cattell, 118; Review by L. S. Marks of F. Bedell's *The Airplane* (New York, 1920), 119–120; "The Washington conference on the history of science" by L. Thorndike, 122 [First

paragraph: "The conference upon the History of Science, initiated by the American Historical Association at its annual meeting a year ago in Cleveland, proved such a success that the program committee devoted another session to the subject this December at Washington. Simultaneously the History of Science Section, which has recently been formed under the auspices of the American Association for the Advancement of Science, was meeting in Chicago, thus demonstrating the widespread interest in this promising field. This widespread interest was further evidenced at Washington by the variety of learned occupations represented by the speakers, who included, in addition to professors of science and history, a librarian, a college president, and the head of an institution for research."]—February 11: "Reply to Professor Horn" [see above] by F. Cajori, 139.

Science Progress, volume 15, January, 1921: "The $\sqrt{-1}$: a protest" by "Amateur," 456–457 [First few sentences: "Probably modern mathematics differs from past mathematics chiefly in the stress which is now laid upon 'Complex Numbers.' When algebra was first invented numbers were conceived to be signless; then gradually the so-called negative numbers were introduced; then mathematicians went on to separate rational from irrational numbers; and now their pupils are obliged to twist their brains by the consideration of complex numbers. Every book one reads nowadays commences with a series of paragraphs on these different kinds of numbers, and the reader is often obliged to generalize the simplest functions in terms of the last mentioned. Is there really any advantage in all this? And, though I am only an amateur, I should like to maintain that there is no such advantage, and, moreover, that complex numbers do not exist at all—though I am aware that such a statement will expose me to adverse or even contemptuous criticism. To begin with, it may be of course even doubted whether there are such things as negative numbers—and this doubt has been frequently expressed by the greatest experts. Negativeness is not a property of number itself but merely an expression of the fact that a number has been subjected to the inverse operation of addition."]; Review by R. R[oss] of Cajori's *History of the Conceptions of Limits and of Fluxions in Great Britain from Newton to Woodhouse* (London, 1919), 486–487.

Scientific American, volume 123, September 18, 1920: "New concepts of the past century. The change in outlook since classical days, which makes non-Euclidean geometry a possibility" by the Einstein Prize Essay Editor, 276, 286, 288.

Scientific Monthly, volume 12, no. 1, January, 1921: "On the character of primitive human progress" by R. D. Carmichael, 53–61 [First and last paragraphs: "The most remarkable thing among natural processes is the unfolding of the intellect and moral nature of man. Since his emergence from the animal state he has possessed powers comparable to those which he now manifests. Neither history nor speculation can reveal a period in his development when he was not making conquests evincing the same high order of intelligence as that which marks even his later career. In the earliest stages the individual man or the small group in a roving tribe had to approach the problems of life and environment without any effective tradition to guide or sympathetic collaboration with others to inspire. This called for a measure of independence unlike anything manifested by individuals today except in the labors of men of dominating genius. Among ruder peoples, in early times and at the present, the remarkable character of the discovery of truth is signalized by the acceptance of the new vision as something supernormal and sacred, akin to the activity of the gods and directly inspired by them. Though we have ceased to refer it to the supernatural, we ourselves understand it but little better. . . . We may take pleasure in such ancestors as our forefathers showed themselves to be even in the periods of savagery and barbarism through which we have rapidly sketched their development. They stood in the presence of phenomena whose nature was awe-inspiring to the creature that first inquired concerning their meaning. With no traditions to assist, with no previous conquests or discoveries of truth to start them out, with only a dumb and undeveloped sense of instinct of the destiny of man to light the way into a darkness of ignorance more profound perhaps than we can conceive today when so much of the push of the past has already been realized in our individual lives before we come to contemplate philosophically the nature of our environment and our relation to it, they began a career of development to which nothing else in our ken is to be compared."]; "The group-theory element of the history of mathematics" by G. A. Miller, 75–82 [First two paragraphs: "Few mathematical terms suggest such fundamental human cravings as the term group, and few have been more appropriately chosen. Just as human society has led to perplexities which increased with the advance of civilization, so the mathematical group-theory has given rise to problems which became more and more difficult with the advances in the development of mathematics. In both cases the primitive stages are comparatively simple and their history throws important light on the later developments.

"The history of the mathematical group-theory can be conveniently divided into three periods. The first of these extends from the beginning of mathematical history to about 1770 A.D., and may be called the *implicit period* since the group concept was then employed without being explicitly stated. The second, or *specialization period,* extends from about 1770 to about 1870. During this period the theory of substitution groups was founded as an autonomous science and the usefulness of this theory in the study of algebraic equations was emphasized. The third, or *generalization period,* extends from about 1870 to the present day, and is characterized by increased generalizations by abstraction and the explicit use of groups in each of the large domains of mathematics."]

Tohôku Mathematical Journal, volume 18, nos. 3 and 4, December, 1920: "On continuous sets of points, II" (continued) by K. Yoneyama, 205–255; "On the separation theorem for the integrals of the differential equation of the third order" by K. Ôishi, 256–260; "On the distribution of electricity on two mutually influencing spheroidal conductors" by B. Datta, 261–267; "Sur quelques propriétés relatives à certains polygones inscrits à une circonférence" by V. Thébault, 268–272; "On some characteristic properties of curves and surfaces" by S. Nakajima, 272–287; "A note on the closed convex surface" by K. Ôishi, 288–290; "An application of the orthogonal trajectory to a problem of singular solutions of ordinary differential equations of the first order" by T. Matsumoto, 291–294; "On the integral $\int_0^\pi e^x \cos r\theta \sin q\theta, \cos q\theta \, d\theta$" by T. Takeuchi, 295–296; "Über lineare Gleichungen mit unendlich vielen Variabeln" by T. Kubota, 297–301; "Akiyuki Kemmochi and his 'Equal circles on the sphere'" [in Japanese] by T. Hayashi, 302–308; "Some formulas in the theory of interpolation of many independent variables" by S. Narumi, 309–321; "On the use of a table of double entry" by T. Hayashi, 322–326; "Shorter notices and reviews, Miscellaneous notes," 327–338.

Transactions of the American Mathematical Society, volume 22, no. 1, January, 1921: "Arithmetical paraphrases" by E. T. Bell, 1–30; "The construction of algebraic correspondences between two algebraic curves" by V. Snyder and F. R. Sharpe, 31–40; "Concerning certain equicontinuous systems of curves" by R. L. Moore, 41–55; "Fundamental systems of formal modular seminvariants of the binary cubic" by W. L. G. Williams, 56–79; "A property of two $(n + 1)$-gons inscribed in a norm-curve in n-space" by H. S. White, 80–83; "Recurrent geodesics on a surface of negative curvature" by H. M. Morse, 84–100; "On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function" by J. L. Walsh, 101–116; "On functions of closest approximation" by D. Jackson, 117–128.

Transactions of the Royal Society of South Africa, Cape Town, volume 8, 1920, part 2: "Note on unimodular and other persymmetric determinants" by T. Muir, 95–101; "Note on certain determinant identities arrived at by H. v. Koch" by T. Muir, 101–105—Part 3: "Note on a sum of products which involves symmetrically the nth roots of 1" by T. Muir, 173–178; "Additional note on the resolvability of the minors of a compound determinant" by T. Muir, 229–233—Part 4: "Second note on the determinant of the sum of two circulant matrices" by T. Muir, 293–296.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel and Otto Dunkel.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2896. Proposed by the late L. G. WELD.

A circle is inscribed within a triangle. In each of the three spandrels between this circle and the vertices another circle is described; in each of the three spandrels between these last circles

and the vertices another circle; and so on *ad infinitum*. Show that the ratio of the sum of the areas of all the circles to the area of the triangle is

$$\frac{\Sigma}{\Delta}=\frac{\pi}{4}\frac{\Delta}{S^2}\left[\frac{1}{\sin\frac{1}{2}A}+\frac{1}{\sin\frac{1}{2}B}+\frac{1}{\sin\frac{1}{2}C}-2+\sin\tfrac{1}{2}A+\sin\tfrac{1}{2}B+\sin\tfrac{1}{2}C\right].$$

2897. Proposed by PAUL CAPRON, U. S. Naval Academy.

Discuss the conditions under which the angles made by two circles on a sphere have the same measures as the distances between their poles.

2898. Proposed by J. W. CLAWSON, Ursinus College.

Four straight lines determine four triangles. It is well known that the circumcenters of these triangles lie on a circle and that the circumcircles intersect this circle in a point, called the Wallace point. It is also well known that the orthocenters of the four triangles lie on a straight line, which is perpendicular to the line on which lie the middle points of the three diagonals of the quadrilateral determined by the four given straight lines.[1]

Prove that the centroids of the four triangles lie on a parabola whose axis is parallel to the mid-diagonal line; and that the distance from the Wallace point to the mid-diagonal line is two-thirds of the distance from the Wallace point to the axis of the parabola.

2899. Proposed by NORMAN ANNING, University of Michigan.

A, B, C, and P are any four coplanar points. P describes a sextant about A when the line AP turns about A through $+60°$. Show that P moves in a closed curve when it describes sextants in succession *either* about $A, B, A, B, A, \cdots$ *or* about $A, B, C, A, B, C, \cdots$.

PROBLEMS—NOTES.

13. Professor C. N. MILLS, of Heidelberg University, Tiffin, Ohio, proposed the following problem: "A, D and C are telegraph poles at equal intervals by the side of a railroad; t_1 and t_2 are the tangents of the angles which AD and DC subtend at any point B on the road; t is the tangent of the angle which DB makes with DC. Show that $2/t = 1/t_1 - 1/t_2$." This relation may be written down at once from the familiar results[2] that $1/t = (\cot A - \cot C)/2$ where $A = \angle BAC$, $C = \angle BCA$; $1/t_1 = 2\cot B + \cot A$, where $B = \angle ABC$; and $1/t_2 = 2\cot B + \cot C$. If s_1, s_2 and s are the sines of the angles whose tangents are t_1, t_2 and t, respectively, it may be readily shown[3] that $s_1 = (a\sin B)/d$, $s_2 = (b\sin C)/d$, $s = (a\sin C)/d$, where a, b, c have the usual significance in connection with the triangle ABC, and $d = \sqrt{2a^2+2c^2-b^2}$. ARC.

14. Professor I. A. BARNETT, of the University of Saskatchewan, proposed the problem: "Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\left(\frac{1}{2}\right)^7."$$

This result is given as example 13 on page 319 of W. E. Johnson, *Treatise on Trigonometry*, London, 1889, and as example 5 on page 119 of E. W. Hobson, *A Treatise on Plane Trigonometry*, second edition, Cambridge, 1897. The solu-

[1] Steiner, *Annales de Mathématiques Pures et Appliquées*, April, 1828.—EDITOR.

[2] See, for example, J. Casey, *A Treatise on Plane Trigonometry*, Dublin, 1888, pp. 174–175; S. L. Loney, *Plane Trigonometry*, Cambridge, 1893, p. 246.

[3] S. L. Loney, *l.c.*, p. 240.

tion follows readily from the preceding theory where it is shown that

$$2^{\frac{1}{2}(n-1)} \sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \cdots \sin\frac{(n-2)\pi}{2n} = 1 \qquad \text{when } n \text{ is odd,}$$

and

$$2^{\frac{1}{2}(n-1)} \sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \cdots \sin\frac{(n-1)\pi}{2n} = 1 \qquad \text{when } n \text{ is even.}$$ ARC.

15. In the discussion of problem 2819 [*1920*, 134, 187–190] concerning the envelope of the system of circles inscribed in a triangle with a given base, *AB*, and a given vertical angle, *C*, it is remarked that, if we regard the variable triangle on one side of the base only, the locus of the centers of the system of circles is an arc of a circle through *A* and *B*, and whose center bisects the arc, on the opposite side of *AB*, of the circumscribing circle. This locus was noticed by John Turner, in *The Mathematician*, no. 5, 1749, p. 312, in the course of his answer to the problem he had proposed the previous year: One side of a triangle, together with the radii of its circumscribing and inscribed circles being given, to construct the triangle geometrically. The locus was also discussed by N. Fuss,[1] in 1794; by J. Bonnycastle,[2] in 1818; by D. Cresswell,[3] in 1819; and by Holleben and Gerwien,[4] in 1831, along with the corresponding locus for the escribed circle in *C*. These loci, and that for a second escribed circle, were called for by J. Luby[5] in 1833.

It is concerning the locus of the incenter that the philosopher Herbert Spencer wrote in his *Autobiography* (New York, 1904), volume 1, pp. 187–188:

"When seventeen I hit on a geometrical theorem of some interest. This remained with me in the form of an empirical truth; but during the latter part of my residence in Worcester, responding to a spur from my father, I made a demonstration of it; and, now that it had reached this developed form, it was published in *The Civil Engineer and Architect's Journal* for July, 1840. It is reproduced in Appendix B. I did not know, at the time, that this theorem belongs to that division of mathematics at one time included under the name 'Descriptive Geometry,' but known in more recent days as 'The Geometry of Position'—a division which includes many marvellous truths. Perhaps the most familiar of these is the truth that if to three unequal circles anywhere placed, three pairs of tangents be drawn, the points of intersection of the tangents fall in the same straight line—a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe: the fact that apparently unrelated circles should in every case be held together by this plexus of relations, seeming so utterly incomprehensible. The property of a circle which is enunciated in my own theorem, has nothing like so marvellous an aspect, but is nevertheless sufficiently remarkable."

The article containing "my own theorem," [6] and a long-winded proof, is reproduced on pages 606–608 of his *Autobiography*, volume 1. The above quotation indicates that Spencer was innocent of knowledge of even the elementary ideas

[1] *Nova Acta Acad. Sc. Imp. Petrop.*, vol. 10 (1792), 1797, pp. 124–125.

[2] J. Bonnycastle, *Elements of Geometry*, sixth edition, London, 1818, pp. 310–311.

[3] D. Cresswell, *A Supplement to the Elements of Euclid*, Cambridge, 1819, pp. 256–257; second ed., 1825, pp. 227–228.

[4] H. v. Holleben and P. Gerwien, *Geometrische Analysis*, vol. 1, 1831, p. 43; vol. 2, 1832, p. 53.

[5] J. Luby, *The Elements of Geometry . . . Also a variety of problems and theorems . . . with analysis*, Dublin, 1833, third part, p. 57.

[6] Discussed, as we have seen, about a hundred years before. References have been given to only a few of the numerous earlier discussions of the theorem and its extensions.

of geometry.[1] This and other instances of such ignorance are set forth by J. S. Mackay in "Herbert Spencer and mathematics," *Proceedings of the Edinburgh Mathematical Society*, 1907, volume 25, pp. 95–106. ARC.

PROBLEMS—SOLUTIONS

2800 [1920, 31]. Proposed by A. M. HARDING, University of Arkansas.

If $x + y + z = xyz$, show that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}.$$

SOLUTION BY ARNOLD DRESDEN, University of Wisconsin.

Except for values of x and y for which $xy = 1$, the relation (1) $x + y + z = xyz$ is equivalent to $z = (x + y)/(xy - 1)$. Direct substitution of this value for z shows that the formula

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2} \tag{2}$$

holds for all values of x, y, except $x = \pm 1$; $y = \pm 1$; $xy = 1$, for which values one or the other of these equations loses meaning.

It is readily seen that an infinite number of other formulas will hold true as a consequence of (1), viz.,

$$\Sigma \tan (n \tan^{-1} x) = \Pi \tan (n \tan^{-1} x), \text{ for } n = 1, 2, \cdots, \tag{3}$$

the sum and product to be extended over x, y, and z. For since $x + y + z = xyz$, it follows that $-\tan^{-1} x - \tan^{-1} y = \tan^{-1} z$, for some determination of $\tan^{-1} z$; in virtue of this relation we have then $-n \tan^{-1} x - n \tan^{-1} y = n \tan^{-1} z$ and therefore:

$$-\tan (n \tan^{-1} z) = \tan (n \tan^{-1} x + n \tan^{-1} y) = \frac{\tan (n \tan^{-1} x) + \tan (n \tan^{-1} y)}{1 - \tan (n \tan^{-1} x) \cdot \tan (n \tan^{-1} y)};$$

i.e., $\Sigma \tan (n \tan^{-1} x) = \Pi \tan (n \tan^{-1} x)$.

It would be interesting to know the general solution of the functional equation:

$$\sum_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} f(x_i),$$

under the restriction $\Sigma x_i = \Pi x_i$.

Also solved by NORMAN ANNING, T. M. BLAKSLEE, S. M. BERG, J. A. BULLARD, H. N. CARLETON, E. S. HAMMOND, C. N. MILLS, E. J. OGLESBY, H. L. OLSON, J. L. RILEY, ARTHUR PELLETIER, H. S. UHLER, W. R. WARNE, and C. C. WYLIE.

2802 [1920, 31]. Proposed by WARREN WEAVER, California Institute of Technology.

Consider two circles, each of radius k, with centers at (0, 0) and $(k', 0)$ respectively, where k' is less than k. Through the point $(k', 0)$ draw a ray making an angle θ with the positive x-axis. Call the intersection of this line with the first circle A, and with the second circle B. Extend the line through the point $(k', 0)$ in the opposite direction, and call the intersection of this extension with the first circle A', and with the second circle B'. Prove that the sum of the two segments AB and $A'B'$ is independent of k, and depends only upon k', i.e. the shift of the circles, and θ.

SOLUTION BY U. G. MITCHELL, University of Kansas.

Let the points A, B, A', B' have coördinates (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), respectively. By the conditions of the problem the points $A = (x_1, y_1)$ and $A' = (x_3, y_3)$ are on the circle $x^2 + y^2 = k^2$, the points $B = (x_2, y_2)$ and $B' = (x_4, y_4)$ are on the circle $(x - k')^2 + y^2 = k^2$, and all four of the points A, B, A', B' are on the line $y = m(x - k')$, where $m = \tan \theta$.

[1] His boasting in the preface to the American edition, in 1876, of *Inventional Geometry: a series of problems*, by his father W. G. Spencer, will be recalled.

Since for a given circle and a given exterior point the product of any secant through the point and the exterior segment of the secant is equal to the square of the tangent from the point to the circle we have $A'B \cdot AB = x_2^2 + y_2^2 - k^2$ and $A'B \cdot A'B' = (x_3 - k')^2 + y_3^2 - k^2$. Hence,

$$AB + A'B' = \frac{x_2^2 + y_2^2 - k^2 + (x_3 - k')^2 + y_3^2 - k^2}{A'B}.$$

By use of $A'B = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$ and the relations given above this equation readily reduces to $AB + A'B' = 2k'/\sqrt{1 + m^2}$, which is independent of k and dependent upon k' and θ only, as was to be shown.

Also solved by T. M. BLAKSLEE, PAUL CAPRON, OLIVE C. HAZLETT, C. N. MILLS, E. J. OGLESBY, ARTHUR PELLETIER, C. P. SOUSLEY, C. E. STROMQUIST, and H. S. UHLER.

2804 [1920, 32]. Proposed by T. H. GRONWALL, Washington, D. C.

Show that for $|x| < 1$

$$\frac{1}{\sqrt{1 - x^4}} \int_0^x \frac{dx}{\sqrt{1 - x^4}} = x + \sum_1^\infty \frac{3 \cdot 7 \cdots (4n - 5)(4n - 1)}{5 \cdot 9 \cdots (4n - 3)(4n + 1)} x^{4n+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1 - x^4}} \right)^2 = x^2 + \sum_1^\infty \frac{3 \cdot 7 \cdots (4n - 5)(4n - 1)}{5 \cdot 9 \cdots (4n - 3)(4n + 1)} \frac{x^{4n+2}}{2n + 1}.$$

SOLUTION BY O. S. ADAMS, Coast and Geodetic Survey.

The general forms for such integrals may be derived and the formulas in the statement will then be special cases of the derived forms. We then state the problem as follows:

For $|x| < 1$, s a positive integer,

$$\frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} = x + \sum_1^\infty \frac{\frac{s + 2}{2} \cdot \frac{3s + 2}{2} \cdots \frac{(2n - 1)s + 2}{2}}{(s + 1)(2s + 1) \cdots (ns + 1)} x^{ns+1},$$

$$\left(\int_0^x \frac{dx}{\sqrt{1 - x^s}} \right)^2 = x^2 + \sum_1^\infty \frac{\frac{s + 2}{2} \cdot \frac{3s + 2}{2} \cdots \frac{(2n - 1)s + 2}{2}}{(s + 1)(2s + 1) \cdots (ns + 1)} \frac{2x^{ns+2}}{ns + 2}.$$

Let

$$\phi(x) = \int_0^x \frac{dx}{\sqrt{1 - x^s}};$$

then

$$\frac{d}{dx} \phi(x) = \frac{1}{\sqrt{1 - x^s}},$$

and

$$2\phi(x) \frac{d}{dx} \phi(x) = \frac{2}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}}.$$

By integration,

$$\phi^2(x) = 2 \int_0^x \left[\frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} \right] dx.$$

Therefore, if the first series is multiplied by $2dx$ and integrated from 0 to x, the second series will be obtained.

Let

$$y = \frac{1}{\sqrt{1 - x^s}} \int_0^x \frac{dx}{\sqrt{1 - x^s}};$$

then

$$\frac{dy}{dx} = \frac{1}{1 - x^s} + \frac{(s/2)x^{s-1}}{(1 - x^s)^{3/2}} \int_0^x \frac{dx}{\sqrt{1 - x^s}} = \frac{1}{1 - x^s} + \frac{(s/2)x^{s-1}y}{1 - x^s};$$

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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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that the lines parallel to the plane of vibration were distinctly sharper than those at right angles to the plane.

2. In this paper Professor Stromquist derives certain statistical formulas, in particular those for the standard deviation and the coefficient of correlation, that result from a given correlation, or double entry, table by adding new individuals to the table, by translating individuals in the table, and by superimposing one correlation table upon another.

3. Mr. Fitch discussed the trajectories due to a flow of water from an orifice subject to constant angle and constant kinetic energy.

4. Professor Light gave the geometrical conditions that must be fulfilled when the extraneous loci are cusp-loci, tac-loci, and singular solutions.

5. Professor Sisam discussed some properties of algebraic correspondences between two given algebraic curves of which at least one is rational.

6. Mr. Kitchen brought out, among other good points, the fact that high school students do not know how to draw conclusions from definite statements.

7. Experimental work on the rate of thermal expansion of glass from room temperature to 750 degrees Centigrade has brought out the relation between this and other thermal properties. Professor Pietenpol showed how the expansion of glass is of particular importance in its relation to the annealing of glass, and that a determination of the rate of expansion at high temperatures may be used as an exact method of determining the suitable annealing temperature.

8. Mr. McNatt took up the derivation of the equation of the catenary, and some of the properties of the equation were applied to the solutions of problems arising in connection with cables used in mines.

9. Professor Sperry gave a proof of a well known theorem that the average value of the ratio of the weight of the observed value of an unknown to its adjusted value for a series of unknowns is equal to the number of unknowns divided by the number of observations. Instead of the usual proof by undetermined coefficients, certain transformations were effected by means of determinants. This proof is believed to be superior in directness and simplicity.

G. H. Light, *Secretary-Treasurer.*

RATIONAL TRIANGLES AND QUADRILATERALS.

By L. E. Dickson, University of Chicago.

1. **The questions treated.** The chief object of this paper is to make a material simplification in Kummer's classic investigation of rational quadrilaterals. Incidentally it is shown that every rational triangle may be formed by juxtaposing two rational right triangles, so that it suffices to know Diophantus's complete solution of $x^2 + y^2 = z^2$ in rational numbers. From the latter will be deduced all solutions in integers, a problem usually treated independently of the former problem of the rational solutions. For most equations the two problems are essentially distinct.

2. **Rational solutions of $x^2 + y^2 = z^2$.** Diophantus (*Arithmetica*, II, 8) took in effect $y = x(m/n) - z$, where m/n is a fraction in its lowest terms [the value of $(y + z)/x$]. Thus

$$x = \frac{2mnz}{m^2 + n^2}, \qquad y = \frac{xm}{n} - z = \frac{(m^2 - n^2)z}{m^2 + n^2}.$$

Hence *the sides of any rational right triangle are proportional to*

$$(1) \qquad 2mn, \qquad m^2 - n^2, \qquad m^2 + n^2,$$

where m and n are relatively prime positive integers.[1] Diophantus spoke of the right triangle with the sides (1) as that *formed from m, n.*

3. **Integral solutions of $x^2 + y^2 = z^2$.** We shall prove that *all positive integral solutions of $x^2 + y^2 = z^2$ are given without duplication by*

$$(2) \qquad x = 2mnl, \qquad y = (m^2 - n^2)l, \qquad z = (m^2 + n^2)l, \qquad m > n > 0,$$

where m and n are relatively prime integers and not both odd, while l is a positive integer.[2]

When m and n are both odd, the numbers (1) rearranged are the doubles of the numbers $\frac{1}{2}(m^2 - n^2)$, mn, $\frac{1}{2}(m^2 + n^2)$, which are the sides of the right triangle formed from the integers $\frac{1}{2}(m + n)$, $\frac{1}{2}(m - n)$. Hence we may restrict attention to the case in which m and n are relatively prime and not both odd, so that one is even and the other odd. Thus the last two numbers (1) are odd; any common divisor of those two would divide their sum $2m^2$ and their difference $2n^2$, and hence divide m^2 and n^2, which are relatively prime. Hence the last two numbers (1) have no common divisor > 1. Thus if their products by the same irreducible fraction a/b are both integers, they must be divisible by b, whence $b = 1$. Hence integers are the only rational numbers whose products by all the numbers (1) give integers, when m and n are relatively prime and not both odd. By Diophantus's result in § 2, all integral solutions (like all rational solutions) are products of the numbers (1) by rational numbers. To complete the proof of our present theorem, it remains only to show that the integral solutions (2) contain no duplicates.

Suppose that the numbers (2) coincide with

$$2MNL, \qquad (M^2 - N^2)L, \qquad (M^2 + N^2)L,$$

---

[1] To give another interesting proof, we note that, if θ is an angle of any rational right triangle, $\sin\theta$ and $\cos\theta$ are rational and hence $t = \tan\frac{1}{2}\theta = \sin\theta/(1 + \cos\theta)$ is rational. Conversely, if t is rational, also

$$\sin\theta = \frac{2t}{1 + t^2}, \qquad \cos\theta = \frac{1 - t^2}{1 + t^2}$$

are rational. The first mathematical article in this MONTHLY [*1894*, 6–11] was one on this subject by the present writer, who was co-editor of the MONTHLY from 1902 to 1908.

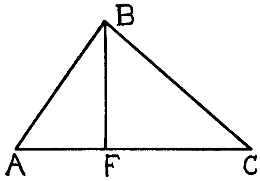
[2] The proof by Kronecker, *Vorlesungen über Zahlentheorie*, 1901, p. 35, is open to the serious objection raised by the writer in "Fallacies and misconceptions in Diophantine analysis," *Bulletin of the American Mathematical Society*, April, 1921. Moreover, it is not proved that l is an integer.

where M and N are relatively prime positive integers. By division,

$$\frac{m^2 \mp n^2}{2mn} = \frac{M^2 \mp N^2}{2MN},$$

whence, by addition, $m/n = M/N$. Since these are irreducible fractions with positive terms, we get $m = M$, $n = N$. Then $l = L$.

4. **Rational oblique triangles.** A triangle is called rational if its sides and area are expressed by rational numbers. Let ABC be a rational triangle whose sides are designated by $a = BC$, $b = AC$, $c = AB$. Since



$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$\cos A$ is rational. Hence AF is rational. Since the area equals $\frac{1}{2}BF \cdot AC$ and is rational, BF also is rational. Hence *every rational oblique triangle may be formed by juxtaposing two rational right triangles.*

By § 2, the sides of any rational right triangle are proportional to

$$2, \quad \frac{m^2 - n^2}{mn}, \quad \frac{m^2 + n^2}{mn};$$

and the sides of any second rational right triangle are proportional to

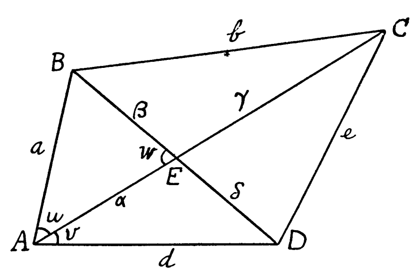
$$2, \quad \frac{M^2 - N^2}{MN}, \quad \frac{M^2 + N^2}{MN}.$$

Juxtaposing these right triangles, we see that the sides of any oblique rational triangle are proportional to

$$\text{(3)} \quad \frac{m^2 + n^2}{mn}, \quad \frac{M^2 + N^2}{MN}, \quad \frac{m^2 - n^2}{mn} \pm \frac{M^2 - N^2}{MN} = \frac{(mN \pm nM)(mM \mp nN)}{mnMN},$$

with the upper or lower signs according as the component right triangles do not or do overlap. This result was stated by Euler[1] in a posthumous fragment, but the portion of his paper which contained his proof is missing.

5. **Quadrilaterals with rational sides and diagonals.** Following Kummer,[2] we first prove that *the segments of the diagonals are rational.* By



$$b^2 = a^2 + AC^2 - 2a \cdot AC \cdot \cos u,$$

$\cos u$ is rational. Similarly, $\cos v$ and $\cos (u + v)$ are rational. Hence $\sin u \cdot \sin v$ and $\sin^2 u = 1 - \cos^2 u$ are rational. By division,

[1] *Comm. Arith. Coll.*, vol. 2, 1849, p. 648; *Opera Postuma*, vol. 1, 1862, p. 101.

[2] *Journal für reine und angewandte Mathematik*, vol. 37, 1848, pp. 1–20.

we see that $\sin u/\sin v$ is rational. Then by

$$\frac{a}{\beta}=\frac{\sin w}{\sin u}, \qquad \frac{d}{\delta}=\frac{\sin w}{\sin v}, \qquad \frac{\beta}{\delta}=\frac{a}{d}\cdot\frac{\sin u}{\sin v},$$

β/δ is rational. Thus $1+\beta/\delta = BD/\delta$ is rational. Thus δ and β are rational. Similarly, the segments α and γ of the other diagonal are rational.

The next step is much simpler than that by Kummer, who separated two cases and defined ξ algebraically, but not trigonometrically as here. *The ratios of the sides of any triangle ABE are rational if and only if*

$$(4) \qquad c = \cos w, \qquad \xi = \frac{\sin w}{\sin u}(1+\cos u)$$

are rational. These are necessary conditions for the rationality of the ratios of the sides in view of the law of cosines and $\sin w/\sin u = a/\beta$. They are sufficient conditions since

$$\frac{1}{\xi}=\frac{1-\cos u}{\sin w \sin u}, \qquad \xi+\frac{\sin^2 w}{\xi}=\frac{2\sin w}{\sin u}=\frac{2a}{\beta}, \qquad \xi-\frac{\sin^2 w}{\xi}=\frac{2\sin w}{\sin u}\cdot\cos u,$$

$$\alpha = a\cos u+\beta\cos w, \qquad \frac{\alpha}{\beta}=\frac{a}{\beta}\cos u + c,$$

so that

$$\frac{a}{\beta}=\frac{1}{2}\left(\xi+\frac{1-c^2}{\xi}\right), \qquad \frac{\alpha}{\beta}=\frac{1}{2}\left(\xi-\frac{\sin^2 w}{\xi}\right)+c=\frac{(\xi+c)^2-1}{2\xi}.$$

Applying this result to the four triangles with the common vertex E, we see that there must be rational numbers ξ, η, x, y, c such that $|c|<1$ and

$$(5) \quad \frac{\alpha}{\beta}=\frac{(\xi+c)^2-1}{2\xi}, \quad \frac{\gamma}{\beta}=\frac{(\eta-c)^2-1}{2\eta}, \quad \frac{\delta}{\alpha}=\frac{(x-c)^2-1}{2x}, \quad \frac{\delta}{\gamma}=\frac{(y+c)^2-1}{2y}.$$

The product of the first and third or second and fourth left members is δ/β. Hence must

$$(6) \qquad \frac{(\xi+c)^2-1}{2\xi}\cdot\frac{(x-c)^2-1}{2x}=\frac{(\eta-c)^2-1}{2\eta}\cdot\frac{(y+c)^2-1}{2y}.$$

Hence for any set of rational solutions ξ, η, x, y, c of (6) for which $|c|<1$, we obtain a quadrilateral the ratios of whose sides and diagonals are all rational, since (5) are then consistent and give rational ratios for α, β, γ, δ, while, as shown above,

$$\frac{a}{\beta}=\frac{\xi^2+t}{2\xi}, \qquad \frac{b}{\beta}=\frac{\eta^2+t}{2\eta}, \qquad \frac{e}{\gamma}=\frac{y^2+t}{2y}, \qquad \frac{d}{\alpha}=\frac{x^2+t}{2x},$$

where $t=1-c^2$. Evidently we may take $\beta=1$. Thus we may assign any rational values to ξ, η, c, with $|c|<1$, and seek the rational values of x for

which the quadratic equation (6) for y has rational roots, i.e., for which its discriminant

$$\{\alpha x^2 - 2c(\alpha + \gamma)x - \alpha t\}^2 + 4t\gamma^2x^2 \tag{7}$$

is a rational square. Only tentative methods are known for making such a quartic function of x equal to a rational square. From one such value of x others may be found by Euler's method.[1] Starting from simple values found by inspection, Kummer deduced various special rules for forming quadrilaterals with rational sides and diagonals.

6. **Rational quadrilaterals.** A quadrilateral is called rational if its sides, diagonals and area are all expressed by rational numbers. The area of $ABCD$ is

$$\tfrac{1}{2}(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha) \sin w.$$

Hence to the conditions in § 5 we must annex the condition that also $\sin w$ be rational (Kummer, *loc. cit.*). The rational solutions of $\sin^2 w + c^2 = 1$ are (§ 2)

$$\sin w = \frac{2\lambda}{\lambda^2 + 1}, \qquad c = \frac{\lambda^2 - 1}{\lambda^2 + 1},$$

where λ is rational. Hence to find all rational quadrilaterals we proceed as in § 5, but restrict c to numbers of the form $(\lambda^2 - 1)/(\lambda^2 + 1)$. For instance, we may assign arbitrary rational values to ξ, η, λ, and seek the rational numbers x for which the quartic function (7) is a rational square.

7. **Quadrilaterals formed by juxtaposing four triangles.** The fact that every rational oblique triangle may be formed by juxtaposing two rational right triangles (§ 4) suggests that a similar attempt be made for quadrilaterals. Waiving the requirement of rational area, we seek quadrilaterals whose sides and diagonals shall be rational and hence (§ 5) also the segments of the diagonals. As the first component triangle AEB take one whose sides a, α, β are measured by any rational numbers satisfying the necessary inequalities. Then

$$a^2 = \alpha^2 + \beta^2 - 2\alpha\beta c, \qquad c = \cos w, \tag{8}$$

determines c rationally. In the second component triangle BEC, we have given side β and angle $180° - w$, and seek rational values of γ and b such that

$$b^2 = \beta^2 + \gamma^2 + 2\beta\gamma c. \tag{9}$$

In view of (8), we know the solution $b = a$, $\gamma = -\alpha$. Hence we set

$$\gamma = -\alpha + z, \qquad b = a + kz, \qquad z \neq 0,$$

where z and k are to be found rationally. Then (9) reduces by means of (8) to $(k^2 - 1)z = 2\beta c - 2\alpha - 2ak$. If $k^2 = 1$, either $a + z = b$ or $b + z = a$, whereas a, b, and $z = \alpha + \gamma = AC$ are sides of a triangle. Hence

$$z = \frac{2\beta c - 2\alpha - 2ak}{k^2 - 1}, \tag{10}$$

[1] L. E. Dickson, *History of the Theory of Numbers*, vol. 2, 1920, p. 639 *seq.*

where k is an arbitrary rational number distinct from ± 1. In the third component triangle CED, we have given side γ and angle w, and seek rational solutions e and δ of

$$e^2 = \gamma^2 + \delta^2 - 2\gamma\delta c.$$

For $\delta = -\beta + x$, $e = b + lx$, this reduces by means of (9) to

$$x = \frac{-2\gamma c - 2\beta - 2bl}{l^2 - 1}. \tag{11}$$

The fourth component triangle AED has two given sides α and δ and the included angle $180° - w$. The condition for rational closure is that $\alpha^2 + \delta^2 + 2\alpha\delta c$ be the square of a rational number d. Replacing δ by $x - \beta$, we get

$$X^2 + t\alpha^2 = d^2, \qquad X = x + \alpha c - \beta, \qquad t = 1 - c^2.$$

The complete solution in rational numbers, found by writing $d = X + m$, is

$$X = \frac{t\alpha^2 - m^2}{2m}, \qquad d = \frac{t\alpha^2 + m^2}{2m},$$

where m is a rational number $\neq 0$. Comparing the resulting value of x with its former value (11), we obtain a single condition on our rational parameters a, α, β, k, l, m, in terms of which all the remaining quantities are expressible rationally. While the present method is more natural than Kummer's and explains intuitively why any method must involve a rational condition of closure [(6) in Kummer's method], his method has the advantage of the symmetry due to his simultaneous consideration of the four component triangles.

8. **Parallelograms with rational sides and diagonals.** The component triangles are equal in pairs. To apply Kummer's method we need consider only the first two fractions (5), which must be equal since $\gamma = \alpha$. Hence shall

$$f \equiv \xi - \frac{t}{\xi} + 4c = \eta - \frac{t}{\eta}, \qquad \eta^2 - f\eta - t = 0,$$

$$\xi^2(2\eta - f)^2 = \xi^2 f^2 + 4t\xi^2 = (\xi^2 + 4c\xi - t)^2 + 4t\xi^2.$$

This final sum, which is therefore to be a rational square, may be deduced directly from (7) by removing the factor $\alpha^2 = \gamma^2$, and writing ξ for $-x$.

We may avoid this difficult problem of making a quartic function equal to a rational square by proceeding as in § 7, where we now require only equations (8) and (9). As before, (8) merely serves to determine c. When this value of c is inserted into (9) with $\gamma = \alpha$, we obtain the same result as if we added (8) to (9):

$$a^2 + b^2 = 2(\alpha^2 + \beta^2). \tag{12}$$

Set $\alpha + \beta = h$, $\alpha - \beta = g$. Then (12) becomes

$$a^2 + b^2 = g^2 + h^2. \tag{13}$$

To solve this in integers, set $a + g = mq$, $h + b = nq$, where m and n are relatively prime. Then $a - g = np$, $h - b = mp$. Thus

$$(14)\quad a = \tfrac{1}{2}(mq + np), \quad g = \tfrac{1}{2}(mq - np), \quad h = \tfrac{1}{2}(nq + mp), \quad b = \tfrac{1}{2}(nq - mp).$$

Since m and n are relative prime, these four numbers are integers only in the following cases: mn odd, p and q both even or both odd; just one of m and n even, p and q even. *All integral solutions of* (13) *are given by* (14) *in which* m *and* n *are relatively prime integers, while* p *and* q *are both even or* p, q, m, n *are all odd.*

The last problem is evidently the same as that of finding all triangles whose three sides and one median are rational.

9. **Conclusion.** All of the problems mentioned in this paper have been completely solved except that of a general quadrilateral whose sides and diagonals (and area) are rational. That question reduces to the problem of making a quartic function equal to a rational square. To this same problem may be reduced the solution of various questions[1] relating to triangles and quadrilaterals, as well as many questions in Diophantine analysis. A complete solution of this common outstanding problem is much to be desired.

THE TRIANGLE OF REFERENCE IN ELEMENTARY ANALYTIC GEOMETRY.

By LENNIE PHOEBE COPELAND, Wellesley College.

While the use of the triangle of reference and homogeneous coördinates is common in advanced work in mathematics, comparatively little is done along this line by undergraduates. It seems possible that younger students might find it profitable and interesting to note the behavior and shape, especially at infinity, of some of the well-known curves, when plotted on a triangle of reference. This triangle may be explained very simply,[2] since it is formed by the three lines of reference (Fig. 1) CA, CB and AB or $y = 0$, $x = 0$ and $z = 0$ corresponding respectively to the X and Y axes of the Cartesian system and the "conventional line at infinity." The distances from these lines, numbers proportional to them, or to arbitrary multiples of them, determine the coördinates of any point. The selection of the negative and positive sides of the lines may be made arbitrarily. However, it is generally more convenient to determine them in such a manner that the coördinates of points within the triangle shall be positive. This region

[1] Dickson, *History of the Theory of Numbers*, vol. 2, 1920, pp. 165–224, 497.

[2] G. Salmon, *Treatise on Conic Sections*, fifth edition, London, 1869, chapter 4.

G. Salmon, *Higher Plane Curves*, second edition, Dublin, 1873, chapter 1.

C. A. Scott, *Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry*, London, 1894.

Clebsch-Lindemann, *Vorlesungen über Geometrie*, 2. Auflage, vol. 1, part 1, Leipzig, 1906, p. 119.

O. Veblen and J. W. Young, *Projective Geometry*, vol. 1, Boston, 1910, p. 174.

corresponds to the first quadrant determined by the Cartesian axes, those portions of the plane outside the triangle corresponding in pairs to the second, third and fourth quadrants. The coördinates of the vertices are $A(1, 0, 0)$, $B(0, 1, 0)$ and C $(0, 0, 1)$. The points of CB are characterized by the fact that x equals zero, and therefore any point will have the coördinates $(0, y, z)$; similarly points on CA and AB have coördinates $(x, 0, z)$ and $(x, y, 0)$ respectively. In general if P is any point not on a side of the triangle, there exist three numbers x, y, z all different from zero such that the projections of P from the vertices on the opposite sides have the coördinates $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$. These numbers are known as the trilinear or homogeneous coördinates of P. When the coördinates are taken proportional to the distances themselves of a point from the sides of the triangle (the arbitrary multipliers equal) the bisectors of the angles of the triangle are the lines $y - z = 0$, $x - z = 0$ and $x - y = 0$, having the relation of harmonic conjugates to the corresponding bisectors of the exterior angles, the lines $y + z = 0$, $x + z = 0$ and $x + y = 0$. The former intersect the opposite sides of the triangle in the points D $(0, 1, 1)$, E $(1, 0, 1)$ and F $(1, 1, 0)$ respectively and meet in the point $P(1, 1, 1)$; the latter intersect the opposite sides produced in three points which lie on the line $x + y + z = 0$. In general any line through C has for its equation $y \pm mx = 0$ and is the harmonic conjugate of the line $y \mp mx = 0$. Similar equations hold for lines through B and A.

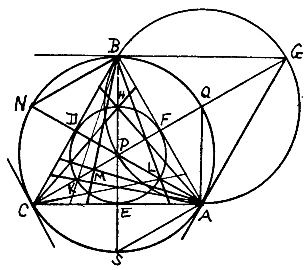


FIG. 1.

The line $x + y + z = 0$ becomes the line at infinity when the triangle of reference is equilateral. Hence it follows in this case that any curve[1] which cuts or touches the line $x + y + 1 = 0$ when referred to Cartesian axes will have infinite branches or points, when referred to this triangle. All lines intersecting on it then become parallel. Thus $NB : 2x + z = 0$ and $SA : 2y + z = 0$ are both parallel to $CF : x - y = 0$. Then if M is the midpoint of CF, the lines BM and AM are the harmonic conjugates of NB and SA or $2x - z = 0$ and $2y - z = 0$. In a similar manner the equations of other lines may be determined. With these facts in mind it is an easy matter to plot the ordinary curves, and although they appear somewhat disguised and distorted it is interesting to examine them from this new point of view. Moreover it is fascinating to see apparently simple functions don unsuspected singularities, while others break in two at unexpected places or appear with their separate branches united. For simplicity we may consider the coördinates as proportional to the distances themselves and the triangle of reference equilateral in the following examples.

The parabola $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$ which lies wholly in the first quadrant and is tangent to the x and y axes becomes the inscribed circle.

[1] The term curve as here used means the graph of an equation. Strictly speaking a given equation represents one curve when plotted on a certain set of axes and a different curve when plotted on others. Also a given curve when referred to different systems of axes does not change although its equation does.

For, writing the equation in homogeneous coördinates

$$x^2 - 2xy + y^2 - 2xz - 2yz + z^2 = 0,$$

it is seen that if $x = 0$, then $(y - z)^2 = 0$, and therefore the line $x = 0$ meets the curve twice at its point of intersection with the line $y - z = 0$, or in other words (Fig. 1) CB is tangent to the curve at D. Likewise CA and AB are shown to be tangents at the points E and F respectively. Further if $x=z$, then $y(y-4z)=0$, and therefore the line $x - z = 0$ cuts the curve in two points, namely the points where $x - z = 0$ intersects the lines $y = 0$ and $y - 4z = 0$. Hence BE cuts the curve at E and H, likewise CF cuts at K and F, and AD at D and L. By similar methods an infinite number of pairs of corresponding points within the triangle can be determined. Now the lines $x + my = 0$, $x + mz = 0$ and $y + mz = 0$ (when m is positive) all intersect the curve in imaginary points. Therefore no part of it lies without the triangle, and the resulting curve can be proved a circle. Here then we see that the two finite tangents $x = 0$ and $y = 0$ remain such, but the tangent at infinity having been brought into the plane is also visible.

The circumscribed circle is obtained by plotting the hyperbola $xy + x + y = 0$ which has a tangent at the origin $x + y = 0$ and the asymptotes $x + 1 = 0$ and $y + 1 = 0$. From its homogeneous equation it is seen that if $x = 0$, then $yz = 0$, or CB cuts the curve at C and B, and similarly CA cuts at C and A, AB at A and B. If $x = z$, then $z(2y + z) = 0$. Hence BE cuts at B and S, and similarly AD at A and N, CF at C and Q. The lines $x + y = 0$, $x + z = 0$ and $y + z = 0$ are tangents at the points C, B and A respectively. Thus the finite tangent is preserved, the two asymptotes become as was to be expected tangents at the points of intersection with $z = 0$, the line at infinity, and the two separate branches of the given hyperbola unite in an unbroken curve, namely a circle.

If we consider the hyperbola $xy = 1$, we see (Fig. 1) that the asymptote $x = 0$ is tangent to the curve at B when plotted on the triangle of reference. Likewise $y = 0$, the second asymptote, becomes a visible tangent at A. $x - y = 0$ cuts the curve at its points of intersection with $x - z = 0$ and $x + z = 0$, that is at P (1, 1, 1) and G (1, 1, -1). Other points obtained in a similar manner show the resulting curve to be a circle, the portion APB corresponding to that portion of the hyperbola in the first quadrant and AGB to that in the third. It is interesting to note that the parabolas $y^2 = x$ and $x^2 = y$ when referred to this triangle become equal circles through P having CA and CB, respectively, for chords.

Now let us consider the hyperbola $4xy = 1$ which is tangent to the line $x + y + 1 = 0$. From its homogeneous equation we observe (Fig. 2) that $x = 0$ and $y = 0$ are tangents to the curve at B and A respectively, while $z = 0$ cuts it at these same points; $x + y = 0$ in imaginary points; $x - y = 0$ at the points where it intersects the lines $2y - z = 0$ and $2y + z = 0$, namely M and infinity; $x - z = 0$ at the points where it intersects the lines $z = 0$ and $4y - z=0$, namely B and T; likewise $y - z = 0$ at A and R; and in general if m is positive

$x + my = 0$ cuts the curve in imaginary points; $x - my = 0$ in two harmonic points one inside the triangle and the other outside; $x + mz = 0$ in two points one of which is B and the other the point where $x + mz = 0$ meets $4my + z = 0$, etc. Hence the curve falling entirely within the regions corresponding to the first and third quadrants, with one point at infinity, takes the form of a parabola.

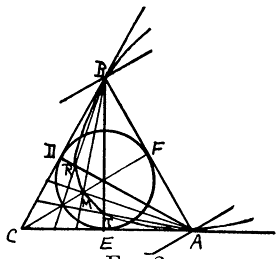


FIG. 2.

If the equation of the hyperbola is taken as $x^2 - y^2 = 1$ or $x^2 - y^2 = z^2$ then $x = 0$ cuts it in imaginary points; $y = 0$ at E and infinity; $z = 0$ at F and infinity; $y - mz = 0$ (m positive) at its points of intersection with the lines $x + z\sqrt{m^2+1} = 0$ and $x - z\sqrt{m^2+1} = 0$ giving an infinity of points inside the triangle and a corresponding infinity on the other side of the line CB, these points corresponding to those in the first and second quadrants respectively. The lines $y + mz = 0$ also cut the curve in a set of points outside the triangle in the regions corresponding to the third and fourth quadrants. The tangents of the original hyperbola $x - z = 0$ and $x + z = 0$ become a tangent at E and an asymptote respectively, while the former asymptotes $x - y = 0$ and $x + y = 0$ become one a tangent and the other an asymptote. Hence the resulting curve is a hyperbola united at one of the points where the original hyperbola broke, and breaking at one new point, namely where the line $x + y + 1 = 0$ cuts $x^2 - y^2 - 1 = 0$.

Similarly a circle about the origin in the ordinary system of coordinates referred to this triangle becomes a parabola, ellipse or hyperbola according as its radius equals, is less than or greater than, $\frac{1}{2}\sqrt{2}$.

If we refer the cubical parabola to an equilateral triangle of reference, we have in homogeneous coördinates $yz^2 - x^3 = 0$ (Fig. 3). Hence the line $x - z = 0$ cuts the curve twice at B and once at P; $x + z = 0$ twice at B and once at G: $x - y = 0$ at C, P and G: $x + y = 0$ at C; $y - z = 0$ at P; $y + z = 0$ at G; $x = 0$ twice at B and once at C; $y = 0$ three times at C, the point of inflexion. Likewise $z = 0$ cuts the curve three times at B, and as this is the only one of the lines through B ($z \pm mx = 0$) which has three intersections with the curve at this point, we see that the multiple point at B must be a cusp and the line at infinity $z = 0$ a cuspidal tangent.

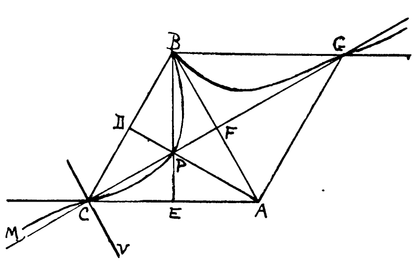


FIG. 3.

Other cubics and quartics such as the Witch of Agnesi, the Cissoid of Diocles, the Conchoid of Nicomedes and the tricuspidal quartic are interesting to plot on the triangle of reference. Moreover by varying the form of the triangle or the parameters of any given equation it may be shown that the resulting graph may assume any one of its projective forms. Thus metrical properties are changed but projective properties are preserved, and hence, in any case, the behavior at infinity may be noted.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

8. Francœur Describes a King.

Francœur, whose chief lines of interest lay in the fields of applied mathematics, was born in 1773 and died in 1849. He was on the staff of the École polytechnique and the Lycée Charlemagne and was a member of the Académie des sciences. In 1824 he went to Aix-les-Bains for the cure, and during the same period Charles Felix, the king of Sardinia from 1821 to 1831, was also taking the waters. Francœur was not one of the greatest mathematicians of France, but he was much greater as a mathematician than Charles Felix was as a king. Among the letters which Francœur wrote to his wife, who was spending the summer near Paris, is a long one in which he gossips about the events of the day and gives his impressions of the royal family. It would be interesting to know the impression, if any, that Francœur made upon the king; at any rate Francœur made considerable impression on the world, while Charles Felix made none.

Portions of the letter are as follows:

Aix in Savoy, Thursday, 29 July, 1824.

Since the post leaves tonight, I want to tell you some of the latest news of our city. The king has come; his arrival, announced long in advance, was awaited. They had stripped the hills near by of holms and pines for the purpose of planting them along by the houses. Between the trees were small blue flags bearing the arms of Savoy. The citizens also covered the walls with garlands of pines and with leaves. The roads were sanded anew and were very clean, and the public administration had done its work well. It is today that the king made his visit to Aix.

The king dined in a house prepared for his reception. He was accompanied by the queen and by her sister the duchess of Chablais. I have never seen persons with more uninteresting faces (plus pauvres visages); they seemed to me good enough people but without any brains. The king has a figure somewhat like that of the duke of Angoulême; his mouth gapes, showing his upper teeth and giving him a stupid appearance. His wife is, they say, forty-four years old. I should take her to be at least fifty-five, she seems so weak and ugly. Her sister is very old. The officers are young and newly chosen, or else they are very old serviteurs. This, without any exaggeration, is a fair picture of the court. During the dinner one could enter and pass about the table, but it was not in this way that I saw them. Instead, it was at a kind of assembly which was organized and which the king visited. All the ladies and gentlemen who take the waters gathered in the hall and he made a tour of the room, addressing remarks right and left with much politeness.

Francœur then proceeds to describe the coolness of the French to the king, remarking that

this nation is French and is unable to celebrate in behalf of a sovereign [of a territory] from which it has been separated for twenty years, and who has neither its manners nor its language.

He speaks of one of the visiting officers who, at the reception, raised his hat and cried "Vive le Roi," and of the fact that there was no response from anyone, all of which gives an interesting view of the feelings of the French people with respect to the political situation of the time. The rest of the letter is of a personal nature and has little general interest. It is not without satisfaction, however, to consider the patent fact that few people now living have ever heard of this monarch of a century ago, while the number of those who know the works of Francœur on geodesy, astronomy, and mechanics, has not lessened materially as the generations have come and gone.

9. Le Verrier and the Cost of Living.

It is consoling, in these days when the cost of living has risen far more rapidly than academic stipends, and when the family budget has assumed a new interest in university circles, to know that others, and those far greater than ourselves, have had to face the same unpleasant problem. It does not seem right that the man, or one of the two men, whose computations led to the location of a new planet without ever having seen it, should have been disturbed by the ancient *res angusta domi,* but such was the case with no less a genius than Urbain-Jean-Joseph Le Verrier (1811–1877). He was urged by Arago to devote his attention to the disturbances in the motions of the planets, and it was suggested to him that these perturbations might be caused by the presence of an undiscovered member of the solar family. He thereupon undertook the work, and in 1846 announced that his calculations showed the presence of such a planet at a specified position in the heavens, a statement at once verified by Galle, a Berlin astronomer. The discovery brought high honors to Le Verrier,—the grand cross of the Legion of Honor, a professorship in the Faculté des sciences, and the directorship of the observatory at Paris. As is well known, John Couch Adams made the same discovery independently, but Le Verrier was the first to announce it, the information being made public on September 23, 1846.

Of a considerable number of letters of Le Verrier in my collection, extending over a period of nearly thirty years, the most interesting one, from a personal standpoint, was written while he was hard at work upon his Neptune computations. It is dated five months before the discovery was announced, and is as follows:

Paris, April 18, 1846

Monsieur le Ministre,

My Father, Receveur des droits de succession at Paris, left on his death a widow, my mother, who has no income except from a pension of six hundred francs; and a daughter, Mademoiselle Léontine Le Verrier, absolutely without any money. I venture to solicit for her [the mother] a Bureau for the distribution of stamped paper at Paris. I have entire faith that my request will not fail so far as concerns the former chief under whom my Father worked, if only I have the good fortune of obtaining a favorable word from you.

My justifications in preferring this request to you, Monsieur le Ministre, are that I am connected with the giving of instruction in a school which has the honor of counting you among its former pupils, and that I have carried on certain astronomical investigations which savants have been pleased to recognize favorably, besides which I have recently been elected a member of the Academy of Sciences.

Receive, Monsieur le Ministre, the homage of my respectful regards.

U. J. Le Verrier
Member of the Institute.

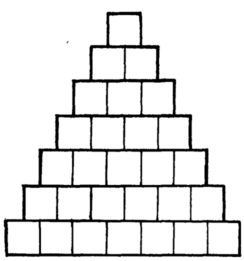
His mother was then receiving a pittance of 600 francs a year, $42 at the present rate of exchange, or $120 at that time; and he, although already well known in astronomy, and a member of the Institute, found it necessary to beg for a position which could not possibly have paid much if any more. Such were some of the difficulties which faced Le Verrier in the darkness that just preceded the dawn which brought him a world-wide recognition.

THE FORMULA $\frac{1}{2}a(a+1)$ FOR THE AREA OF AN EQUILATERAL TRIANGLE.

By G. A. MILLER, University of Illinois.

Among the interesting facts connected with the formula $\frac{1}{2}a(a+1)$ for the area of an equilateral triangle whose side is a are the following: It is remarkably inaccurate, the figures relating thereto which appear in various well known histories of mathematics are misleading, and an attempt to explain why it fails to yield correct results has been called "the first mathematical paper of the Middle Ages which deserves this name."

As regards the first of these points it may be sufficient to direct attention to the fact that in the work of Ahmes, written about 1700 B.C., the area of an isosceles triangle seems to have been found by multiplying one-half the base by one of the equal sides instead of by the altitude. This fact has been regarded as noteworthy, but when it is observed that more than two thousand years later Roman surveyors were taught to calculate the area of a special isosceles triangle by multiplying the numerical measure of half the base by a number which is even larger than the numerical measure of another side there seems to be sufficient ground for being surprised.



The present note does not aim to give a sketch of the history of this interesting formula, which sheds much light on the lack of geometric insight of the Roman surveyors during the first few centuries of the Middle Ages, appearing in such writings as Boethius's Geometry and Gerbert's Geometry. The main object of this note is to supplement statements and figures relating to this formula which appear in some of our best known modern histories of mathematics. The full benefit of the note can be secured only by those who consult the statements and figures to which references will be furnished but an effort has been made to make it instructive also to others.

The mathematics connected with the formula $\frac{1}{2}a(a+1)$ for the area of an equilateral triangle whose side is a is very elementary. If a given line segment is divided into a equal parts and if squares are arranged thereon as in the adjoining figure it is obvious that the number of these squares is the triangular number $\frac{1}{2}a(a+1)$, and that as a increases the totality of these squares approaches the area of an isosceles triangle whose altitude is equal to the base. Hence it results that if the formula $\frac{1}{2}a(a+1)$ were used to calculate the area of a given isosceles triangle whose altitude and base are both equal to a it would represent this area more and more closely as the unit of measure of the base is diminished, and the approximation could thus be made arbitrarily close by taking the unit of measure sufficiently small.

On the other hand, when the area of an equilateral triangle whose side is a

is computed by this same formula the area thus found will approach a limit, as a increases, whose ratio is to the true area as 2 is to $\sqrt{3}$. That is, the error in this case is always larger than 15 per cent.

The reason for referring here to these obvious facts is that Gerbert, who died in 1003 as Pope Sylvester II, attempted to explain in a letter to Adelbold why the area of an equilateral triangle whose side is 7 when computed by the formula $\frac{1}{2}a(a + 1)$ was too large. He based his arguments on the fact that squares lying only partly within the equilateral triangle were counted according to this formula as if they were entirely within the triangle. The correctness of this part of the explanation in this particular case can be established by consulting the given figure since an equilateral triangle on the base of this figure has its vertex very slightly above the base of the uppermost square.

One of the objects of the present note is to direct attention to the fact that in the three editions of volume I of Cantor's *Vorlesungen über Geschichte der Mathematik*, 1880, 1894, 1907, pages 744, 815 and 866 respectively, the figure relating to this triangle is inaccurate. What is still more important is the fact that the corresponding figure found in various other histories is still more misleading since it represents according to the explanations in the text an isosceles triangle whose base is equal to the altitude while the text itself relates to an equilateral triangle. This fact can be verified by consulting either edition of Cajori's *History of Elementary Mathematics*, 1896 or 1917, page 132, or Günther's *Geschichte der Mathematik*, 1908, page 249. Unless the inaccuracy of these figures is noted the reader is apt to draw incorrect conclusions in regard to the merits of Gerbert's attempted explanation, which was called inaccurate by M. Chasles in his well known *Aperçu Historique*, 1875, page 506, but has been called correct in each of the mathematical histories to which reference was made above.

In view of the fact that we are now living in an age of numerous mathematical papers it may be of special interest to note here that H. Hankel[1] called Gerbert's letter to which we referred "the first mathematical paper of the Middle Ages which deserves this name." The interest which such an assertion awakens is reflected in the fact that F. Cajori quotes this statement in both editions of his *History of Mathematics*, 1894 and 1919, as well as in both editions of his *History of Elementary Mathematics*, 1896 and 1917. This high epithet of Gerbert's letter is based on the attempted explanation to which we referred and hence the appropriateness of this epithet is necessarily called in question if this explanation is regarded as inaccurate or trivial.

The triviality of the explanation results directly from the figure given above since it is obvious that the uppermost square lies above the vertex of the equilateral triangle on the same base whenever the integer $a > 7$. This triviality becomes the more obvious if it is noted that in the first part of the letter in question Gerbert speaks of a triangle whose side is 30 and his explanation would fail entirely in this case. If this triviality is granted it follows directly that the letter in question does not merit the epithet "the first mathematical paper of the Middle Ages which deserves this name."

[1] In his *Geschichte der Mathematik*, 1874, p. 314.

The present note deals admittedly with questions whose satisfactory treatment would require a large amount of space. It is hoped, however, that a few points have become perfectly clear and that these may throw light on others. In the first place, there seems to be no room for doubt in regard to the inaccuracy of the figures appearing in various well and favorably known histories of mathematics which aim to present Gerbert's explanation of the formula $\frac{1}{2}a(a+1)$ for the area of an equilateral triangle whose side is a. A correct figure appears in Bubnov's *Gerberti Opera Mathematica*, 1899, Tab. I. In the second place, it should be clear that Gerbert's attempted explanation of the formula in question exhibits too little mathematical insight and is too trivial to merit the epithet which has been so widely attributed to it. While the time from the beginning of the Middle Ages to the end of the tenth century does not exhibit much creditable work in mathematics it does present some noteworthy advances, especially in algebra and trigonometry.

NOTE ON THE PRIME DIVISORS OF THE NUMERATORS OF BERNOULLI'S NUMBERS.

By E. T. BELL, University of Washington.

1. Using the even-suffix notation for Bernoulli's numbers, $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $\cdots B_{12} = -691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $\cdots$, as in Lucas, *Théorie des Nombres*, Chap. XIV, we shall prove the following

THEOREM. *If p is an odd prime which does not divide $4^r - 1$, the numerator of B_{2pr} is divisible by p.*

Hence for $r = 1$ we have a result due to John Couch Adams:[1]

COROLLARY. *If $p > 3$ is a prime, the numerator of B_{2p} is divisible by p.*

Both of these are useful as checks in numerical work, also they have a certain theoretical interest in some parts of arithmetic. Another observation due to Adams (quoted by Lucas, p. 435), states[2] that if p is an odd prime divisor of q and not a divisor of the denominator of B_{2q}, the numerator of B_{2q} is divisible by p. A comparison of this result and that which we shall establish shows that in numerical work one can often be applied with less labor than the other.

2. The proof depends upon the known fact that for $q > 0$ an integer,

$$I_q \equiv 2^{2q-1}(2^{2q} - 1)B_{2q}/q$$

is an integer.[3] Assume this for a moment; write $q = pr$, where p is an odd prime, and put $B_{2q} = N_{2q}/D_{2q}$, N_{2q}, D_{2q} being the numerator and denominator respectively of B_{2q}. Then

$$I_{pr} = 2^{2pr-1}(2^{2pr} - 1)N_{2pr}/prD_{2pr}.$$

[1] *Scientific Papers of John Couch Adams*, vol. 1, 1896, pp. xliv, 430.—EDITOR.

[2] J. C. Adams, Crelle's *Journal*, vol. 85, 1878, p. 269; *Scientific Papers*, vol. 1, p. 430.—EDITOR.

[3] Compare *Encyklopädie der Mathematischen Wissenschaften*, vol. II–1, 2–3, 1899, p. 183.—EDITOR.

By Fermat's theorem

$$(2^{2r})^p - 2^{2r} = M(p), \text{ (a multiple of } p),$$

and hence

$$(2^{2r})^p - 1 = M(p) + (2^{2r} - 1),$$

the right-hand member of which, and therefore also the left, is a multiple of p when and only when $2^{2r} - 1$ is divisible by p. Obviously p cannot be a divisor of 2^{2pr-1} (excluding the trivial case $p = 1$); and hence since I_{pr} is an integer it follows that if p is not a divisor of $2^{2r} - 1$, then p must divide N_{2pr}; which is the theorem.

3. It doubtless is easy in many ways to show that I_q is an integer. We give the following for its suggestiveness: the simple remark that the coefficients in the k-polynomials are integers, when combined with less obvious properties of the elliptic integrals than that which is used here, leads to a rich and unexplored field for the Bernoulli and Euler numbers. This is particularly the case when the symbolic calculus of Blissard[1] (and Lucas) is applied to the formulas furnished by the theory of transformation.

Indicating in the usual manner the modulus of the elliptic function $\operatorname{sn} x$ by k and writing $\operatorname{sn}(x, k)$, we have for the modulus unity $\operatorname{sn}(x, 1)$, and it is easy to show (cf. Cayley, *Elliptic Functions*, p. 59) that $\operatorname{sn}(x, 1) = -i \tan ix$, where $i = \sqrt{-1}$, and therefore

$$2ix \operatorname{sn}(ix, 1) = -2x \tan x.$$

But, as may readily be seen on expanding by Maclaurin's theorem, the coefficient of $(-1)^n x^{2n+1}/(2n+1)!$ $(n \geqq 0)$ in the development of $\operatorname{sn}(x, k)$ is of the form

$$s_0 + s_1k^2 + s_2k^4 + \cdots + s_nk^{2n},$$

in which $s_0, s_1, \cdots, s_n$ are positive integers, and hence their sum S is a positive integer.

On the other hand it is well-known (cf. Lucas, *loc. cit.* p. 262) that the coefficient of $(-1)^n x^{2n}/(2n)!$ $(n \geqq 0)$ in the development of $2x \tan x$ is $2^{2n}G_{2n}$, where G_{2n} is the $2n$th Genocchi number[2] defined by

$$G_{2n} = 2(1 - 2^{2n})B_{2n}.$$

Hence, equating coefficients of like powers of x in the two developments, we find

$$S = \frac{(-1)^{n+1}4^n}{n+1} G_{2n+2} = \frac{(-1)^n 2^{2n+1}(2^{2n+2} - 1)}{n+1} B_{2n+2};$$

and therefore on writing $q = n + 1$, we have, in the notation of § 2, $I_q =$ the integer $(-1)^n S$.

[1] J. Blissard, *Quarterly Journal of Mathematics*, vols. 6–9, 1863–1867.—EDITOR.

[2] A. Genocchi, *Annali di Scienze Mathematische e Fisiche*, vol. 3, 1852, p. 395.—EDITOR.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTIONS.

43. Is any rapid special method known for the evaluation of the Sylvestrian determinant met with so often in elimination by the dialytic method? It would seem that there must be, both on account of its interesting shape, and of its frequent occurrence; yet no text that I am familiar with deals with this form in detail. Will some colleague who is a specialist in this field and knows the periodical literature give us some information on this matter?

44. Is there any known formula for the co-factor of the element a_{ij} in the "binomial" determinant here shown?

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ C_1^2 & 1 & 0 & \cdots & 0 & 0 \\ C_2^4 & C_1^4 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{n+1}^{2n} & C_n^{2n} & C_{n-1}^{2n} & \cdots & C_1^{2n} & 1 \end{vmatrix}$$

Calling the co-factors A_{ij}, it is plain that $A_{ii} = 1$, and $A_{ij} = 0$, $(i > j)$. The co-factors of the zero elements, $(i < j)$, are undoubtedly expressible in a formula; the writer would like to know if any reader of the MONTHLY has met with such a formula anywhere.

DISCUSSIONS.

Professor MacNeish shows how the intersections of two conic sections with given foci and directrices may be found by the use of ruler and compass, in case they have a focus in common. The problem reduces to special or to limiting cases of the problem of Apollonius.

Perpetual calendars have been discussed in the MONTHLY by Roman [*1915*, 241] and Morris [*1921*, 127]. Mr. Franklin calls attention in the second discussion below to a formula obtained by Zeller enabling one to calculate directly the day of the week on which any date will fall.

I. THE INTERSECTIONS OF TWO CONIC SECTIONS WITH A COMMON FOCUS.

By H. F. MACNEISH, College of the City of New York.

The determination of the points of intersection of two conic sections having given foci and directrices is in general a problem of the fourth degree, and cannot be solved by ruler and compass. If, however, the two conics have a common focus the problem reduces to one of the second degree. The geometric solution

is here made to depend on the Problem of Apollonius[1]: To construct a circle tangent to three given circles (in particular cases the circles may reduce to points or lines).

The problem may be applied to the determination of the intersection of orbits in astronomy in the case of ellipses and parabolas; and to the determination of the location of artillery by the sound of the discharge in the case of hyperbolas.[2]

Case 1. Two Parabolas. Draw a circle passing through the common focus F of the parabolas, and tangent to the two directrices. There will be two such circles; their centers C_1, C_2 will be the intersections of the parabolas.

Proof. C_1, C_2 are both equally distant from the focus and the directrix of each parabola.

Case 2. Parabola and Ellipse. Let the common focus be F and the other focus of the ellipse E. With E as center and radius equal to m, the major axis[3] of the ellipse, describe a circle E. Construct circles each passing through F and each tangent to E and to the directrix of the parabola. Their centers C_1, C_2 will be the intersections desired.

Proof. C_1 and C_2 lie on the parabola, since they are equidistant from F and the directrix. Since the line EC_1 passes through T_1, the point of tangency of circles E and C_1, $EC_1 + C_1F = EC_1 + C_1T_1 = ET_1 = m$; therefore C_1 lies on the ellipse; a similar proof holds for C_2. There will be either two points of intersection which may coincide or none.

Case 3. Parabola and Hyperbola. Let the common focus be F and the other focus of the hyperbola H. With H as center and radius equal to t, the transverse axis[3] of the hyperbola, describe a circle H. Construct circles each passing through F and each tangent to H and to the directrix of the parabola. Their centers will be the intersections desired.

Proof. Similar to the preceding case. There will in general be four intersections.

Case 4. Two Ellipses. Let F be the common focus, E and E' the other foci, m and m' the corresponding major axes. With centers E and E' and radii m and m' respectively draw circles E and E'. The point F is interior to both circles. Construct circles through F tangent to E and E'. Their centers are the intersections of the ellipses.

Proof. Similar to case 2. There are in general two intersections.

Case 5. Ellipse and Hyperbola. Let F be the common focus, E the other focus and m the major axis of the ellipse, H the other focus and t the transverse axis of the hyperbola. Draw a circle E with center E and radius m, and a circle H with center H and radius t. Construct circles through F tangent to H and E. Their centers will be the intersections of the conics.

Proof. Let one of the circles found, C_1, with center C_1 be tangent to E at T_1

[1] Solutions of some of the cases are indicated in *Methods and theories for the solution of problems of geometrical construction*, by J. Petersen, Copenhagen, 1879, problems 181, 187, 238, 276, 401, 402, 403.

[2] See H. F. MacNeish, *School Science and Mathematics*, October, 1918, p. 626. Cf. *1921*, 36.

[3] This length may easily be constructed, since foci and directrices are given.

and to H at T_1'. Then

$$EC_1 + C_1F = EC_1 + C_1T_1 = ET_1 = m,$$
$$HC_1 - C_1F = HC_1 - C_1T_1' = HT_1' = t;$$

and C_1 lies on both curves. There are in general four such intersections.

Case 6. Two Hyperbolas. The construction and proof in this case are entirely similar to those of Cases 4 and 5. There are in general four intersections.

II. An Arithmetical Perpetual Calendar.

By Philip Franklin, Princeton University.

In connection with the discussion of perpetual calendars given by Doctor Morris in the Monthly, *1921*, 127, attention is called to a formula given by Christopher Zeller[1] which enables one to obtain by merely arithmetical operations the day of the week on which any given date falls. It is thus an arithmetical perpetual calendar, giving the same information as the mechanical ones described by Doctor Morris.

$$w = \left[\frac{c}{4}\right] - 2c + \left[\frac{y}{4}\right] + y + \left[\frac{(m+1)26}{10}\right] + d,$$

where c is the number of the century,
y is the number of the year in the century,
m is the number of the month,[2]
d is the day of the month,

and the number of the day of the week to be found is the remainder obtained by dividing w by 7. $[X]$ means the greatest integer in X.

E.g., for March 4, 1921, we have

$$w = \left[\frac{19}{4}\right] - 2 \times 19 + \left[\frac{21}{4}\right] + 21 + \left[\frac{(3+1)26}{10}\right] + 4$$
$$= 4 - 38 + 5 + 21 + 10 + 4 \equiv 6 \pmod{7},$$

giving the sixth day of the week, or Friday; while for February 22, 1921, we have

$$w = \left[\frac{19}{4}\right] - 2 \times 19 + \left[\frac{20}{4}\right] + 20 + \left[\frac{(14+1)26}{10}\right] + 22$$
$$= 4 - 38 + 5 + 20 + 39 + 22 \equiv 3 \pmod{7},$$

giving the third day of the week, or Tuesday.

As noted by Zeller, to prove the formula correct, we have merely to check it for one date and notice that it gives the proper changes when we increase any of the numbers on which it depends. The reader will find this statement easy to verify if he uses the facts given in the article referred to above.

[1] *Acta Mathematica*, vol. 9, 1887, pp. 131 f.

[2] January and February are counted as the 13th and 14th months of the preceding year.

RECENT PUBLICATIONS.

REVIEWS.

Plane Algebraic Curves. By HAROLD HILTON. Oxford, at the Clarendon Press, 1920. 8vo. 16 + 388 pages. Price 28 shillings.

Preface: "Though the theory of plane algebraic curves still attracts mathematical students, the English reader has not many suitable books at his disposal. Salmon's classic treatise supplied all that could be desired at the time of its appearance, but the last edition was published some forty years ago, and has been long out of print. It seemed therefore as if a new book on the subject might be useful, if only to bring some more recent developments within the reach of the student.

"In the preparation of this volume I have made frequent use of the books written by Salmon, Basset, Wieleitner, Teixeira, Loria, &c. But most of the contents and examples are extracted from a very large number of mathematical periodicals. With the exception of the list at the end of Ch. XX, I have not attempted to give systematic references. In fact, in a field which has attracted so many workers, it would be almost impossible to trace the steps by which particular results have reached their present form. In some cases I cannot even remember whether a result is my own or not; but Chapters IX, XI, XVII, and XVIII contain most of my own contributions to the subject. The solutions are mine for the most part, even in the case of examples derived from other authors.

"In a book dealing with so wide a subject I can hardly hope to escape the criticism that I have included just that material which happens to interest myself, and have excluded other matter of equal or greater importance. I have not seriously dealt with problems of enumeration, such as 'How many conics touch five given conics?' I have treated all curves with the same degree and singularities as forming a single type, and have not attempted to subdivide the type by considering all their possible positions relative to the line at infinity. I have not given the properties of 'special plane curves,' unless they are representative of some general type, such as, for example, Cassinian curves, into which any quartic with two unreal biflecnodes can be projected. I have not included any discussion of curves of degree n for special values of n other than 2, 3, or 4. A thorough discussion of quintic curves would be very welcome, but at present the difficulties seem insuperable. At any rate very little work has been published on their properties. The reader will doubtless detect other important omissions. But on the whole I have tried to cover the limited ground I have selected with reasonable completeness.

"No one can really master a branch of mathematics except by working at it himself. I make no apology, therefore, for the long lists of examples. The reader can select from them few or many, as he pleases. I have given hints for solution in most cases. I hope that these will be of real assistance to the student, setting him on the right track if he is in difficulties, enabling him to check the accuracy of his results, and giving him a guarantee that the examples are not of unreasonable difficulty."

Contents—

Pioneers of Progress: Kepler. ("Men of Science" series, edited by S. Chapman.) By W. W. BRYANT. London, Society for Promoting Christian Knowledge, 1920. 62 pages + portrait frontispiece of Kepler. Cloth. Price 2 shillings. We have referred already (*1921,* 133) to the volume on Archimedes, by Sir Thomas L. Heath, in this admirable little series of biographies.

Last two paragraphs: "Kepler's fame does not rest upon his voluminous works. With his peculiar method of approaching problems there was bound to be an inordinate amount of chaff mixed with the grain, and he used no winnowing machine. His simplicity and transparent honesty induced him to include everything, in fact he seemed to glory in the number of false trails he laboriously followed. He was one who might be expected to find the proverbial 'needle in a haystack,' but unfortunately the needle was not always there. Delambre says, 'Ardent, restless, burning to distinguish himself by his discoveries he attempted everything, and having once obtained a glimpse of one, no labour was too hard for him in following or verifying it. All his attempts had not the same success, and in fact that was impossible. Those which have failed seem to us only fanciful; those which have been more fortunate appear sublime. When in search of that which really existed, he has sometimes found it; when he devoted himself to the pursuit of a chimera, he could not but fail, but even then he unfolded the same qualities, and that obstinate perseverance that must triumph over all difficulties but those that are insurmountable.' Berry, in his *Short History of Astronomy*, says 'as one reads chapter after chapter without a lucid, still less a correct, idea it is impossible to refrain from regrets that the intelligence of Kepler should have been so wasted, and it is difficult not to suspect at times that some of the valuable results which lie embedded in this great mass of tedious speculation were arrived at by a mere accident. On the other hand it must not be forgotten that such accidents have a habit of happening only to great men, and that if Kepler loved to give reins to his imagination he was equally impressed with the necessity of scrupulously comparing speculative results with observed facts, and of surrendering without demur the most beloved of his fancies if it was unable to stand this test. If Kepler had burnt three quarters of what he printed, we should in all probability have formed a higher opinion of his intellectual grasp and sobriety of judgment, but we should have lost to a great extent the impression of extraordinary enthusiasm and industry, and of almost unequalled intellectual honesty which we now get from a study of his works.'

"Professor Forbes is more enthusiastic. In his *History of Astronomy*, he refers to Kepler as 'the man whose place, as is generally agreed, would have been the most difficult to fill among all those who have contributed to the advance of astronomical knowledge,' and again *à propos* of Kepler's great book, 'it must be obvious that he had at that time some inkling of the meaning of his laws—universal gravitation. From that moment the idea of universal gravitation was in the air, and hints and guesses were thrown out by many; and in time the law of gravitation would doubtless have been discovered, though probably not by the work of one man, even if Newton had not lived. But, if Kepler had not lived, who else could have discovered his Laws?'"

Contents—Chapter I: Astronomy before Kepler, 5–12; II: Early life of Kepler, 13–18; III: Tycho Brahe, 19–27; IV: Kepler joins Tycho, 28–34; V: Kepler's laws, 35–51; VI: Closing years, 52–57; Appendix I: List of dates, 59; II: Bibliography, 60; Glossary, 61–62.

Carnegie Institution of Washington. Year Book No. 19, 1920. Published by the Institution, Washington, U. S. A., January, 1921.

In referring to the publications of the year ending October 31, 1920, the president remarks that "three may be cited by reason of their diversity in subject-matter and by reason of special contemporary interests they have aroused." He then proceeds (page 13): "Attention was invited in the report of a year ago to the publication of Volume I of Professor DICKSON's *History of the Theory of Numbers*. Volume II (octavo, pp. xxv + 803) of this work has appeared during this year. It is devoted to what is now called 'Diophantine Analysis,' cultivated alike by the ancient, the medieval, and the modern schools of mathematicians. It is remarkable as the branch of mathematics which has the greatest number of devotees; and its history shows well how the higher developments in science are evolved, in general, out of amateurism and dilettantism. Hence the desirability of commending both these latter stages while at the same time urging individuals to linger in neither. It is especially noteworthy in the volume in question that the French statesman Fermat (1601–1665) should be one of the most prominent of the many famous names which adorn this sort of analysis. It has turned out, in fact, that he is more distinguished for his *Opera Mathematica* than for his high conduct as councillor for the parliament of Toulouse."

BENJAMIN BOSS, director of the department of Meridian Astronomy, makes report on pages 201–207, and Director G. E. HALE gives a summary of the year's work at Mount Wilson Observatory.

GEORGE SARTON makes report (pages 383–385) concerning his work in the History of Science,

July, 1919, to August, 1920, under the headings: (*a*) work in Europe; (*b*) history of science; (*c*) Leonardo studies; (*d*) history of physics in the nineteenth century; (*e*) the new humanism.

In the report on Mathematical Physics by F. R. MOULTON, reference is made to the volume on *Periodic Orbits* (*1920*, 472), and to three papers on optical subjects published in *Visual Education*, Chicago. He gives the full program of his colloquium lectures before the American Mathematical Society in 1920, and he notes that "a chapter on numerical integration of differential equations has been written for a volume which is being prepared for the Smithsonian Institution by Professor E. P. ADAMS."

FRANK MORLEY makes report for Mathematics as follows: "Professor COBLE has published his memoir on "The ten nodes of the rational sextic and of the Cayley symmetroid" (*Amer. Jour. Math.*, vol. 41, no. 4, pp. 243–265, October 1919). He has completed a memoir on Double binary forms and the closure property,[1] which develops some new points of view and many new instances.

"Two other researches of Professor Coble which are in progress may be mentioned: In the first, the modular functions of genus 3 are used to obtain a system of irrational invariants of the ternary quartic which can be identified with a similar system arising from the set of seven points in a plane. This connects the rational invariants of the quartic with the invariants of a finite collineation group in 15 variables and would indicate that the complete system of the quartic consists of not more than 17 members.

"The second research connects his discovery that the symmetroid can be transformed by Cremona transformations into only a finite number of distinct types with the fact discovered by Schottky that the symmetroid arises from the modular functions of genus 4. Cremona transformations of the symmetroid are induced by the integer linear transformations of the modular functions, when reduced modulo 2."

Tables of the Digamma and Trigamma Functions. By Eleanor Pairman. (Tracts for Computers, edited by Karl Pearson, no. 1). Cambridge University Press, 1919. 8vo. 19 pages. Price 3 shillings.

The digamma function is $d/dz \log\Gamma(1+z)$ and the trigamma function $d^2/dz^2 \log \Gamma(1+z)$. Their computation facilitates the summation of series of the form

$$S = \sum_{i=1}^{\infty} \frac{a_0 + a_1 i + a_2 i^2 + \cdots + a_{n-2} i^{n-2}}{(p_1 i + q_1)(p_2 i + q_2) \cdots (p_n i + q_n)},$$

where the a's, p's and q's are numerical quantities, and any number of pairs of factors in the denominator may be equal. In work on the torsion flexure of aeroplane propeller blades it was found necessary to sum a large number of series of this form.

"It was not discovered until the present tables were almost completed that a certain portion of the work had been already performed. Gauss (*Werke*, vol. 3, pp. 161, 162 $\cdots$) gave tables of $d[\log \Gamma(1+z)]/dz$ correct to 18 decimal places for values of z between 0 and 1, the increment in z being .01. Professor G. N. Watson (*Report of the British Association*, 1916, pp. 125, 126) gives tables of the same function correct to 13 decimal places for all integers and halves of odd integers from 0 to 100. These two valuable tables have been made use of as a check on the present 8-figure tables. So far as has been discovered there are no other tables of $d^2 [\log \Gamma(1+z)]/dz^2$."

Prefatory note by K. Pearson: "During the course of the past five years the Department of Applied Statistics in the University of London (University College) has carried out a great deal of computing work of one kind or another bearing on special war problems of a physical character. Its members have been struck by the absence of any simple text-book for the use of computers and still more by the absence of obviously necessary auxiliary tables. The present series of *Tracts for Computers* will endeavour to fill this gap as far as it lies in our power. It will not concern itself with the higher mathematical theory, but solely with the practical difficulties of the computer, or rather such difficulties as we have met with in our own experience. The first tract

[1] A memoir by Professor Coble, entitled "Multiple binary forms with the closure property," was published in *Amer. Jour. Math.*, vol. 43, January, 1921, pp. 1–19.—EDITOR.

will be followed not only by others containing recently computed tables or by the republication of old tables at present very inaccessible, but by tracts dealing with interpolation, quadrature, mechanical integration, calculating machines, tabling machines, and bibliographies of memoirs, and of tables having special value to the practical computer. In regard to the present tract, giving the values of the digamma and trigamma functions, we should ourselves have been saved many weeks of work had it been in existence four years ago. Further, we believe it will be of help not only in many physical problems other than those we have had to deal with ourselves, but to the schoolmaster who grasps the urgent importance of teaching practical mathematics to the modern school boy. The table of logarithms is not the only table that a schoolboy should learn to handle. In most modern computing laboratories a table of logarithms is very rarely used—and when used it is generally one to 10 or 14 figures[1] where multiplications are necessary which exceed the range of the ordinary multiplying machine. Nowadays the schoolboy ought to be practised in computing, and this practice should run parallel with his algebraic work. He should be exercised in the use of tables which are not becoming obsolete like the smaller tables of logarithms. He comes at a fairly early stage to the summation of series and he is liable to regard certain series as unsummable because be has not approached them numerically, just as he unfortunately regards certain integrals as unintegrable, because he is not introduced at a quite early stage to graphical, mechanical and numerical methods of quadrature. The present tract covers a very wide class of numerically summable series, and we can conceive no better practice than the schoolmaster could provide for his pupils by teaching them to sum all such series by tabular aid. If the pupil be asked at the same time to compare the result obtained by summing directly 15 to 20 terms of the series set (using tables of logarithms if he likes!), he will have learnt during the process a good deal of the practical value of logarithms, of tests for convergency, of partial fractions, of interpolation and of the value of tabular aids to the computer. He will further have realised that 'proportional parts' are neither the sole nor necessarily adequate method of entering a table;—a belief not indeed infrequently found to dominate the post-graduate as well as the school boy mind and probably arising from the same limitation of experience—the very words 'mathematical tables' being treated as synonymous with the smaller tables of common and trigonometrical logarithms."

On the Construction of Tables and on Interpolation. Part I: Uni-variate Tables; Part II: Bi-variate Tables. By Karl Pearson. (Tracts for Computers, edited by K. Pearson, nos. 2, 3). Cambridge University Press, 1920. 8vo. 54 + 70 pages. Price $3\frac{3}{4}$ + $3\frac{3}{4}$ shillings.

Prefatory note: "These tracts do not profess to be a complete treatise on the construction of mathematical tables, still less a full mathematical treatment of interpolation. They put together some of the practical processes, which have been found of service in the Biometric Laboratory, and state some of the difficulties which have arisen in very heavy recent computations and I would draw the attention of the pure mathematician to the necessity for their solution. As far as I am aware, but I have not made a wide search of the literature, the bi-variate central difference formulae provided are novel. They are those which naturally arise, however, when we come to deal with tables of double entry in practice.

"The main doctrine insisted on is that in ordinary mathematical tables accuracy would be gained if the tabulation of first differences were replaced by the tabulation of first central differences, and that in bi-variate tables the tabulation of the two first central differences of both variates is in the bulk of cases the sole method by which the material can be reduced within the bounds of possible publication."

On pages 62–70 of Tract no. 2, there is an annotated list of fifty works and memoirs dealing with interpolation from 1624 down to the present.

Tables of the Logarithms of the Complete Γ-function to twelve Figures. (Tracts for Computers, edited by Karl Pearson, no. 4.) Cambridge University Press. 4to. 1921. 14 pages + portrait frontispiece of Legendre. Price $3\frac{3}{4}$ shillings.

[1] "And where in seeking the antilog. the school boy's knowledge of the process is idle!"

This table of log Γa, from $a = 1.000$ to $a = 2.000$ is a facsimile reproduction of the one given on pages 490–499, volume 2, of Legendre's *Traité des Fonctions Elliptiques,* Paris, 1825. This table was also reproduced in O. Schlömilch's *Analytische Studien,* 1848, p. 183f. A seven-figure abridgement is given in *Smithsonian Physical Tables,* seventh revised edition, 1920, pp. 62–63. A six-figure abridgment is given in B. Williamson, *Integral Calculus,*[1] 1884, p. 169. A four-figure table for $a = 1.01$ to 2.00 is given in B. O. Peirce, *A Short Table of Integrals,* 1899. There is a very brief table, for $a = 1.0$ to 1.9, on page 30 of E. Janke and F. Emde's *Funktionentafeln mit Formeln und Kurven,* 1909. A ten-figure table for $a = 1.005$ to 2.000 for intervals 0.005 is given by G. N. Watson in *Report of the . . . British Association . . . 1916,* pp. 123–124. But earlier than any of these was a table to twenty figures given in 1813 by Gauss[2] (*Werke,* vol. 3, pp. 161, 162; vol. 10_1, p. 375), for $a = 1.00$ to 2.00. Legendre's table is the only one of these referred to in the pamphlet under review.

A seven-figure table was given on pages 58–61 of *Tables for Statisticians and Biometricians* edited by K. Pearson (Cambridge University Press, 1914). It has been found however that for many purposes especially in the construction of tables of other functions, it was needful to work with at least ten figures.

A ten-place table of $10 + \int_0^x \log_{10} \Gamma(1 + t)dt$, for $x = .01$ to 1.00 for intervals 0.01, was given by G. N. Watson (l.c., p. 124).

R. C. ARCHIBALD.

Specimen Answers of College Candidates in Plane Geometry written at the Examinations in June, 1920. (Document no. 99, April 1, 1921), New York, College Entrance Examination Board, 1921. 8vo. 22 pages. Price 25 cents.

Preface: "The following specimen answers, with the accompanying general suggestions to candidates, have been prepared for publication under the editorship of the chief reader in plane geometry, with the co-operation of the other readers. The editor desires to acknowledge hereby his appreciation of the indispensable assistance of his colleagues, at the same time accepting personal responsibility for such numerous imperfections as have doubtless resulted from his failure to give full and exact expression to their convictions.

DUNHAM JACKSON."

On pages 3–6 there is a general introductory commentary: on page 7 the paper set; and on pages 8–22, three answer books, one marked 80, another 60, and the third 55. The marks for each question and the reasons therefor are indicated.

Suggestions for Students of Mathematics. Mathematics and Life Activities. (Brown University, Bulletin of the Department of Mathematics, Number one). Providence, R. I., March, 1921. 8vo. 8 pages.

Foreword: "This Bulletin is intended primarily for students taking an introductory mathematical course in college.

"A second Bulletin will set forth the facilities and opportunities offered at Brown for pursuing the study of mathematics—especially for its own sake. This will include details regarding the

[1] This table is given in C. B. Davenport, *Statistical Methods,* second edition, New York, 1904, pp. 126–127; and in W. P. Elderton, *Frequency Curves and Correlation,* London, 1906, pp. 166–167.

[2] The *Encyklopädie der mathematischen Wissenschaften,* vol. II–1, 2, 3, 1899, p. 170, incorrectly attributes this table to Nicolai. On the other hand, the table of digamma functions, attributed to Gauss in Tract no. 1, reviewed above, was *not* by him, but computed by Nicolai under Gauss's direction.

contents of courses given in the University for the training of teachers in colleges and secondary schools."

Contents—"I: What benefits should be derived from the study of mathematics? II: Suggestions as to methods of studying mathematics. III: Mathematics and activities subsequent to college years—A. Occupations for which concentration in mathematics is desirable; B. Occupations for which concentration in mathematics combined with other subjects is desirable; C. Fields of work in which mathematical training or some knowledge of mathematics is desirable. IV: Departmental Directions."

A limited number of these pamphlets are available for distribution to those interested.

The Teaching of Geometry. By ARCHIBALD HENDERSON. *The University of North Carolina Record.* Extension series no. 39, October, 1920. 49 pages. Price 50 cents.

Headings of sections: Introduction, 3–4; The aims and results of geometrical study, 5–8; The problem of instruction (Text, teacher, pupil), 9–14; Analysis versus synthesis, 15–21; The basic problems of construction, 22–27; The problem of research, 28–45; Procedure in attacking geometrical problems, 45–48; Bibliographical note, 48–49.

NOTES.

Professor A. L. Candy's article in this MONTHLY (*1920*, 195–199) entitled "A mechanism for the solution of a equation of the *n*th degree" is reproduced in abridged form, and in Spanish, in *Revista Matemática Hispano-Americana,* December, 1920, pp. 308–309.

Reference may be given to two articles in *Monatshefte für Mathematik und Physik,* volume 30, Vienna, 1920 (216 pages). One is "Papierstreifenkonstruktion einer durch konjugierte Durchmesser gegebenen Ellipse" by K. Mack (pages 103–104); the other "Die Verallgemeinerung der Feuerbachschen Sätze" by L. Klug (pages 131–152). It is pointed out that Mack's construction is essentially that given in De La Hire, *Sectiones Conicæ,* Paris, 1685, pp. 198–199.

A series of articles, by B. LEFEBURE, published in *Revue des Questions Scientifiques* has been collected and issued in book form with the title: *Notes d'histoire des mathêmatiques* (*Antiquitê et moyen âge*) (Louvain, Société Scientifique de Bruxelles, 11 rue des Récollets, Louvain, 1920; 8vo; 154 pages). The articles dealt with numeration and the origin of our numerals, and the history of mathematics in antiquity and in the middle ages till after the contributions of Arabian science.

An elaborate volume *Cicero: a Biography* by Torsten Peterssen (University of California Press, 1920, 5 + 699 pages) was issued as one of the series "Semicentennial Publications of the University of California, 1868–1918." The following paragraph based on information in Cicero's *Tusculans* occurs on page 173:

"The grave of the great mathematician Archimedes was supposed to be in Syracuse, but the Syracusans had neglected it, did not know where it was, and even denied its existence.[1] Cicero was familiar with a description in verse of the tomb, to the effect that it contained a globe and a cylinder. In company with some prominent citizens of Syracuse, he went to a burial place by one of the gates, and there, after some search, he found a small column with a globe and cylinder. It was almost covered with weeds. When these had been removed, the verses remembered by Cicero were found on the front of the pedestal, but only the beginnings of the lines were still legible. 'To think,' exclaims Cicero, with the amateur's delight, 'that I, an Arpinate, should find the grave of Archimedes, the most famous citizen of Syracuse, when his fellow-citizens knew nothing about it!' In a small way, the incident is indicative of a greater movement; the Greeks were yielding their places to the Romans in nearly every sphere; in the next generation, largely as a result of Cicero's literary activity, Rome would produce authors far superior to their Greek contemporaries. As for Archimedes, Cicero seems to have studied mathematics and may have had some faint understanding of his greatness."

In the *Scientific American Monthly* for March, 1921, volume 3, pages 196–198, is one of the essays submitted in competition for the prize of five thousand dollars offered by the *Scientific American* [1921, 191]. It is entitled "The quest of the absolute; an essay on modern developments in theoretical physics." The author is Dr. F. D. MURNAGHAN, associate in applied mathematics, Johns Hopkins University. The following editorial comment is added in connection with publication of the article:

"The Judges and the Einstein Editor have no hesitation in pronouncing the essay of Dr. Murnaghan, presented herewith, to be, for the man who is equipped to read it with full understanding, altogether the most illuminating of the essays submitted in the contest, if not indeed the most successful discussion of comparable length that has appeared anywhere. The Judges were agreed that Dr. Murnaghan's essay was of doubtful value before a general audience, and that in the presence of an essay such as Mr. Bolton's appeared to be it could not properly claim the prize; but it is so very good of its kind that they clung to the last moment to the possibility of its being the best; and only allowed it to be eliminated from their consideration after the most searching examinations of Mr. Bolton's work had shown that it was all it seemed to be.

"This criticism of Dr. Murnaghan's work makes it plain that it demands publication, and equally plain that the place for it is here, rather than in the *Scientific American* proper. By all means it deserves the distinction of being the first of the competing essays to appear in the *Scientific American Monthly*, and we hasten to give it this distinction.—Editor."

About a year ago (*1920*, 218) we listed the thirteen volumes of the great edition of Euler's *Opera Omnia* which had been published at that time. Two more volumes have now been issued from the press making five published since the outbreak of the war. They are : Institutiones Calculi Integralis, part 3, edited by Engel and Schlesinger; Abhandlungen aus der Integralrechnung, 2 volumes, edited by Gutzmer and Liapounoff; Abhandlungen aus der Arithmetik, 2 volumes, edited by Rudio. The volume on Algebraische Abhandlungen, edited by Rudio, Stäckel, and Krazer, is practically complete; and the volume on Artilleriewesen, edited by Scherrer is in galley proof.

[1] In William Forsyth's *Life of Marcus Tullius Cicero*, London, 1864, vol. 1, p. 51, the following lines are quoted:

"When Tully paused amidst the wreck of time
On the rude stone to trace the truth sublime;
Where at his feet, in honored dust disclosed,
The immortal sage of Syracuse reposed."

Who wrote these lines? The incident here mentioned occurred about 75 B.C. when Cicero was 32 years of age. Archimedes died 212 B.C.—EDITOR.

The Euler Commission of the Swiss Society of Naturalists issued to its subscribers, in December, 1920, a statement[1] including the following:

"It has been possible, in spite of all difficulties to prepare five volumes of the Euler edition during the past years. Being convinced however that to many subscribers at the present time it would be a great hardship to receive such a number of volumes at once, the Euler Commission permits itself to make a present of four of these volumes and to ask for the subscription price to the fifth volume only. This applies to all subscribers, not only to private persons but also to the academies and other learned societies, as well as to the libraries.

"Pray do not conclude from this action that the financial position of the Euler undertaking is brilliant. Altogether otherwise, we look to the future with grave misgivings. Not only are the costs of composition increased more than ten fold over those before the war, but we meet also earlier unsuspected difficulties on account of the low value of money in many states. A continuation of the undertaking can therefore only be possible if all of our subscribers remain faithful to us and if we are successful in finding yet others.

"We earnestly beg you therefore, to continue to retain your highly prized good will towards the great Swiss work of the Euler Edition and to help us to bring to a happy conclusion the undertaking which has been commenced."

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 43, no. 1, January (published March), 1921: "Multiple binary forms with the closure property" by A. B. Coble, 1–19; "Einstein's, theory of gravitation: Determination of the field of light signals" by E. Kasner, 20–28; "Note on Einstein's equation of an orbit" by F. Morley, 29–32; "A one-to-one representation of geodesics on a surface of negative curvature" by H. M. Morse, 33–51; "Conjugate systems with indeterminate axis curves" by E. P. Lane, 52–68.

Annals of Mathematics, second series, volume 22, no. 3, March, 1921: "The asymptotic expansion of the Sturm-Liouville functions" by F. H. Murray, 145–156; "On the conformal mapping of a region into a part of itself" by J. F. Ritt, 157–160; "Conjugate nets R and their transformations" by L. P. Eisenhart, 161–181; "The applications of modern theories of integration to the solution of differential equations" by T. C. Fry, 182–211.

Educational Administration and Supervision, volume 7, no. 2, Feburary, 1921: 'Subject matter courses in mathematics for the professional preparation of Junior High School teachers" by P. M. Symonds, 61–76.

L'Enseignement Mathématique, volume 21, nos. 3–4 (published December, 1920): "Sur un théorème de cinématique" by C. Cailler, 163–169; "Généralisation des coordonnés polaires. Applications" by E. Jablonski, 170–175; "Sur les systèmes de nombres bicomplexes" by L.-G. Du Pasquier, 175–183; "Développement d'une puissance quelconque, entière et positive, de cos x ou de sin x en fonction linéaire des cos et sin de multiples de x" by E. Barbette, 184–187; "Analyse indéterminée du p^{me} degré sur les sommes de puissances égales des nombres" by E. Barbette, 188–191; "Congrès international des mathématiciens. Strasbourg, 22–28 septembre 1920," 192–209; "Les travaux de la Section de Mathématiques et d'Astronomie de l'Association française pour l'Avancement des Sciences," 209–215; "Société mathématique suisse," 215–229; "Chronique," 229–231; "Notes et documents," 232–236; "Bibliographie," 236–243; "Bulletin bibliographique," 243–250.

Grinnell Review, Grinnell College, volume 16, March, 1921: "Vindicating Euclid and Newton" [review of Girolamo Saccheri's *Euclides Vindicatus*, translated by G. B. Halsted (Chicago, 1920) and of F. Cajori's *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse* (Chicago, 1919)] by R. B. McClenon, 379.

Literary Review, published by New York Evening Post, volume 1, January 15, 1921: "The rudeness of poets" by Christopher Morley, 7 [First paragraph: "The poet who has not learned how to be rude has not learned his first duty to himself. By 'poet' I mean, of course, any imaginative creator—novelist, mathematician, editor, or a man like Herbert Hoover. And by 'rude' I mean the strict and definite limitation which, sooner or later, he must impose upon his sociable instincts. He must refuse to fritter away priceless time and energy in the random

[1] *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1921, pp. *52–53*.

genialities of the world. Friendly, well-meaning, and fumbling hands will stretch out to bind the poet's heart in the maddening packthread of Lilliput. It will always be so. Life, for most, is so empty of consecrated purpose, so full of palaver, that they cannot understand the trouble of one who carries a flame in his heart, and whose salvation depends on his strength to nourish that flare unsuffocated by crowding and scrutiny."]; "Science histories—*History of the Theory of Numbers.* Vol. I: Divisibility and primality; Vol. II: Diophantine analysis. By L. E. Dickson. Carnegie Institution of Washington" by R. D. Carmichael, p. 9 [Quotations: "There is in many quarters a growing realization of the importance of the history of science, not only to the progress of science itself but also to the general advancement of civilization. It is certain that there is no better way to foster interest in the search for and discovery of truth than by a widespread and accurate knowledge of the way in which it has been ascertained in times past and has yielded unexpected values of essential importance. Again, it is certain that there is no greater incentive and support to the arduous duties of research than a clear conception of the way in which other thinkers have met and overcome the difficulties hindering earlier progress. . . . Whatever may be our judgment as to the ultimate relative importance of the various ends to be served by a history of science, we must recognize that the purposes of general culture cannot be met until we have first brought together the detailed facts in elaborate summaries prepared for the specialist. At the present time we do not have an adequate literature in any single body of science for serving any one of the four fundamental ends of scientific history," namely, "to enrich the general culture and intellectual life of cultivated people, to help the progress of science, and hence of human betterment, by a more widespread appreciation of its problems and the services rendered by it, to enable a scientific worker quickly to orient himself in a chapter of a science so as to proceed most readily to its detailed mastery, to enable a scientific worker to ascertain with completeness what has already been attained in a given subject."]

Nature, volume 106, January 27, 1921: "The space-time hypothesis before Minkowski" by E. H. Synge, 693 [First two paragraphs: "It is, perhaps, not generally realized that the theory of space and time, to which Minkowski was led on experimental grounds, had been formulated on general principles sixty-five years previously by Hamilton, the Irish mathematician. The point is, however, of interest, not merely as a question of priority, but for the insight it affords into the philosophic basis of the theory, as well as for the useful mathematical methods it suggests.

"It is curious, therefore, that there should be a lack of recognition that the world of Minkowski is in all points identical with the system of quaternions of Hamilton, and that the latter mathematician specifically regarded this system as a four-dimensional expression of space and time, in which space bears to time the relation which $\sqrt{-1}$ bears to unity, time being the scalar part of the quaternion."]

Proceedings of the National Academy of Sciences of the U. S. A., volume 6, no. 11, November, 1920: "Seminvariants of a general system of linear homogeneous differential equations" by E. B. Stouffer, 645–648; "Some new methods in interior ballistics" by A. G. Webster, 648–659; "The permanent gravitational field in the Einstein theory" by L. P. Eisenhart, 678–682; "A simplified method for the statistical interpretation of experimental data" by G. A. Linhart, 682–684.

Revue de l'Université de Bruxelles, Bruxelles, no. 4, January, 1921: "Les mathématiques dans la biologie (La coquille du nautile)" by D'A. W. Thompson, 1–19 [Lecture delivered at the university, November 26, 1920, by the distinguished professor at the University of St. Andrews (cf. *1918*, 189, 192, 238). Last three paragraphs: "On a appelé la mathématique la servante des sciences physiques, mais elle est aussi leur reine. Par le nombre, l'ordre et la position, elle nous met sur la voie de la connaissance exacte, la voie de la vérité scientifique; ces trois termes, le nombre, l'ordre et la position, nous fournissent les premières grandes lignes d'un croquis de l'univers. Par le compas et par l'équerre, par le cercle et le carré, on nous fait mieux comprendre, comme dit le vieux charpentier dans le poème de Verhaeren: 'Les lois indubitables et fécondes, qui sont la règle et la clarté du monde.'

"Les mathématiques ne sont pas seulement une science avec ses lois, elles nous fornissent une langue—et on a dit que c'est la seule langue que le physicien puisse parler. Et un grand mathématicien écossais, qui étudia le rayon de miel il y a près de deux siècles, en a tiré la leçon que la perfection de la beauté mathématique est telle, que tout ce qui est le plus beau et le plus régulier est en même temps le plus utile et le plus excellent.

"Hier soir, sur un rayon de la bibliothèque de M. Paul Héger, j'ai mis la main sur un des ouvrages d'Henri Poincaré,—et comme vous le savez bien, même après avoir maintes fois lu ses

écrits, on y trouve toujours quelque chose de frappant et de nouveau. A la première page que j'ai ouverte, il compare la réalité objective avec l'harmonie que l'intelligence humaine croit découvrir dans la nature; et en dernière analyse il arrive à la conclusion que cette harmonie, qui s'exprime par les lois mathématiques, est la seule réalité objective, la seule vérité que nous puissons atteindre. Et en ajoutant que l'harmonie universelle du monde est la source de toute beauté, Henri Poincaré, mathématicien, arrive à la même conclusion laquelle Henri Fabre, naturaliste, est parvenu—c'est-à-dire que dans le Nombre on trouve *le pourquoi et le comment des choses*, et que l'on s'imagine y voir *la clef de voûte de l'Univers*."]

Revue Genéralé des Sciences, volume 32, January 15, 1921: "Electricité et géométrie, après les théories récentes" by L. Bloch, 5–11.

Revue Scientifique, volume 59, January 22, 1921: "Sir Norman Lockyer: la découverte de l'hélium et la température des étoiles" by H. Deslandres, 51.

Science, new series, volume 53, February 18, 1921: "A brief historical consideration of the metric system" by L. C. Karpinski, 156–157; "The history of science and the American Association for the Advancement of Science" by F. Cajori, 163–164 [Last paragraph: "In the judgment of the present writer, the dignified and logical procedure for those interested in the History of Science is, therefore, to withdraw altogether from organized historical work in connection with the American Association for the Advancement of Science until such time when the council and general session will be ready to welcome them into the Association as a separate Section."]—March 4: "Human nature as a repeating factor: that thrice told tale" by W. W. Campbell, 211–212 [First sentence: "The following comments on Professor Wood's 'Thrice told tale,' *Science*, January 14, 1921, are based upon my long experience in showing celestial objects through a great telescope to tens of thousands of Saturday night visitors, and in explaining photographs of star clusters, the milky way, spiral nebulae, etc. to thousands of others."]; "Galileo and Wood" by A. G. Webster, 212-213—March 11: "Musical notation" by R. P. Baker, 235–236 [Letter; first three sentences: "While musical notation is not a matter of great scientific interest, reform presumably is. The desirability of the changes advocated by Professors Huntington and Hall [*1921*, 35, 225] may be admitted. This leaves the space available for briefly discussing the cost."]; Review by L. C. Karpinski of A. Mieli's *Gli Scienziati Italiani* (Rome, 1921), 237–238 [cf. *1921*, 173]; "The Einstein solar field and space of six dimensions" by E. Kasner, 238–239.

Texas Mathematics Teachers' Bulletin, volume 6, no. 2, February 10, 1921: "The National Committee on Mathematical Requirements" by E. R. Hedrick, 7–15; "Quantity or quality" by Mary Campbell, 16–17; "The slide rule" by A. E. Cooper, 18–26 [A photograph of a ten-inch Keuffel & Esser slide rule is on a fly-leaf. This may be cut out and pasted on pieces of thin wood.]; "Einstein's relativity and gravitation theories" by P. M. Batchelder, 27–34; "Brown Mathematical Prizes for freshmen" by H. J. Ettlinger, 35–36; "Some elementary principles of Non-Euclidean geometry" by Ethel Burch, 37–44; "Why study mathematics?" by Arnold Dresden, 45–54 [Reprinted from *School and Society;* see this Monthly, *1921*, 83]; "A mathematician in love" from *Boston Transcript*, 55.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 51, nos. 11–12, published December 1, 1920: "Zum Mathematikunterrichte am deutschen Gymnasium" by A. Weise, 257–262; "Die Schwingungsformel der oszillatorischen Entladung im Unterricht" by K. Hahn, 262–264; "Die Schwingungsdauer der oszillierenden Entladung im Unterricht" by W. Hillers, 264–273; "Kleine Mitteilungen," 273–276; "Aufgaben-Repertorium," 276–280; "Bücherbesprechungen" and "Zeitschriftenschau," 284–288.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

The Mathematics Club of Connecticut College, New London, Conn. [*1918*, 270, 460; *1920*, 28.]

The general plan of the Club is to have at each meeting two papers. In 1919–20 the first paper was biographical and the second on any topic of general

interest. The papers are planned not to exceed twenty minutes each. After the formal papers have been read and discussed, each member who during the preceding month has read an article of interest, directly or indirectly related to mathematics, is expected to report on it. This does not exclude mathematical jokes and puzzles. This closing feature has at many meetings been most valuable and stimulating, and the only difficulty that arises is in finding time to give all a chance to report. The plan for 1920–21 is quite similar.

The Club is run entirely by its undergraduate officers. The average attendance is about 15.

The programs for 1919–20 were as follows:

October 13, 1919: Social meeting of old and prospective members at the home of Professor David D. Leib.

November 3: "Euclid" by Margaret Paul '21. Discussion of Greek mathematics. "Mathematics requirements in several women's colleges" by Louise Avery '21.

December 1: "Not 10 but 12" by Justine McGowan '20; "Descartes" by Augusta O'Sullivan '22.

January 12, 1920: "Applied mathematics" by Miss Elizabeth C. Wright, bursar of the College; "Russian peasant method of multiplication" by Barbara Ashenden '21; "Fermat" by Ruth Wells '23.

March 1: "Newton and Leibnitz" by Marcia Langley '23; "How x came to be used for the unknown quantity" by Verna Kelsey '23.

April 26: "Modern American mathematicians" by Elizabeth Hall '22; "Codes and ciphers" by Dorothy S. Wheeler '22.

May 26: Business meeting; election of officers for ensuing year.

June 1: Picnic.

The officers for 1920–21 are: President, Dorothy Pryde '21; secretary, Dorothy Wheeler '22; treasurer, Marcia Langley '23; chairman program committee, Ruth E. Wells '23.

The programs for 1920–21 are as follows:

October 8, 1920: Social meeting of old and prospective members at the home of Professor Leib.

November 8: "Mathematical ability tests on high-school students" by Augusta O'Sullivan '22; "Some impossible problems" by Barbara Ashenden '21.

December 6: "Alice in Wonderland in the land of Math." by Marion Lyon '21 (Reading); "The number 9" by Marcia Langley '23.

January 10, 1921: "Trisecting an angle and duplicating a cube" by Ruth Wells '23; Current articles on "Fleury, the blind calculator"; "The number 13 in Wagner's life" by Alice Boehringer '23.

February 7: Valentine party.

March 7: "Mathematical recreations" by Professor Ray E. Gilman, Brown University.

(Reported by Professor Leib.)

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Md.
[*1918*, 357; *1919*, 305.]

The club met every two weeks from December to May in 1919–20. In order to give each member an opportunity to participate, it was decided to have several short papers presented at each meeting. The papers were, with few exceptions, prepared and presented by the students. Visiting students from the science departments frequently took part in the discussion—an evidence of desirable correlation of related subjects. The programs were as follows:

1. "Theorems concerning a triangle and its related circles" by Deldee Groff '20; "The nine-point circle" by Loretta Whelan '20.
2. "A simple ruler-and-compass construction for a regular pentagon, and discussion of the figure arising therefrom" by Jean Burke '20 and Mildred Trueheart '22.
3. "Zeno's paradoxes" by Ruth Marshall '20; "The origin and early history of the conic sections" by Alice Langerhausen '20.
4. "Trisection of an angle and squaring of a circle by use of the quadratrix" by Mildred Grafflin '20.
5. "Geometrical fallacies, *e.g.* All triangles are isosceles" by Dorothy Biscoe '22 and Rose Diggs '22; "The prismatoid formula" by Virginia Gallup '21.
6. "The life and work of Newton" by Elsie Keith '21; "Calculation of the velocity of escape from the earth's attraction" by Mildred Brown '21.
7. "Applications of mathematics in statistical work at the Johns Hopkins School of hygiene" by Agnes Bacon, Wellesley College.
8. Joint meeting of the Mathematics and Science Clubs. "Einstein's theory of relativity" by Professor Florence P. Lewis.

In accordance with the custom of the club, refreshments, prepared by the speakers of the evening, were served at each meeting during the period for informal discussion. The annual picnic took place after the final examinations.

(Reported by Professor Lewis.)

THE MATHEMATICAL CLUB, Harvard University, Cambridge, Mass.
[*1918*, 196, 449; *1919*, 262; *1920*, 480.]

The officers for 1920–21 are: President, Instructor Rudolph E. Langer; secretary-treasurer, Instructor Bancroft H. Brown; Faculty adviser, Professor Oliver D. Kellogg.

The programs for 1920–21 were as follows:

October 6, 1920: "Cubics and biquadratics" by Professor William F. Osgood.
October 20: "Plane cubic curves" by Charles A. Spoerl '22.
November 3: "A neglected chapter of analytic geometry" by Professor Kellogg.
November 17: "Dynamics of the billiard ball" by Instructor Charles A. Rupp.
December 15: "'Necessary,' 'sufficient,' and 'necessary and sufficient' conditions" by Instructor Carl A. Garabedian.

January 5, 1921: "The mathematics of investment" by Dr. James S. Taylor, Massachusetts Institute of Technology.

January 19: "Mathematical education in Sweden" by Dr. Einar Hille.

March 2: "The dog and duck problem"[1] by Professor Henry P. Manning of Brown University.

(Reported by Instructor Brown.)

THE JUNIOR MATHEMATICS CLUB, University of Minnesota, Minneapolis, Minn. [*1918*, 312; *1919*, 209.]

The Mathematical Club at the University of Minnesota meets at irregular intervals. Some of the meetings are of the club as a whole and some are meetings of the Graduate Section (g.s.) only. Recent programs are as follows:

March 22, 1920: (g.s.) "A class of orthotomic cubics" by Instructor Robert M. Mathews.

May 3: (g.s.) "Dynamic balancing" by Professor Burt L. Newkirk.

May 17: Annual banquet.

May 24: (g.s.) "Pressure in water due to wave motion" by Professor G. C. Priester.

October 25: "Mathematics at the University of Edinburgh" by Professor Raymond W. Brink.

November 8: (g.s.) "Functions of infinitely many variables" by Professor William L. Hart.

November 22: "The geometric representation of complex variables" by Instructor Gladys Gibbens.

January 17, 1921: "Non-Euclidean geometry" by Professor William H. Bussey.

(Reported by Professor Bussey.)

THE MATHEMATICAL CLUB OF ROCKFORD COLLEGE, Rockford, Ill. [*1918*, 188, 409: *1920*, 77.]

The officers for 1920–21 are: President, Margaret Dodd '21; vice-president, Anna Foster '21; secretary-treasurer, Frances Regan '22.

The following programs were given during the year.

September, 1920: Election of officers.

October: Initiation of new members. "Mathematical conundrums and catch problems, spiced with poetry."

November: A semi-humorous treatment of one, two and three dimensional spaces.

December: (Open meeting) "The fourth dimension" by Professor Bessie I. Miller.

January, 1921: (Open meeting) "The Einstein theory" by Professor Miller.

(Reported by Miss Dodd.)

THE PENTAGRAM, University of Texas, Austin, Texas. [*1918*, 273; *1919*, 364; *1920*, 321.]

The following is the report for the first two terms of 1920–21:

[1] Compare this MONTHLY, *1921*, 54–61, 91–97, 278–279, 281-282.—EDITOR.

Officers: President, Paul Boner Gr.; vice-president, Jessie M. Fouts '21; secretary-treasurer, Dr. Jessie M. Jacobs, instructor in mathematics; executive council, Dr. Goldie P. Horton, instructor in mathematics, and Renke G. Lubben '21.

There are twenty-seven student members and sixteen faculty members.

October 7, 1920: Election of officers.

October 21: "Non-euclidean geometry" by Professor Robert L. Moore.

November 4: "Periodic phenomena and periodic functions" by Paul Boner Gr.

November 20: Social meeting and initiation of new members at the home of Dr. Horton.

December 2: "The number e" by Mary Cook '23.

January 13, 1921: "Numbers" by Professor Milton B. Porter.

January 27: "Complex numbers" by Sophie Anderson '22; "Matrices" by Ruth Peden '22.

February 10: "The vibrating string" by Jessie M. Fouts '21; "Laplace's equation" by L. Vernon Robinson '22.

February 24: "One of the most practical applications of actuarial mathematics" by Mr. C. P. Rockwell, state actuary of Texas.

(Reported by Dr. Jacobs.)

NOTES.

At some colleges it seems best to form clubs for both mathematics and science students.

Professor Clyde S. Atchison, of Washington and Jefferson College, Washington, Pa., writes (February 4, 1921):

"One year ago two senior honor men in the department of mathematics, with the coöperation of the head of the department, initiated a movement for the organization of an honorary fraternity, limited in membership, for those upperclassmen especially interested in mathematics, physics, and chemistry. We now have a live organization, in which mathematics occupies its full place. Occasionally open meetings are held."

At Baldwin-Wallace College, Berea, O., according to Professor O. L. Dustheimer, about one third of the student body were present at a mathematical gathering early in January, 1921, at which questions were propounded, such as:

"Arrange the figures 1 to 9 inclusive, so their sum will be 100.

"John found $5.00; what was his gain per cent.?

"When is a number divisible by 9?

"If a melon 20 inches in diameter is worth 20 cents, what is one thirty inches in diameter worth?

"At 4 per cent., what would be the amount due last Christmas on $1.00 put at interest at the beginning of the Christian Era, to be compounded annually?

"If 6 cats eat 6 rats in 6 minutes, how many cats will it take to eat 100 rats in 100 minutes?"

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2900. Proposed by I. A. BARNETT, University of Saskatchewan.

AB is the diameter of a circle and Q_0 any point on the circumference; $Q_1, Q_2, Q_3, \cdots$ are the points of bisection of the arcs $AQ_0, AQ_1, AQ_2, \cdots$; to prove that the product of the chords of the circle $BQ_1, BQ_2, BQ_3, \cdots, BQ_n$ is equal to $OA^n \cdot (AQ_0/AQ_n)$, O being the center of the circle.

2901. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Given the length of the base of a variable triangle and the positions of the feet of the altitudes on the other two sides, find (*a*) the locus of the vertex opposite the base; (*b*) the locus of the foot of the altitude on the base.

2902. Proposed by C. N. MILLS, Tiffin, Ohio.

Find the locus of a point the feet of perpendiculars from which, on the sides of a triangle, lie on a straight line.

Note by the EDITORS: It is well known that the circumscribed circle of the triangle is at least part of the locus, by virtue of the theorem of William Wallace, *Mathematical Repository,* March 1799. No proof of this result is called for in this problem.

2903. Proposed by A. A. BENNETT, University of Texas.

Given the base of a triangle in position and length, the length of the median to the base, and the difference of the base angles; find a simple ruler and compass construction for the triangle.

2904. Proposed by N. P. PANDYA, Amreli, Kathiawad, India.

Pairs of tangents to a conic intersect on a fixed straight line; find the locus of the middle points of the chords of contact.

2905. Proposed by the late ARTEMAS MARTIN.

A circle, radius a, is drawn on a circular slate radius r; a second circle, radius a, is drawn on the slate so as to intersect the first. If a third circle, radius a, be drawn on the slate, what is the probability that it will intersect the other two?

2906. Proposed by ELIJAH SWIFT, University of Vermont.

Given any number of five digits, reverse the order and subtract the smaller of the two numbers thus formed from the larger. Show that if told the last three digits of this difference, we can find the first two and give a simple rule for determining them.

2907. Proposed by G. E. RAYNOR, Princeton University.

Sum the following series:

$$(a)\ \sum_0^\infty \frac{1}{(2n+1)(3n+1)}; \qquad (b)\ \sum_0^\infty \frac{1}{(3n+1)(3n+2)}.$$

PROBLEMS—NOTES.

16. Skeleton Division. This note may be regarded as a continuation of note 2 (*1921*, 37). In *Nieuw Tijdschrift voor Wiskunde,* volume 8, 1920–21, Professor F. Schuh, of Delft, proposed (page 64) two arithmetical puzzles; one of them is herewith reproduced.

"The following calculation represents the development of an irreducible fraction to a repeating decimal:

$$
\begin{array}{rl}
\text{XXXXXX)} & \text{XXXXXXX}\ (\text{XX.X}\overline{\text{XXXXXXXXX}} \\
 & \underline{\text{XXXXXX}} \\
 & \text{XXXXXXX} \\
 & \phantom{\text{X}}\underline{\text{XXXXXX}} \\
 & \phantom{\text{X}}\text{XXXXXXX} \\
 & \phantom{\text{X}}\underline{\text{XXXXXXX}} \\
 & \phantom{\text{XX}}\text{XXXXXXX} \\
 & \phantom{\text{XXX}}\underline{\text{XXXXXX}} \\
 & \phantom{\text{XXX}}\text{XXXXXXX} \\
 & \phantom{\text{XXX}}\underline{\text{XXXXXXX}} \\
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 & \phantom{\text{XXXXX}}\text{XXXXXXX} \\
 & \phantom{\text{XXXXX}}\underline{\text{XXXXXXX}} \\
 & \phantom{\text{XXXXXX}}\text{XXXXXXX} \\
 & \phantom{\text{XXXXXX}}\underline{\text{XXXXXXX}} \\
 & \phantom{\text{XXXXXXXXXXX}}\text{XXXXXX} \\
 & \phantom{\text{XXXXXXXXXXX}}\underline{\text{XXXXXX}} \\
 & \phantom{\text{XXXXXXXXXXX}}\text{XXXXXX}
\end{array}
$$

Crosses indicate digits—all of which are possible. The repeating digits are shown by a line over the crosses. What irreducible fraction has been developed?"

ARC.

17. Curves of Pursuit (Compare *1921*, 54–61, 91–97, 184–185). I see that the AMERICAN MATHEMATICAL MONTHLY has been interesting itself in early examples of the Curve of Pursuit. Perhaps some of its readers may like these notes on four examples of it which appeared in England before 1750.

The general problem of the form of the curve, the pursued travelling in a straight line, is discussed in T. Simpson's *New Treatise of Fluxions,* London, 1737, p. 170, that is five years after the appearance of Bouguer's memoir.

A problem on it, by Robert Heath, appeared in *Ladies Diary,* 1742 and 1743 (Leybourn's reprint, vol. 1, p. 323), in the following form: "Admiral Vernon

sailing on a south course, from Jamaica to Carthagena, sees Don Blass right before him steering west, along the shore. He now continually bears directly upon him in a right line; when coming up with him, it appears that the Don had sailed 8 leagues during the chace, and that the said admiral was 7 leagues distant from him when the chace began: Now, supposing each ship's motion to be uniform during the whole chace, to find from thence the distance sail'd by Admiral Vernon?"

The following problem was propounded by Thomas Perryam in *Gentleman's Diary*, 1749: "A Captain of a Privateer seeing a Merchant Ship at S.S.E. sailing due West, continually bears directly upon her in a right Line; and coming up with her, it appear'd that the Merchant's Ship sail'd 30 Miles during the Chace, and that the said Privateer was 21 Miles distant from her when the Chace first began: Now supposing each Ship's Motion to be uniform during the whole Chace, To find from thence the Distance sail'd by the said Privateer, its East and West Departure, and also their Difference of Latitude when they were North and South of each other." A solution by fluxions was given in 1750.

Another question on the subject was propounded by John Ash in *Ladies Diary*, 1748, (Leybourn's reprint, vol. 2, p. 15) in the following form: "A spider, at one corner of a semi-circular pane of glass, gave uniform and direct chase to a fly, moving uniformly along the curve before him; the fly was 30° from the spider at the first setting out, and was taken by him at the opposite corner. What is the ratio of both their uniform motions[1]?" On this question the editor said "Mr. Landen sent us a true method [of solution]; but the calculus being so operose, it was not wrought out. And no method appearing to us yet elegant enough for a place, it will be next year before we shall have time to catch the solution." No further remarks on it appear in subsequent numbers of the *Diary*.

It would seem then that before 1750 such questions had become sufficiently familiar in England to appear in popular journals.

W. W. Rouse Ball.

Trinity College, Cambridge.
March 1, 1921.

18. Radius of the sphere circumscribing a tetrahedron. A member of the Association proposed the following problem which during the past century has been frequently solved: "Find, in terms of the lengths of the edges of a tetrahedron, the radius, R, of the circumscribing sphere." Denoting the pairs of opposite edges of the tetrahedron by a, a_1; b, b_1; c, c_1, L.N.M.Carnot derived[2] in 1806 the relation

$$\begin{aligned}4R^2(a_1^4a^2 + a^4a_1^2 + b_1^4b^2 + b^4b_1^2 + c_1^4c^2 + c^4c_1^2 + a^2b_1^2c_1^2 + c^2a_1^2b_1^2 \\ + b^2a_1^2c_1^2 + a^2b^2c^2 - a^2b^2b_1^2 - b^2c^2b_1^2 - b^2a_1^2b_1^2 - a^2a_1^2b_1^2 - b^2b_1^2c_1^2 \\ - c^2b_1^2c_1^2 - a^2b^2a_1^2 - a^2c^2a_1^2 - b^2c^2c_1^2 - a^2c^2c_1^2 - a^2a_1^2c_1^2 - c^2a_1^2c_1^2) \\ + 2b^2c^2b_1^2c_1^2 + 2a^2c^2a_1^2c_1^2 + 2a^2b^2a_1^2b_1^2 - a^4a_1^4 - b_1^4b^4 - c_1^4c^4 = 0.\end{aligned}$$

[1] Compare the notes on problem 2801 below.—Editor.

[2] *Mémoire sur la Relation qui existe entre les distances respectives de cinq points quelconques pris dans l'espace.* Paris, 1806, p. 11.

In 1847 Brassine gave,[1] without proof, the formula:

$$R = \frac{1}{24\cdot V}\sqrt{(aa_1+bb_1+cc_1)(aa_1+bb_1-cc_1)(aa_1+cc_1-bb_1)(cc_1+bb_1-aa_1)},$$

or

$$R = \frac{1}{6\cdot V}\sqrt{p(p-aa_1)(p-bb_1)(p-cc_1)},$$

where V is the volume of the tetrahedron, and $2p = aa_1 + bb_1 + cc_1$. That is, six times the product of the volume of the tetrahedron and of the radius of the circumscribed sphere is numerically equal to the area of a triangle whose sides are of lengths aa_1, bb_1, cc_1. This result was published in 1821 by Crelle.[2] In 1752 Euler gave in effect, the following expression,[3] in modern notation, for V:

$$\frac{1}{12}\sqrt{\begin{aligned}&[a^2a_1^2(b^2+b_1^2+c^2+c_1^2) - a^2a_1^2(a^2+a_1^2) + b^2b_1^2(c^2+c_1^2+a^2+a_1^2)\\ &\quad - b^2b_1^2(b^2+b_1^2) + c^2c_1^2(a^2+a_1^2+b^2+b_1^2) - c^2c_1^2(c^2+c_1^2)\\ &\quad - a^2b^2c^2 - a^2b_1^2c_1^2 - b^2c_1^2a_1^2 - c^2a_1^2b_1^2].\end{aligned}}$$

Killing and Hovestadt state this relation[4] in the form

$$\begin{aligned}144V^2 = {}&(a^2+b^2+c^2+a_1^2+b_1^2+c_1^2)(a^2a_1^2+b^2b_1^2+c^2c_1^2) - 2a^2a_1^2(a^2+a_1^2)\\ &- 2b^2b_1^2(b^2+b_1^2) - 2c^2c_1^2(c^2+c_1^2) - a_1^2b^2c^2 - a^2b_1^2c^2 - a^2b^2c_1^2\\ &- a_1^2b_1^2c_1^2.\end{aligned}$$

Baltzer gave[5] the determinant form:

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1\\ 1 & 0 & a^2 & b^2 & c^2\\ 1 & a^2 & 0 & c_1^2 & b_1^2\\ 1 & b^2 & c_1^2 & 0 & a_1^2\\ 1 & c^2 & b_1^2 & a_1^2 & 0\end{vmatrix},$$

and Joachimsthal, and Dostor the following:[6]

$$576V^2R^2 = -\begin{vmatrix} 0 & a^2 & b^2 & c^2\\ a^2 & 0 & c_1^2 & b_1^2\\ b^2 & c_1^2 & 0 & a_1^2\\ c^2 & b_1^2 & a_1^2 & 0\end{vmatrix} = -\begin{vmatrix} 0 & aa_1 & bb_1 & cc_1\\ aa_1 & 0 & cc_1 & bb_1\\ bb_1 & cc_1 & 0 & aa_1\\ cc_1 & bb_1 & aa_1 & 0\end{vmatrix}.$$

[1] *Nouvelles Annales de Mathematiques*, vol. 6, 1847, p. 227; a demonstration by de Perrodil is given on pages 396–398 of the same volume.

[2] A. L. Crelle, *Sammlung mathematischer Aufsätze und Bemerkungen*, Berlin, vol. 1, p. 117.

[3] *Novi comment. acad. sc. Petrop.*, vol. 4 (1752–53), 1758, p. 159.

[4] W. Killing und H. Hovestadt, *Handbuch des mathematischen Unterrichts*, Leipzig, vol. 2, 1913, p. 421.

[5] R. Baltzer, *Théorie et Applications des Déterminants*. Traduit de l'allemand. Paris, 1861 p. 206.

[6] F. Joachimsthal, Crelle's *Journal*, vol. 40, 1850, p. 33. G. Dostor, *Eléments de la Théorie des, Déterminants*, deuxième éd., Paris, 1883, pp. 281–282; also *Nouvelles Annales de Mathématiques*, vol. 32, 1873, p. 374.

Other derivations of expressions for the radius of the circumscribed sphere were given by Legendre, *Eléments de Géométrie*, 8e éd., Paris, 1809, pp. 302–304 of note V; by Baltzer, *Die Elemente der Mathematik*, Leipzig, volume 2, 1860, pp. 348–349; and by G. Holzmüller, *Elemente der Stereometrie*, Leipzig, volume 2, 1900, pp. 228–231. ARC.

PROBLEMS—SOLUTIONS.

2801 [1920, 31; 1921, 54–61, 91–97]. Proposed by A. S. HATHAWAY, Houston, Texas.

A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $k : 1$, determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

I. Additional Remarks by the Proposer.

If we let $d\sigma$ denote differential along the apparent path of the dog, we shall have

$$\left(\frac{d\sigma}{ds}\right)^2 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = MQ^2/a^2,$$

whence $d\sigma/dt = MQv/a$, where $v = ds/dt$ is the real velocity of the duck.

When $k > 1$, $MQ \geqq ka - a$, and so the apparent velocity of the dog within or on the edge of the pond is never less than $(k - 1)v$.

When $k = 1$, the apparent velocity of the dog is never zero within the pond, nor on the edge except at P. In fact the dog will approach P as limiting point.

When $k < 1$, MQ will be zero at the point where the ray $\sin\theta = k$ intersects the circle CKP. Let α be this value of θ and let w be the distance of Q from this point. Then

$$w^2 = r^2 + a^2\cos^2\alpha - 2ar\cos\alpha\cos(\theta - \alpha),$$

and

$$w\,dw/ds = -k(r + a\cos\alpha)[1 - \cos(\theta - \alpha)],$$

which is never positive, and is zero only when $\theta = \alpha$. Therefore w is always decreasing (except when $\theta = \alpha$), and the apparent path of the dog is a spiral converging to the point where the apparent velocity is zero, $\theta = \alpha$, $r = a\cos\alpha$.

It is interesting to note that if the speed of the dog be less than that of the duck, there is one position of starting where the relative positions of the two will remain unchanged, and that this is the limiting goal of the dog from whatever position he starts within the pond.

A number of interesting results may be deduced by determining entrances and exits on fixed curves. Thus on $r = ma\cos\theta$, exits and entrances are divided by the ray $(m - 2)\sin\theta + k = 0$, exits over the shorter arc.

II. Remark by H. P. Manning, Providence, R. I.

Mr. Morley has made a slight mistake (*1921*, 60) in regard to the cusps and inflections of his integral curves. The substitutions on page 55 lead first to an irrational equation for dv/dp, and the second derivative of v for a curve of pursuit is zero only when the integral curve crosses that part of the cubic which lies to the left of the point for which $p = c$. Between the two parts of the cubic all integral curves are concave downwards, but at any point outside of the cubic they curve in opposite directions; and so from a cusp the two branches curve away in opposite directions, one less steep and the other steeper than the slope $-2c$, and the cusps are all ordinary cusps, and not of the rhamphoid type. Indeed, one such cusp is at the point $(0, 1)$, where one branch corresponds to the curve of pursuit and the other to the "curve of flight."

III. Remarks and Historical Notes by H. P. Manning, and R.C. Archibald, Brown University.

We have remarked before (*1921*, 92) that the earliest reference then found, to a curve of pursuit where the pursued moved on a circle, was in Ficklin's problem of 1859. Mr. Ball has noted

above that a similar problem, that of a spider and a fly, was proposed, but not solved, in the *Ladies Diary* more than a century earlier. It appears now that this *Diary* problem suggested the following[1]: "To find the nature of the curve, described by a body giving uniform and direct chase to another, supposing the body pursued to move uniformly in the periphery of a given circle." The "solution," published in 1751, leads to Professor Hathaway's equations.

Using Professor Hathaway's notation and figure (*1921*, 94) and regarding the problem as that of a dog and duck, we can describe this "solution" as essentially the following:

First, the two fundamental equations, Professor Hathaway's (3) and (4), are derived.

The velocity with which the dog gains on the duck is equal to his own velocity diminished by the component along QP of the duck's velocity. This gives us at once equation (3).

To get equation (4) take the component along CQ of the dog's velocity. CQ being z, we have

$$dz/ds = k \sin KCQ = k(a \cos \theta - r)/z.$$

But

$$z^2 = r^2 - 2ar \cos \theta + a^2,$$

whence

$$z\,dz = (r - a \cos \theta)dr + ar \sin \theta d\theta,$$

or

$$(a \cos \theta - r)(k + dr/ds) = ar \sin \theta d\theta/ds,$$

which gives at once equation (4).

Ratios are used instead of the trigonometric functions, and x and y are used for r and PK $(= a \cos \theta)$.

The equation in x and y, equivalent to Professor Hathaway's (5), contains the expression $\sqrt{a^2 - y^2}$. It is proposed to develop this in a series and then to use "the method of resolving fluxional equations," a reference being given to Simpson's *Fluxions*. For the particular case of the spider and fly problem it is found that, velocity of the spider : velocity of the fly = 1.16 : 1.

The discussion concludes with the following note: "When the velocity of the body pursued is the greater of the two, the required curve will be a spiral converging continually nearer and nearer to the circumference of a circle concentric with the given one; whose radius is to that of the given one in the ratio of the lesser velocity to the greater."

As a further bibliographical note, reference may be given to problem 2971, proposed by the late Artemas Martin in *Educational Times*, volume 22, 1869, p. 141. It was as follows: "Show that the solution of the famous 'curve of pursuit problem' when the object pursued moves in the circumference of a circle and the pursuer starts from the center, can be made to depend upon the solutions of the differential equations

$$d\theta = -\frac{dt}{n - \cos \phi} \quad (1), \qquad tdt = r\{d(t \sin \phi) - ntd\phi\} \quad (2);$$

where r is the radius of the circle, t the distance the two objects are apart at any time during the motion, ϕ the angle t makes with a tangent to the circle, and θ the arc described by the pursued object from the commencement of the motion, supposing the pursuer to move n times as fast as the pursued." A solution of this problem, by James McMahon, appeared in *Mathematical Questions and Solutions from the "Educational Times"* volume 51, 1889, p. 159.

Captain Henri Brocard, of Bar-le-Duc, France, has kindly drawn our attention to solutions of the following problems (21, 22) proposed by Dr. W. Kapteyn in *Wiskundige Opgaven met de Oplossingen*, new series, volume 10, pp. 50–52, 1907: "21. A point C moves on a circle of radius equal to unity. Another point, situated originally at the center O of the circle, moves with the same velocity as the point C on a curve whose tangent passes constantly through this point. Prove that the radius of curvature at any point M of this curve is equal to the segment [measured from C] intercepted on the radius OC by the normal in M." 22. "Consider the radius of curvature ρ [at M], of the curve referred to in the previous question, as a function of the distance, p, of the origin from the tangent to this curve [at M]. Form the differential equation connecting ρ and p."

Of the first of these problems three solutions were published. In one of them, by the proposer, equations very similar to those employed by Mr. Morley (*1921*, 55) are derived.

The problems were reprinted in *Nouvelles Annales de Mathématiques*, February, 1907, pp. 95, 476; a solution of the first was given on page 173 by M. d'Ocagne, who refers to his paper "Centre de courbure des courbes de poursuite" in *Bulletin de la Société Mathématique de France*, volume 11, 1883, pp. 133–134.

[1] J. Turner, *Mathematical Exercises*, London, 1750–1752; no. 2, 1751, p. 32; no. 3, 1751, pp. 77–80. The problem and "solution" are reprinted in T. Leybourn's *Mathematical Questions proposed in the Ladies' Diary*, vol. 2, 1817, pp. 15–17.

2805 [1920, 32]. Proposed by C. N. MILLS, Brookings, S. Dakota.

Derive the expression for volume

$$v = \int\int\int \rho^2 \sin \phi d\rho d\phi d\theta.$$

In Byerly's *Integral Calculus*, page 183, revised edition, is a method of revolution, and in Czuber's *Integralrechnung*, page 200, is a method using the Jacobian determinant.

Required, a simple method one might use in developing the volume integral in polar coordinates.

SUGGESTION BY PAUL CAPRON, U. S. Naval Academy.

Draw a figure to show an octant of a sphere, as it might be of the earth with the North Pole atop, from Long. 0° to Long. 90° W. and from Lat. 0° to Lat. 90° N. The angle θ will correspond to the longitude, ϕ to the complement of the latitude. Show two meridians $d\theta$ apart and two parallels $d\phi$ apart. On the rectangular area so bounded build up an element of volume by extending through its corners four radii, projecting a distance $d\rho$ to a concentric spherical surface, on which they mark the corners of another rectangular base.

The sides of the inner rectangular base are $\rho \sin \phi d\theta$, $\rho d\phi$, $\rho \sin (\phi + d\phi)d\theta$, $\rho d\phi$; the sides of the outer base are the same, with $(\rho + d\rho)$ in place of ρ.

This point in the discussion should be reached at just about the end of hour. Conclude by leaving to the individual students the task of proving, each to his own satisfaction, that the neglected infinitesimals are of order higher than the first. Serve out enough applications to furnish a diversion of interest and start something useful and fascinating at the next lecture.

Also answered by T. M. BLAKSLEE, A. R. NAUER, H. L. OLSON, D. H. RICHERT, and ELIJAH SWIFT.

2806 [1920, 32]. Proposed by R. E. MORITZ, University of Washington.

An anthropologist told me recently that large numbers of Russian peasants, whose knowledge of numbers is limited to multiplication and division by 2, employ the following method of multiplication which they were taught by a priest.

(1) Write the two numbers to be multiplied in the same horizontal line.

(2) Multiply the first number by 2, and write the product under the number so multiplied.

(3) Divide the second number by 2, discarding the remainder 1 when it occurs, and write the quotient under the number so divided.

(4) Treat the product and quotient thus obtained in the same manner as the original numbers. Continue this process until the quotient 1 is obtained.

(5) Strike out all the numbers on the left for which the corresponding numbers on the right are even.

(6) Add the remaining numbers on the left. Their sum is the required product.

Problem: Prove that this rule is correct.

SOLUTION BY P. R. RIDER, Washington University.

The method depends on the fact that any number can be written in the binary scale of notation, that is, as a sum of positive integral powers of 2. To express a number in this way we divide successively by 2 until the quotient is 1. This 1 will be the first digit, beginning at the left, and the remainders (always either 1 or 0) will be the other digits expressing the number. That is, the remainder after the rth division by 2 will be the digit in the rth place, counting from the right, in the binary scale expression of the number, or the coefficient of 2^{r-1} in the expansion of the number in positive integral powers of 2. (See Todhunter's *Algebra for the use of Schools and Colleges*, chapter 29.) Thus, the multiplication process considered consists essentially in expressing one number in positive integral powers of 2 and multiplying the other number by this expression.

For instance, suppose that the numbers to be multiplied by the method are a and b, a heading the first column and b the second. Then the second column is the work of developing b in powers of 2, and the numbers in the first column are the terms of this development, each multiplied by a. For, if a number in the rth line of the second column is odd, there will be a remainder of 1, and

the corresponding digit in the expression of b in the binary scale is 1, giving the term of value $1 \times 2^{r-1}$ in the development of b in powers of 2. If, on the other hand, the number in the rth line of the second column is even there will be no remainder, and the corresponding digit in the binary scale expression of b is 0, giving the term of value $0 \times 2^{r-1}$ in the development of b. But the number in the rth line of the first column is $2^{r-1} \times a$, and since all numbers in the first column that are opposite even numbers in the second column are stricken out, the sum of the remaining numbers will be precisely $b \times a$.

A numerical example will make this much clearer. The work for the multiplication of 14 and 83 would appear (except for the figures in parentheses) as follows:

$(2^0 \times 14 =)$	* 14	83
$(2^1 \times 14 =)$	* 28	41
$(2^2 \times 14 =)$	56	20
$(2^3 \times 14 =)$	112	10
$(2^4 \times 14 =)$	*224	5
$(2^5 \times 14 =)$	448	2
$(2^6 \times 14 =)$	*896	1
	1162	

The number 83 expressed in the binary scale of notation would be 1010011 (*i.e.*, $83 = 2^1 + 2^4 + 2^1 + 2^0$). Thus the sum of those parentheses marked with an asterisk is $(2^6 + 2^4 + 2^6 + 2^0) \times 14$, or 83×14.

In presenting a similar discussion Professor U. G. MITCHELL cited the article in this MONTHLY, *1918*, 139–142, by Professor R. C. ARCHIBALD, entitled: "The binary scale of notation, a Russian peasant method of multiplication, the game of nim, and Cardan's rings." Many references are there given to the literature of the history and discussion of the binary scale and its applications.—EDITORS.

Also solved by T. M. BLAKSLEE, B. A. BERNSTEIN, PAUL CAPRON, CARL GUNDERSEN, W. H. HAYS, A. M. KENYON, THEODORE LINQDUIST, ROSCO LAMONT, H. F. MACNEISH, L. C. MATHEWSON, H. L. OLSON, ARTHUR PELLETIER, W. B. PIERCE, D. H. RICHERT, H. S. UHLER, and C. C. WYLIE.

2822 [1920, 185]. Proposed by A. M. HARDING, University of Arkansas.

Show that the sum of the series:

$$1 + 3\cdot 2 + 5\cdot 2^2 + 7\cdot 2^3 + \cdots + (2n-1)2^{n-1}$$

(to n terms) is $3 - 2^n + (n-1)2^{n+1}$.

SOLUTION BY LOUIS O'SHAUGHNESSY, Virginia Polytechnic Institute.

Set

$$S = 1 + 3\cdot 2 + 5\cdot 2^2 + 7\cdot 2^3 + \cdots + (2n-3)2^{n-2} + (2n-1)2^{n-1}.$$

Then

$$2S = 2 + 3\cdot 2^2 + 5\cdot 2^3 + \cdots + (2n-5)2^{n-2} + (2n-3)2^{n-1} + (2n-1)2^n.$$

Hence,

$$S = -1 - 2\cdot 2 - 2\cdot 2^2 - 2\cdot 2^3 - \cdots - 2\cdot 2^{n-2} - 2\cdot 2^{n-1} + (2n-1)2^n,$$

$$= -1 - \sum_2^n 2^n + (2n-1)2^n = -1 - (2^{n+1} - 4) + (2n-1)2^n = 3 - 2^n + (n-1)2^{n+1}.$$

Also solved by T. M. BLAKSLEE, H. N. CARLETON, P. J. DA CUNHA, E. B. ESCOTT, R. M. GINNINGS, H. HALPERIN, HARRY LEVY, L. C. MATHEWS, H. L. OLSON, ARTHUR PELLETIER, A. V. RICHARDSON, ETHELDRED A. WILLMOTT, and C. C. WYLIE.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

At Defiance College, O., Mr. E. H. VANCE was appointed professor of mathematics in January, 1921.

Professor EVA CHANDLER, of Wellesley College, retired in June, 1920, as professor emeritus. She was instructor of mathematics at Wellesley, 1879–1888; associate professor, 1888–1912; professor, 1912–1920.

Dr. GERTRUDE I. MCCAIN, professor of mathematics at Oxford College for Women, O., will teach at the summer school of Hunter College, New York City, in 1921.

Mr. R. L. WILDER, of Brown University, has been appointed instructor of mathematics at the University of Texas.

Mr. D. E. WHITFORD, of Brown University, has been appointed instructor of mathematics at the University of Rochester.

Instructor H. T. STETSON, of Harvard University, has been promoted to an assistant professorship.

Professor FRANK MORLEY, of the Johns Hopkins University, is now in England on leave of absence for the second semester of the current year.

At the University of Alberta, Dr. S. D. KILLAM, associate professor of applied mathematics, has been promoted to a full professorship, and Assistant Professor J. W. CAMPBELL (*1920*, 336, 498) has been promoted to an associate professorship.

W. F. PICKERING, assistant professor of astronomy and director of the Mandeville Station in Jamaica of the Harvard Astronomical Observatory, has been granted leave of absence for one year. He went first to England. Later he will probably study volcanic craters in the neighborhood of the Mediterranean Sea.

Miss RACHEL T. EASTERBROOKS, of Brown University, has been appointed instructor of mathematics and assistant in physics at Hood College, Frederick, Md.

Professor W. W. RANKIN, of the University of North Carolina, who has been on leave of absence for the last two years acting as instructor at Columbia University, has been appointed in charge of the department of mathematics at Agnes Scott College, Decatur, Ga.

Professor JOSEPH LIPKA, of Massachusetts Institute of Technology, is to spend the year 1921–1922 in Europe and Great Britain. He expects to work for some months with Professor LEVI-CIVITA in Rome.

At Union College, Mr. A. D. SNYDER of Lafayette College, Easton, Penn., and Mr. RALPH BENNETT have been appointed instructors of mathematics in place of Mr. COMSTOCK and Mr. TERWILLIGER (cf. *1921*, 234).

At Westminster College, New Wilmington, Pa., Associate Professor J. V. MCKELVEY of Iowa State College has been appointed professor of mathematics.

Associate Professor W. A. MANNING, of Stanford University, has been promoted to a professorship of mathematics.

Dr. M. C. FOSTER, of Yale University, has been made instructor of mathematics at the university.

The following appointments and promotions were made in the autumn of 1920:

At the Carnegie Institute of Technology, Pittsburgh, Mr. J. B. ROSENBACH, formerly of the University of Illinois, and Mr. C. H. KOHLER were appointed instructors of mathematics in the School of Science and Engineering. Mr. J. R. EVERETT, Dr. GEORGE HESS, and Mr. H. F. KOHL were appointed instructors of mathematics in the School of Applied Industry.

At the Georgia School of Technology, Atlanta, Ga., Mr. D. L. STAMY was promoted from an instructorship of mathematics to an assistant professorship. The following were appointed instructors of mathematics: Mr. W. W. ELLIOTT, Mr. P. L. ARMSTRONG, Mr. J. L. DRISCOLL, and Mr. L. H. HOLLER.

Miss IDA WHITAKER was appointed instructor of mathematics at the Municipal University of Akron, O.

At the Colorado School of Mines, Golden, Mr. G. W. GORRELL, formerly of DePauw University, was appointed professor of mathematics and Mr. W. R. HALE, formerly of Broadus College, was appointed assistant professor.

At Phillips University, East Enid, Okla., Mr. F. E. KNOWLES, formerly of the Oklahoma Agricultural College, was made professor of mathematics and physics. Mr. W. M. REEVES, formerly of Cotner College, was appointed associate professor of mathematics. Mr. A. J. HARGETT resigned to become professor of mathematics in Texas Christian University.

At Henry Kendall College, Tulsa, Okla., Dr. W. E. HOWARD was appointed in charge of the department of mathematics. Mr. O. S. DUFFENDICK, who had been in charge of the departments of mathematics and physics and who is during the present year on leave of absence, will return next year to take charge of the department of physics.

Mr. C. J. STOWELL was appointed in charge of the department of mathematics at McKendree College, Lebanon, Ill.

Mr. ALFRED DOOLITTLE, instructor of astronomy at the Catholic University of America, Washington, D. C. since 1913, died February 23, 1921. Son of the late Charles Leander Doolittle [*1919*, 178], and a brother of the late Eric Doolittle [*1920*, 438], he was born at Ontario, Ind., June 14, 1867. After receiving his A.B. from Lehigh University in 1887, he was instructor of mathematics and astronomy at Lehigh, 1889–91, assistant in the Nautical Almanac Office, 1892–97, instructor of mathematics and director of the astronomical observatory of The Catholic University of America, 1898–1901, 1906–1913. He was a piece-work computer at the Nautical Almanac Office from 1897 until his death.

Don TOMÁS DE AZCÁRATE, director of the Marine Observatory of San Fernando, Spain, since 1903, died on January 25, 1921, at the age of seventy-one years. He published at least one volume entitled *Anales del Instituto y Observatorio de Marina de San Fernando,* and had charge of the preparation of the *Almanaque Nautico.*

WILHELM JULIUS FÖRSTER, dean of German astronomers, died on January 18, 1921. He was born December 16, 1832. He joined the staff of the Berlin Observatory in 1855, became its director ten years later, and retired in 1904. "He

was one of the first two secretaries of the Astronomischer Gesellschaft, and served his country and the scientific world faithfully both there and in many other works. He was an active supporter of the Geodetisches Institut, and in connection both with that and with astronomical conferences he had a world wide circle of friends who will mourn his loss" (*The Observatory,* April, 1921).

MAGNUS NYRÉN, for the forty years 1868–1908 a worker at the Pulkowa Observatory, Russia, died January 16, 1921. Born in Sweden February 21, 1837, his earliest astronomical work was at the Upsala Observatory. He was the author of a large number of important papers (see "Poggendorff"). "After his retirement he lived in Stockholm, and last year published a catalogue of the proper motions of 633 stars in the new Pulkowa fundamental catalogue" (*Observatory,* March, 1921).

GUSTAV WILHELM LUDWIG STRUVE, younger brother of K. H. Struve whose death we have recently recorded (*1920,* 438) died November 4, 1920. Born October 20, 1858, he was astronomer in the Pulkowa Observatory 1880–1894. In 1894 he was made extraordinary professor and in 1897 ordinary professor of astronomy and geodesy at the University of Charcow. His best known work was on the constant of precession, but he took an active part in the Russian triangulation.

OSKAR LESSER, of the Klingeroberrealschule in Frankfurt a.M., Germany, who died September 23, 1920, was born October 4, 1867. He was the author of several books dealing with elementary mathematics through the calculus. For example: *Hilfsbuch für den geometrischen Unterricht an höheren Lehranstalten,* 1902; *Die Infinitesimalrechnung im Unterricht der Prima,* 1906 (second edition, 1911); and *Graphische Darstellungen im Mathematikunterricht der höheren Schulen,* 1908. There were also numerous editions and forms of K. Schwab and O. Lesser's *Mathematisches Unterrichtswerk,* of from one to three volumes each, 1909–1915. The editor of *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* has very kindly sent to us proof sheets of a five-page article on Oskar Lesser by G. Wolff. It concludes with a list of 19 titles of his publications.

Professor E. L. MARK, director of the Zoölogical Laboratory at Harvard University, has just retired from active teaching, and been appointed professor emeritus, after forty-four years of service on the staff of the University. In 1871 he received the degree of A.B. from the University of Michigan and taught mathematics there during the next year. He then served for a year as astronomer of the United States Northwest Boundary Survey. In 1873 he went to Europe. Returning in 1877 he became instructor in zoölogy at Harvard University.

At Cornell University Professor VIRGIL SNYDER has been granted leave of absence for the academic year 1921–1922. For the second term of the year he has been awarded, from the Heckscher Foundation for the promotion of research at Cornell University, a grant in support of research in the theory of algebraic surfaces. To Professor H. S. VANDIVER has been awarded a grant from the same Foundation in support of investigations on the theory of algebraic numbers. The Heckscher Foundation was established in 1920 by a gift to Cornell University of five hundred thousand dollars. The donor was August Heckscher, the New

York capitalist, miner, and manufacturer, born at Hamburg, Germany, in 1848. The plan for the administration of the Foundation was published in *Science*, October 15, 1920.

Of the five thousand dollars set aside for distribution in 1921 by the Committee on Grants of the American Association for the Advancement of Science, the following amounts were assigned for work in the fields of mathematics and astronomy: one hundred and fifty dollars to Professor SOLOMON LEFSCHETZ, of the University of Kansas, in support of his work in algebraic geometry; two hundred dollars to Dr. SEBASTIAN ALBRECHT, of Dudley Observatory, Albany, N. Y., in support of his investigation of the variation of wave-length of lines in different types of stellar spectra; and two hundred dollars to Miss CAROLINE E. FURNESS, of Vassar College Observatory, for assistance in the measurement and reduction of photographic plates.

Sir GEORGE GREENHILL was last March elected a corresponding member of the section in mechanics of the Academy of Sciences of the Institute of France (see *1920*, 384; *1921*, 238).

On April 20, 1921, Professor OSWALD VEBLEN was elected for the ensuing year a member of the executive committee of the Division of Physical Sciences of the National Research Council (cf. *1920*, 190).

In April, 1921, Professor G. A. MILLER, of the University of Illinois, was elected a member of the National Academy of Sciences in the Section of Mathematics. The other thirteen members of the Section are: G. D. BIRKHOFF, G. A. BLISS, H. F. BLICHFELDT, OSKAR BOLZA, L. E. DICKSON, EDWARD KASNER, E. H. MOORE, W. F. OSGOOD, W. E. STORY, E. B. VanVLECK, OSWALD VEBLEN, H. S. WHITE, and E. J. WILCZYNSKI. A. R. FORSYTH, DAVID HILBERT, M. E. C. JORDAN, C. F. KLEIN, JOSEPH LARMOR, C. E. PICARD, and VITO VOLTERRA are foreign associates of the Academy. Professor A. E. KENELLY, of Massachusetts Institute of Technology, was also elected a member, in the Section of Engineering.

In April, 1921, G. D. BIRKHOFF, of Harvard University, was elected a member of the American Philosophical Society. Among the members of this Society are the following mathematicians and astronomers: E. W. BROWN, W. W. CAMPBELL, L. E. DICKSON, L. P. EISENHART, H. B. FINE, G. E. HALE, E. O. LOVETT, J. A. MILLER, E. H. MOORE, FRANK MORLEY, F. R. MOULTON, W. F. OSGOOD, H. N. RUSSELL, FRANK SCHLESINGER, T. J. J. SEE, M. B. SNYDER, OSWALD VEBLEN, A. G. WEBSTER, E. B. WILSON, and R. S. WOODWARD. Among the foreign members are JOSEPH LARMOR, C. E. PICARD and VITO VOLTERRA.

Foreigners of the allied nations are now admitted as students in residence (élèves internes) at the Ecole Polytechnique, Paris.

The University of Nebraska offers to graduate students three fellowships in mathematics, of one thousand dollars each, for 1921–22. These fellowships are open to graduates of any good college or university. Appointees will be expected to devote about half their time to teaching.

At the annual meeting in Spokane the last of March the Mathematics Section of the Inland Empire Teachers' Association was reorganized as the Inland Empire Council of Teachers of Mathematics, with the intention of affiliating more closely with the National Council of Teachers of Mathematics. Professor W. C. EELLS of Whitman College, Walla Walla, Washington, was chosen first President of the

new organization. The greater part of the meeting was devoted to a discussion of the report, of the National Committee, on College Entrance Requirements in Mathematics. The sentiment of the teachers present was very favorable to the general features of this report.

At the meeting of the Academy of Sciences of the Institute of France held December 27, 1920, a paper by L. E. DICKSON was presented. It was entitled: "Les polynomes égaux à des déterminants." At the meeting of March 14, 1921, the following papers were presented: "La composition des polynomes" by L. E. DICKSON, and "Sur la position des racines des dérivées d'un polynome" by J. L. WALSH.

At recent meetings of the London Mathematical Society the following papers were communicated by title from the chair. On November 11, 1920: "On the conformal transformations of a space of four dimensions" by H. BATEMAN; "Arithmetic of quaternions" by L. E. DICKSON; and "The group of the linear continuum" by N. WIENER. On January 13, 1921: "Determination of all the characteristic sub-groups of an Abelian group" by G. A. MILLER.

At the spring meeting of The Association of Teachers of Mathematics in New England, held in Boston University, May 7, 1921, the following program was given: "Models for teaching geometry" by A. H. WHEELER, North High School, Worcester, Mass.; "Junior high school mathematics" by J. A. FOBERG, Pennsylvania State Department of Education; "Proportional representation" by E. V. HUNTINGTON, Harvard University; "Teaching arithmetic, algebra, and geometry in elementary and secondary schools, together and concretely" by C. W. ELIOT, president emeritus, Harvard University.

Among the papers presented at meetings of the American Philosophical Society, Philadelphia, April 21–23, were the following: "Further investigations on the relation between terrestrial magnetism and atmospheric electricity" by L. A. BAUER; "Discussion of a kinetic theory of gravitation" and "Some new experiments in gravitation" by C. F. BRUSH; "The atomic theory and ideal numbers" by L. E. DICKSON; "Discussion of the application of the method of the interferometer to the measurement of double stars" by J. A. MILLER; "Recent astronomical explorations in space and in time" by F. R. MOULTON; "The Roger Bacon cipher" by W. R. NEWBOLD.

At the annual meeting of the National Academy of Sciences held in Washington, April 25–26, 1921, the following papers were read: "On the problem of three or more bodies" by G. D. BIRKHOFF; "On the approximate solutions in integers of a set of linear equations" by H. F. BLICHFELDT; "Investigations in algebra and number theory" and "Quaternions and their generalizations" by L. E. DICKSON; "A model of the solar gravitational field" by EDWARD KASNER; "On the problem of steering an automobile around a corner," "On the radiation of energy from coils in wireless telegraphy," and "On the vibrations in gun-barrels" by A. G. WEBSTER. Dr. ALBERT EINSTEIN was present at public sessions of the Academy, and responded to an address of welcome presented to him by President Walcott.

At the April meeting of the San Francisco Section of the American Mathematical Society held at Stanford University on Saturday, April 9, the following papers were presented: "Anharmonic polynomial generalizations of the numbers

of Bernoulli and Euler," "Note on the prime divisors of the numerators of Bernoulli's numbers," "On a general arithmetical formula of Liouville," "Proof of an arithmetical theorem due to Liouville," and "The Bernoullian functions occurring in the arithmetical applications of elliptic functions" by E. T. BELL; "The approximate solutions in integers of a set of linear equations" by H. F. BLICHFELDT; "Euclid of Alexandria and the bust of Euclid of Megara," and "The spread of Newtonian and Leibnizian notations of the calculus" by F. CAJORI; "Normal ternary continued fractions" by P. H. DAVIS; "Autopolar curves and surfaces" by M. W. HASKELL; "On the computation of interest on certain kinds of investments" by D. N. LEHMER; "The hyperspace generalization of the lines on the cubic surface" by D. V. STEED.

At the sixteenth regular western meeting of the American Mathematical Society, held in Chicago, March 25 and 26, 1921, the following papers were presented: "New properties of all functions" by H. BLUMBERG; "Fallacies and misconceptions in Diophantine analysis" and "A new method in Diophantine analysis" by L. E. DICKSON; "Some new formulæ in combinatory analysis" by A. DRESDEN; "The group of motions of an Einstein space" by J. EIESLAND; "A disputed point regarding the nature of the continuum" by W. B. FORD; "Summable infinite determinants" by W. L. HART; "On a general theory of functions" (preliminary communication) by T. H. HILDEBRANDT; "The general theory of approximation by polynomials and trigonometric sums" by D. JACKSON; "A general theory of congruences" by E. P. LANE; "The equivalence of pairs of Hermitian forms" by MAYME I. LOGSDON; "Invariants and vector covariants of linear algebras without the associative law" by C. C. MACDUFFEE; "An overlooked infinite system of groups of order pq^2, p and q being prime numbers" by G. A. MILLER; "On the expansion of powers of trigonometric functions" and "On the summation of a trigonometric power-series" by I. J. SCHWATT; "Generational definition of linear associative hypernumbers" and "On Hamiltonian products" (second paper) by J. B. SHAW; "On non-loxodromic substitution groups in n dimensions" by E. B. VAN VLECK; "Some projective generalizations of geodesics" by E. J. WILCZYNSKI; "Congruences characterized by certain coincidences" by F. E. WOOD.

At the meeting of the American Mathematical Society in New York City, April 23, 1921, the following papers were presented: "A covariant of three circles" by A. B. COBLE; "Most general composition of polynomials" and "Number of real roots by Descartes' rule of signs" by L. E. DICKSON; "Concerning Laguerre's inversion" and "On certain type of system of ∞^2 curves" by J. DOUGLAS; "The Einstein solar field" by L. P. EISENHART; "The kernel of the Stieltjes integral corresponding to a completely continuous transformation" by C. A. FISCHER; "The mathematical theory of proportional representation" (third paper) by E. V. HUNTINGTON; "Topics in the theory of divergent series" by W. A. HURWITZ; "Note on an irregular expansion problem" by D. JACKSON; "Closed connected point sets which are disconnected by the omission of a finite number of points" by J. R. KLINE; "On the geometry of motion in a curved space of n-dimensions" by J. LIPKA; "Hyperspherical goniometry, with applications to the theory of correlation for n variables" by J. MACMAHON; "On the gyroscope" by W. F. OSGOOD; "On the apportionment of representatives" (second paper) by F. W. OWENS; "On the theorems of Green and Gauss" by

V. C. Poor; "On a simple class of deductive systems" by E. L. Post; "Pressure distribution around a breech-block" by J. E. Rowe; "Higher derivatives of functions of functions," "Method for the summation of a general case of a deranged series" and "The sum of a series as the solution of a differential equation" by I. J. Schwatt; "On the location of the roots of polynomials" by J. L. Walsh; "Seven points in space and the eighth associated point" by H. S. White; "A special kind of ruled surface" by J. K. Whittemore; "A new vector method in integral equations" by N. Wiener and F. L. Hitchcock.

Professor Albert Einstein arrived in the United States on April 2. Although he came primarily in the interests of the Zionist movement [*1921*, 191], he has been giving scientific lectures, in German, at various universities. On April 15 he lectured at Columbia University; on April 18, 19, 20, and 21 at the College of the City of New York; on May 3, 4, and 5 at the University of Chicago, and on May 9, 10, 11, 12, and 13 at Princeton University.

Courses in applied mathematics to be offered at the *Massachusetts Institute of Technology* during the year 1921–22 are as follows: Mathematical theory of investment, by Dr. J. S. Taylor, Fourier's series, by Dr. Norbert Wiener, Vector analysis, by Dr. S. D. Zeldin, Statistical mechanics, by Dr. H. B. Phillips, Aeronautics, by Dr. C. L. E. Moore, and Applied mathematics to chemistry, by Dr. F. L. Hitchcock (cf. *1920*, 242).

Additional courses in mathematics offered at Summer Sessions in 1921 (cf. *1921*, 194, 241) are as follows:

University of Colorado, First Term, June 13–July 20; Second Term, July 21–August 27. A review course in mathematics, by Professor G. H. Light, will be given the first term only. The following courses will be given throughout the quarter: College algebra, by Professor Light; Trigonometry, Plane analytic geometry, and Differential and integral calculus, by Professor G. W. Smith, of the University of Kansas; Differential equations, by Professor Abraham Cohen, of The Johns Hopkins University. The following term courses will be given either the first or second term: Teachers' course in mathematics, by Professor Cohen; Theory of equations, by Professor Light. One or more of the following term courses will be given by Professors Cohen and Light to meet the demands of the greatest number: Calculus of variations, Definite integrals, Theory of a complex variable, Elliptic integrals and functions, An introductory course in analysis, Differential geometry, and Series. Each course meets five days a week. Five hours credit is given for a course throughout the quarter; two and one-half hours credit for a term course.

University of Wisconsin, June 28–August 5. By Professor C. N. Slichter: Mechanics, 5 hours, and Applications of calculus, 3 hours. By Professor E. B. VanVleck: Survey of algebra and geometry, 5 hours, Analytic geometry, 5 hours, and Theory of integrals including applications to Beta and Gamma functions, 3 hours. By Professor L. W. Dowling: Plane trigonometry, 5 hours, Projective geometry, 5 hours, and Geometrical aspects of relativity, 3 hours. By Professor Warren Weaver: Theory of equations, 5 hours, Differential equations, 5 hours, and Wave theory, 3 hours. By Professor W. W. Hart:

Algebra, 5 hours, and Teachers' course, 5 hours. By Mr. J. E. Davis: Differential calculus, 12 hours. By Mr. M. H. Ingraham: Integral calculus, 12 hours. By Mr. Frederick Wood: Elementary analysis, 12 hours.

The Summer Meeting of the Association.

The sixth summer meeting of the Association will be held at Wellesley College, Wellesley, Mass., on Tuesday and Wednesday, September 6–7, 1921. It will immediately precede the meeting of the American Mathematical Society to be held at the same place September 7–9. A joint meeting of the two organizations will be held on Wednesday afternoon, when phases of relativity will be the topic for discussion, and there will be a joint dinner Wednesday evening. Detailed programs of the meeting will be mailed to all members of the Association at a later date. Tower Court will be available for the accommodation of attending members and friends at a rate of three dollars per day for room and meals. A separate wing will be reserved for ladies and married couples. The great hall of Tower Court will furnish an attractive meeting place for those in attendance. Several social functions and perhaps an excursion will be arranged. Lake Waban will be available for boating and bathing. Wellesley is fifteen miles west of Boston on the main line of the Boston and Albany Railroad.

Annals of Mathematics

The following notice appearing in the current issue of the *Annals of Mathematics* is commended to the attention of members of the Association: "According to an agreement between the Mathematical Association of America and the editors of the *Annals of Mathematics*, the Association contributes to the support of the *Annals*, and the *Annals* is supplied to individual members of the Association at one half of the regular price. In consequence of this agreement the volume of the *Annals* was increased by 100 pages which are devoted to expository and historical articles in so far as suitable articles of this class are obtainable. Thus far the editors have not received enough such articles to fill the space available, and therefore wish to call the attention of authors to this lack and to the fact that as long as the shortage continues expository or historical articles of sufficient merit will receive prompt publication.

"A number of the expository articles which have already been published are available in separate form and are listed for sale on the inside of the back cover of this number of the *Annals*. The regular subscription price of the *Annals* is $3.00 a volume."

At present 241 members of the Association are subscribers to the *Annals*.

[*Copy sent to the printer, April 23, 1921; printing delayed by a strike till August 15, 1921.*]

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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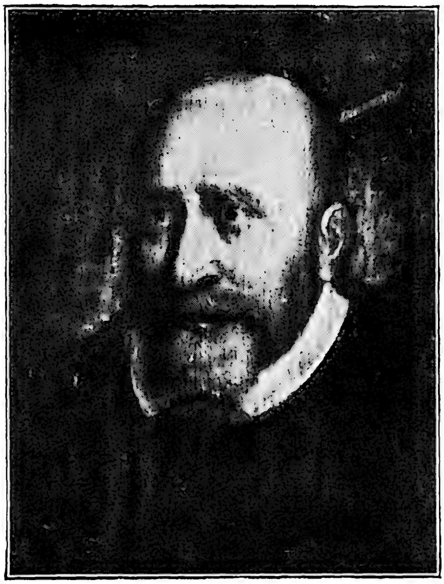
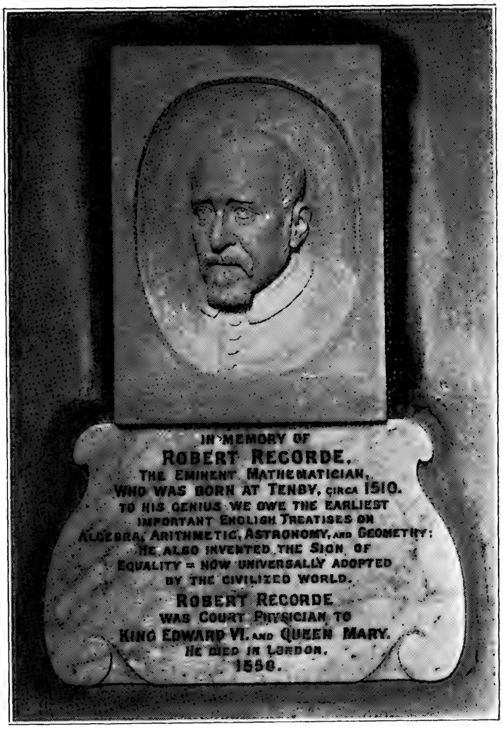


ROBERT RECORDE.

From a hitherto unpublished oil portrait.



TOMBSTONE IN THE PARISH CHURCH AT TENBY.

THE MARCH MEETING OF THE OHIO SECTION.

The sixth regular meeting of the Ohio Section was held at the Chemistry Building, Ohio State University, Columbus, on March 25, 1921, in connection with the meetings of the Ohio College Association. An afternoon and an evening session were held. Professor S. E. RASOR occupied the chair, being relieved by Professor HARRIS HANCOCK for an interval.

Forty-five persons were registered, the following twenty-seven being members of the Association:

R. B. Allen, W. E. Anderson, G. N. Armstrong, C. L. Arnold, C. B. Austin, Grace M. Bareis, W. S. Beckwith, A. A. Bennett, R. D. Bohannan, W. D. Cairns, A. G. Caris, V. B. Caris, E. H. Clarke, O. L. Dustheimer, T. M. Focke, Harris Hancock, H. W. Kuhn, C. N. Moore, C. C. Morris, Amy F. Preston, S. E. Rasor, Hortense Rickard, K. D. Swartzel, T. E. Trott, J. H. Weaver, R. B. Wildermuth, F. B. Wiley.

At the business session, the secretary reported a membership of 68 and 8 institutional members, as against 70 and 8, respectively, last year. The officers elected for this year are: Chairman, Professor B. F. YANNEY, College of Wooster; Secretary-Treasurer, Professor G. N. ARMSTRONG, Ohio Wesleyan University; third member of executive committee, Professor K. D. SWARTZEL, Ohio State University.

There were 32 persons present at the evening dinner and session held at the Ohio Union. The after-dinner reports upon the mathematical situation in twelve institutions were full of information and inspiration.

The following ten papers were presented at the regular meeting:

(1) Chairman's Address: "Functions and functionals" by Professor S. E. RASOR, Ohio State University;

(2) "An inquiry regarding collegiate departments of applied mathematics" by Professor G. N. ARMSTRONG, Ohio Wesleyan University;

(3) "Ballistic tables" by Professor A. A. BENNETT, University of Texas;

(4) "Mathematics in accounting" by Mr. W. E. LANGDON, Secretary of the Ohio Society of Certified Public Accountants, Columbus;

(5) "The problem of three bodies" by Professor E. S. MANSON, professor of astronomy, Ohio State University;

(6) "Mathematics and patent law" by Mr. S. S. DUNHAM, of Kerr, Page, Cooper & Hayward, New York City;

(7) "Mathematical problems in the work of the U. S. Coast and Geodetic Survey" by W. D. LAMBERT and O. S. ADAMS, Geodetic Division, U. S. Coast and Geodetic Survey;

(8) "Statistics, a comparison of the correlation coefficient and best-line deviation methods" by Professor W. D. CAIRNS, Oberlin College;

(9) "Solutions of the equation $\xi^2 + \eta^2 = \zeta^2$, where ξ, η, ζ are quadratic integers" by Professor HARRIS HANCOCK, University of Cincinnati;

(10) "Discussion of the reports of the National Committee on Mathematical Requirements" by Dr. C. N. Moore, University of Cincinnati, Mr. J. C. Boldt, Stivers High School, Dayton, and Miss Marie Gugle, Assistant Superintendent of Schools, Columbus.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. In Professor Rasor's paper the general idea of function was developed by reference to the work of Descartes, Leibniz, Euler, d'Alembert, and Dirichlet. This idea of function was then extended to functions of a line. Continuity, successive derivatives, and successive variations were then defined for them with the proof, by limits and by analogy, of the symmetry of the parameters entering in the second and higher derivatives. Illustrations of these functions of lines were given from the calculus of variations and in a generalization of Taylor's theorem. Euler's equation from the calculus of variations, thought of as a function of a line, was shown to be precisely the Volterra derivative.

2. Professor Armstrong's inquiry regarding collegiate departments of applied mathematics concerned itself with the content, conduct and administration of such courses and departments, and was based not only upon his own opinions but also upon the answers to a questionnaire on the topic, which was submitted to about 85 professors of mathematics in Ohio and other states. The main conclusions were these: There are no generally accepted definitions of pure and applied mathematics, although two types have been long recognized. Very few American universities make even a nominal distinction in their organization; that attempts to separate the two should be avoided, is the opinion of a large majority of mathematicians; what a great many of our institutions have been labeling "applied mathematics" is really "preëngineering mathematics"; there is very little research work in applied mathematics being done here of the kind carried on in certain European universities,—Edinburgh and Göttingen among others; the interest in the applications of mathematics has existed long and is increasing; men with tastes and talents in either line are equally worthy of encouragement and support in our faculties. An attempt was made to put into four groups all courses popularly considered as applied mathematics, and a list of thirty-five such courses was given. The latest to come into college instruction are those based upon the "error curve" and the "compound interest law." Formal instruction in computation is highly desirable.

3. Dr. Bennett discussed in this paper some physical and mathematical features connected with the question of the motion of projectiles. The primary problem for artillery is the range to be obtained. This is fundamentally empirical and is determined by firings. The range under standard conditions being once secured, the computation of changes in range for slight variations in conditions of firing is best derived from physical theory by mathematical methods. The nature and construction of tables was discussed.

4. In his paper Mr. Langdon stated that modern business concerns depend on accounting, in order to keep track of cash, customers' accounts or accounts

receivable, creditors' accounts or accounts payable, notes, material or merchandise, payrolls, equipment, investments, etc., as well as to determine the profit or loss. He listed the different kinds of accounting as: Single entry bookkeeping or memorandum records; double entry bookkeeping; auditing; statistical reports; graphic charts; business systems; cost accounting; income tax reports. He discussed the problems arising in each, dwelling particularly upon: (a) The purposes of an audit, which are, to ascertain the actual financial condition and earnings of an enterprise; to make income tax reports; to detect irregularities or errors; and (b): Cost accounting, which aids in the establishment of correct selling prices; coöperation between sales department and factory; elimination of waste and the preparation of periodical balance sheets, profit and loss, and income statements.

5. Professor Manson's paper consisted of a brief discussion of some points connected with the problem of three bodies, particularly the restricted case where the motion of a body of infinitesimal mass subject to the attraction of two finite masses which are themselves revolving in circles about their common center of mass is considered. It was shown analytically and also by a geometrical construction that if the infinitesimal body is in the plane of the orbits of the two finite masses and forms an equilateral triangle with them it will under proper initial conditions remain in equilibrium in this relative position and so maintain the equilateral configuration. The question of the stability of this solution was briefly discussed. Mention was made of the fact that the six asteroids of the so-called Trojan group seem to be oscillating with respect to these equilateral triangle points of equilibrium; two being about 60° behind Jupiter and the other four about 60° in advance. The fact of the existence of three straight line points of equilibrium; one between the two finite bodies, one between the first finite body and minus infinity and one between the second finite body and plus infinity, was also mentioned.

6. Mr. Dunham's paper contained a brief discussion of the nature of patents for inventions, and the reasons why a knowledge of mathematics beyond the rudiments of the science is a useful part of the patent lawyer's equipment, as a means of checking theories and effects, confirming results, detecting fallacies, etc., and as a means of generalizing. Illustrations having mathematical bearing were drawn from the speaker's experience.

7. The paper by Mr. Lambert and Mr. Adams will be published in the next issue of this MONTHLY.

8. Professor Cairns summarized the Bravais-Galton-Pearson method of finding the correlation coefficient and the regression equations from which, for given values of one or more variables, are found corresponding values of a dependent variable with the least (mean square) deviation of that variable. He contrasted with this the case occurring frequently, where there is no reason to regard one variable as depending on the others in the sense of cause and effect. A closer fit for a line in two variables or a plane in three variables can be obtained by imposing the condition that the mean square of the *normal* distances shall be

a minimum; like results hold for more than three variables. This is a direct generalization of the standard deviation method of comparing different distributions and yields a unique equation of relation between a variable as opposed to the usually inconsistent regression equations of the Pearson method. A short process for obtaining a first approximation for the best line or plane was given.

9. In this paper Prof. Hancock showed if $c_2 = (k^2+l^2)[1-m^2-p^2]$, $c_1 = mk+pl$, $b_1 = k + p(ml - pk)$, $a_1 = l + m(pk - ml)$, $b_2 = [(ml - pk)^2 - k^2][1 - m^2 - p^2]$, $a_2 = [(pk - ml)^2 - l^2][1 - m^2 - p^2]$, ($k$, l, m, p, arbitrary rational integers), be written in the equations: $x^2 - 2a_1x - a_2 = 0$ (a_1, a_2 rational integers), $y^2 - 2b_1y - b_2 = 0$ (b_1, b_2 rational integers), $z^2 - 2c_1z - c_2 = 0$ (c_1, c_2 rational integers), the roots of these three equations, *i.e.*, the algebraic quadratic integers ξ, η, ζ, say, satisfy the equation $\xi^2 + \eta^2 = \zeta^2$.

10. Prof. Moore presented a statement concerning the aims and progress of the reports of the National Committee on Mathematics Requirements. The succeeding discussion, by Mr. Boldt and Miss Gugle, dealt chiefly with the report on junior high school mathematics. In the succeeding general discussion, Prof. Weaver, speaking from experience in vocational and other schools, appealed for attention to the preparation of teachers as well as of students. Prof. Wildermuth emphasized the necessity of holding students to strict account for the work assigned; Professor Kuhn related his experiences with beginning junior high school mathematics in his own family. Further participants in the spirited discussions were Professors Beatty, Bennett, Rasor, and Dustheimer, and Supt. Collicott, and Miss Amy F. Preston of the Columbus Schools.

G. N. ARMSTRONG, *Secretary-Treasurer.*

NEW INFORMATION RESPECTING ROBERT RECORDE.

By DAVID EUGENE SMITH, Columbia University.

Our knowledge of Robert Recorde, the first mathematician of any note to publish works in the English language, is very meager. In general, recent writers have trusted to the books which he wrote and to various biographical notes that have appeared,—the former affording evidence of undoubted value, but the latter being little more than tradition.

Recorde's tombstone in the parish church at Tenby, Pembrokeshire, might be expected to be of assistance, but it turns out to be a modern one, affording no information of value. The inscription is as follows (compare the frontispiece):

IN MEMORY OF
ROBERT RECORDE,
THE EMINENT MATHEMATICIAN,
WHO WAS BORN AT TENBY, CIRCA 1510.
TO HIS GENIUS WE OWE THE EARLIEST
IMPORTANT ENGLISH TREATISES ON
ALGEBRA, ARITHMETIC, ASTRONOMY, AND GEOMETRY;
HE ALSO INVENTED THE SIGN OF

EQUALITY = NOW UNIVERSALLY ADOPTED
BY THE CIVILIZED WORLD.
ROBERT RECORDE
WAS COURT PHYSICIAN TO
KING EDWARD VI. AND QUEEN MARY.
HE DIED IN LONDON,
1558.

It is well known that Recorde was a student at Oxford, becoming a fellow of All Soul's College in 1531. In 1545 he received the degree of M.D. at Cambridge, and he is known to have taught mathematics in private classes, at both Oxford and Cambridge, prior to going to London as physician to Edward VI and Queen Mary. Roger Ascham was the Latin secretary to both the king and the queen at about that time, and it is probable that Recorde's ideas of textbook making were received in part from this great educator. Of the mathematical works which he seems to have written, only four were published.[1] These are well known and need not be mentioned in this connection.

A number of years ago the Reverend Done Bushell, rector at the Harrow School, purchased at a sale in Harrow a small portrait on an oak panel about 12 inches by 14 inches in size. The painting is apparently the work of a sixteenth century artist and is much dimmed by age. In the upper left-hand corner there is the inscription:

"Rob.t Record. M.D.
1556."

although this is so dim from age as not to show in the photographic reproduction given in the frontispiece.

As to the authenticity of the painting there can be no question. The spelling of the name is not unusual, Recorde sometimes using the final "e" and sometimes not. He was not a man of such prominence that his portrait would be painted after his death, and all the evidence goes to show that we have here an authentic painting from life, made in the year 1556, and the only portrait painting known.[2] This portrait is now reproduced by the courtesy of the present owner, W. F. Bushnell, Esq., master at the Rossall School, Fleetwood.[3]

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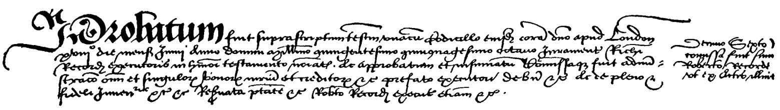
[1] The titles of these works, with the dates of the first editions, are as follows: *The Grounde of Artes*, 1541 (?); *The Castle of Knowledge*, 1551; *The Pathway of Knowledge*, 1551; *The Whetstone of Witte*, 1557.—EDITOR.

[2] W. F. Sedgwick, author of the article on Recorde in the *Dictionary of National Biography*, vol. 47, 1896, states that "the only known portraits of Recorde are woodcuts in the 'Urinal of Physick' and the 'Pathway to Knowledge.'" Consultations of copies of these works in the Library of Congress, in the Surgeon General's Library, Washington, in the Bodleian Library, Oxford, and in the Library of the British Museum, failed to reveal any such portraits. It is true that in the 1548 and 1567 editions of the *Urinal*, a cut two inches high by one and one quarter inches broad pictures the typical doctor. So also in the 1557 edition of the *Pathway* there is a figure (1 x $\frac{1}{2}$ inch) of a student at his desk in the letter, G, of the word geometry. Although such cuts have not uncommonly been treated as portraits, Mr. A. W. Pollard of the department of printed books in the British Museum writes concerning the cuts here: "I do not think that there is any reason whatever to treat any of them as a portrait of Record." Hence the portrait published in connection with Professor Smith's article is probably the only existing portrait of Recorde. —EDITOR.

[3] It will also appear in the writer's forthcoming *History of Mathematics*, volume 1, together with the facsimile of the *probatum* of the will mentioned below.

It has not heretofore been published, but it was used as a basis for the bas-relief upon Recorde's monument at Tenby.

When J. O. Halliwell wrote his *Connexion of Wales with the early science of England* (London, 1840) he embodied a number of statements which, owing to his authority in matters of English History, have been generally accepted, and which were apparently relied upon in the preparation of the article on Recorde in the *Dictionary of National Biography.* Among these are the statements that Recorde's will is in the Prerogative Office and that it was dated June 28, 1558. A search for the will reveals the fact that it is only the official copy that is preserved, and that this has no date. The will was proved, as the facsimile of the official record shows, on June 18, so that Recorde must have died before that date. The *Probatum,* in reduced facsimile, is as follows:



Inasmuch as the will does not seem to have been previously printed, and is so frequently mentioned in articles upon Recorde, it is of sufficient interest and value to deserve publication. Since the official transcript is written in the legal hand of nearly four centuries ago, it is difficult to make out all of the letters, but the following copy is substantially correct:

[In the margin: T(estamentum) Roberti Recorde.] *In the name of god amen.* fforasmuch as nothing is more certaine to man then deathe, and nothing more vncertain, the houre and tyme thereof, therefore knowe yō [= that] mi^{e} [Mr.? me?] Robert Recorde doctor of physicke though sicke in bodye yet whole in mynde thanckis be to god make my last will and testament in manner and fourme following. ffirst I comitt my soule unto thandes [= the hands] of the same allmighti god my only maker and redemer trusting by the merites of his passion to be one of his electe in glorye foreur [= forever]. my bodye as receyved from the earthe I bequeath thither again to be buryed among other christians according to the solempne vsage of the churche, my temporal goods and chattalles I ordre wille and dispoas in manner and fourme following. Secondly, I geve to Arthure hilton vnderm'shall [= undermarshall] of the kings benche whereat I now remayn prisonner xxd Item to his wife other xxd. Item to the gent [?] noix [?]prisonners w^{t} me xxxd. Item I give other xxd to the said Arthure hilton to be by him distributed amonge thofficers according to his discreation. Item to his wif to be distributed amonge her women vj s viij d. Item I give generally to the common gaole of the saide prisonne xl s to be equally distributed amonge the prisonners there. Item I give and bequeathe to myid [?] anone [?] mother and to my father in lawe her husbande xx li. Item I give to my s . . nnte (= servant) John xj li Item I give unto the children of John Battyn xl s to be distributed at the discreation of their saide father. The resideue of all my goods and chattalles moveable and unmoueable reall and psonnal. I give and bequeath

unto my brother Richard Recorde and Robert Recorde his sonne my nephewe whome I make and ordayne my full and hole executorns to thende that they beside my funeralles of the same shall truly and faithfully pay my detts. Whiche are to Nicolas ffulythm [?]citizen and merchanntaill9 [= merchant tailor] of London fiftie poundes to Mr Battyn xl s [?].

Memorand9 thatte saide testator on the morrowe next after the making of his testat aforesaide being then of his parfite mynde and memorye adding to the said testament gave and bequeathed to Alice and Rose Recorde daughters of the saide Richarde Recorde, and also unto Julian Raye all his vtensiles or householde stuff to be egallie diuided betwene them. Item he willed and diuised that Nicolas Adames then being prisonner in the kinges bench shulde have all his bookes concerning the lawes of this Realme at the price of iiij li. Witnes hereunto Richard Corbett George Marten and Richard Thymylby.

The tradition that Recorde was imprisoned for debt is not borne out by any evidence that I have been able to find. A search in the King's Bench Records has thus far been unproductive. There seemed to be a chance to find some evidence of payments to him as royal physician, or as Comptroller of the mines, but this, too, has not materialized, and the Memoranda Rolls and the Exchequer records are equally barren of information. In the printed Privy Council Acts, however, there are various entries relating to him as Comptroller of Mints and Monies in Ireland. There are also several entries about him in the Irish State Papers, from which entries it may be inferred that he was imprisoned owing to some misdemeanor in connection with the mines in Ireland. There is also a mention of the will in the *Index of Wills Proved in the Prerogative Court of Canterbury*, (volume 2, p. 257, London, 1898), but it contains no information of particular value. These sources are given as possible aids to anyone who may be desirous of pursuing the interesting historical investigation further. It may be added that Recorde was not Comptroller of the Mint at Bristol, as is often asserted; but the official documents show that this office was held by one Richard Recorde, presumably the brother who is referred to in the will. The statement that Recorde died "probably not long after making his will, June 28, 1558" (as in the *Dictionary of National Biography*) is also incorrect. The "XVIII" of the Probatum was probably read "XXVIII" by Halliwell or some earlier writer, and the error has been repeated by later biographers.

Editor's Bibliographical Note.

The article on Robert Recorde in the *Dictionary of National Biography*, volume 47, 1896, contains nearly forty references to various writings dealing with his life and works. The following additional references may be given:

M. Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 2, 2d ed., Leipzig, 1900, pp. 477–480, 552, 608, 621, 721, 791.

G. Eneström, "Sur l'algèbre de Robert Recorde (1546)," *Bibliotheca Mathematica*, 1901, series 3, vol. 2, p. 152.

G. Wertheim, G. Eneström, F. Amodeo, *Bibliotheca Mathematica*, series 3, vol. 3, 1902, p. 117; vol. 7, 1907, p. 290; vol. 8, 1908. p. 407.

J. Knott, "Robert Recorde, a pioneer mathematician, astronomer, and physician," *Indian Medical Record*, Calcutta, 1904, vol. 25, pp. 1–3.

D. E. Smith, *Rara Arithmetica*, Boston, 1908, pp. 213–221, 253, 286–288. [Title pages of *The Ground of Artes*, 1558, and *The Whetstone of Witte*, 1557, are here reproduced; also pages from these works showing counter reckoning and the explanation of the use of the sign for equality].

H. Zeitlinger and H. C. S., *Bibliotheca Chemico-Mathematica*, Catalogue 682 of Henry Sotheran & Co., London, 1908, p. 197; (also in *Bibliotheca Chemico-Matematica*, vol. 1, London, 1921, p. 197. There is here, opposite page 200, a facsimile page from *The Castle of Knowledge*, the first English work recognizing the Copernican system.)

L. C. Karpinski, "The Whetstone of Witte (1557)," *Bibliotheca Mathematica*, 1913, series 3, vol. 13, pp. 223–228.

J. Knott, "Robert Recorde," *Nature*, Dec. 7, 1916, vol. 98, p. 268, see also p. 172.

F. P. Barnard, *The Casting-Counter and the Counting-Board. A Chapter in the History of Numismatics and Early Arithmetic.* Oxford, 1916, pp. 254–266.

F. Cajori, *History of Elementary Mathematics with Hints on Methods of Teaching.* Revised and enlarged edition. New York, 1917, pp. 183–188; numerous other page references are given in the index.

F. V. Morley, "Finis coronat opus," *Scientific Monthly*, 1920, vol. 10, 306–308.

ON A DIOPHANTINE PROBLEM.

BY O. D. KELLOGG, Harvard University.

Professor Carmichael has been kind enough to take an interest in, and give some currency to, a problem in Diophantine analysis which I communicated to him some years ago.[1] I stated to him at that time that the maximum value of any of the unknowns that can occur in a solution in positive integers of the equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1 \tag{1}$$

was u_n, where $u_1 = 1$, and

$$u_{k+1} = u_k(u_k+1). \tag{2}$$

His remark, in the review cited, that a complete theory of this equation seemed desirable stimulated me to attempt to reconstruct the proof of my statement, which proof I do not seem to have preserved. The attempt, however, has given rise to a doubt in my mind as to my ever having had a really valid proof, but at the same time to confirm my belief in the accuracy of the statement.

It therefore seems to me proper to make the above confession to such of the readers of the MONTHLY as may have taken an interest in the problem, and to tell what I do know about it, for it has some rather nice aspects. Diophantine problems have at least the conspicuous merit that many phases of them are quite intelligible without profound knowledge of analysis, and for that reason are frequently a stimulus to mathematical interest.

The problem arose in connection with the familiar mapping of the surface of a triangle in the z-plane upon the upper half of the w-plane by means of an analytic function of a complex variable, $w = f(z)$. If the mapping is extended by

[1] See his *Diophantine Analysis*, New York, 1915, p. 115; also his review of Dickson's *History of the Theory of Numbers*, vol. 2, in this MONTHLY, *1921*, 77.

reflections of the triangle in its sides, and thus gives rise to a single valued function, it must be possible to cover the plane with non-overlapping triangles congruent with the given one. That is, if the angles are $a\pi$, $b\pi$, $c\pi$, then a, b, c must be aliquot parts of unity with sum equal to 1. It was an extension of the question of determining what those angles could be which was formulated in the above problem.

In a certain sense a problem may be considered as solved when it is shown how its solutions may be determined by a finite number of trials. For instance, the problem of finding the rational roots of a polynomial equation with integral coefficients may be regarded as settled by the theorem that if m and n are relatively prime, m/n cannot be a root of the equation unless n is a factor of the coefficient of the highest power of the variable present in the equation, and m is a factor of the coefficient of the lowest power. In the case of the problem of finding the positive integral solutions of the equation (1), it is clear that for $n > 1$, 2 is a minimum value for any unknown. Hence all that is needed in order to show that the solutions can be found by a finite number of trials is the determination of a finite upper limit for the numbers which may occur in a solution.

I therefore offer the following "proof" of my statement about the maximum of an x_i. Regarding as identical solutions obtained from each other by a permutation of the values of the x_i, we may fix our attention on those in which the x_i form a monotone increasing sequence. Then x_n will be a maximum when $(1/x_1) + (1/x_2) + \cdots + (1/x_{n-1})$ has "exhausted" 1 to as great a degree as possible, *i.e.*, when $1 - [(1/x_1) + (1/x_2) + \cdots + (1/x_{n-1})]$ has the least possible positive value. The greatest exhaustion of 1 which is possible by one term is obtained by taking $x_1 = 2$. This leaves 1/2. The greatest exhaustion of this 1/2 which is possible by the next term is obtained by taking $x_2 = 3$. The greatest exhaustion of the remaining 1/6 which is possible by the next term is obtained by taking $x_3 = 7$, and so on. In general, if u_k is the maximum value of x_k in the equation (1) with $n = k$, we exhaust 1 best with k terms by leaving $x_1, x_2, x_3, \cdots x_{k-1}$ unchanged, giving to x_k the value $u_k + 1$. This leaves

$$\frac{1}{u_k} - \frac{1}{u_k + 1} = \frac{1}{u_k(u_k + 1)}$$

for the $(k + 1)$th term, *so that* $u_{k+1} = u_k(u_k + 1)$ *is the maximum value of an* x_i *in the equation* (1) *with* $k + 1$ *terms.*

The "proof" is phrased as if entirely valid in order to allow the reader, if so disposed, to criticize it before reading the exposé given now. The logical lapse, of course, consists in the implied assumption that the minimum of a function (here $1 - (1/x_n)$) may be obtained by minimizing it *successively* with respect to each independent variable rather than *simultaneously* with respect to them all.

A few remarks seem worth while. First, it should be observed that the method sketched always gives *a* solution, namely $x_1 = u_1 + 1$, $x_2 = u_2 + 1$, $\cdots$, $x_{n-1} = u_{n-1} + 1$, $x_n = u_n$. Further, while space is lacking to give all the

considerations which tend to substantiate the correctness of the assertion about the maximum x_n, one or two are worth mentioning. It will be observed that x_n is not greater than the least common multiple of all the other x_i, as may be seen by subtracting the sum of their reciprocals from 1, and that it is equal to this least common multiple if they are all relatively prime. It therefore appears plausible that the maximum x_n will occur when the other x_i are all relatively prime. But this is a property of the number sequence $u_k + 1$. To show this, I form the difference equation for $v_k = u_k + 1$:

$$v_{k+1} = v_k^2 - v_k + 1. \tag{3}$$

Iteration shows that v_{k+m} is a polynomial in v_k with constant term 1, so that no divisor of v_k can divide v_{k+m}. The first four values of v_k are 2, 3, 7, 43, so that one might be tempted to guess that he had here a sequence of primes. He would be promptly undeceived, however, on finding that the next number, 1807 has the factor 13. Indeed it appears difficult even to decide whether or not the sequence contains an infinite number of primes. It is, however, a simple matter to show that certain primes never occur as factor in any of the v_k, for instance 5 and 11.

That the solution containing the maximum x_n is far from the symmetric one in which all the x_i are equal ($x_i = n$), is evidenced by the following theorem. *In the solution containing the maximum x_n, no two of the x_i can be equal, except possibly the last two.* For, if $x_i = x_{i+1}$, we have the equation in $n - 1$ unknowns

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{2}{x_i} + \frac{1}{x_{i+2}} + \cdots + \frac{1}{x_n} = 1,$$

which, on dividing by 2 and adding 1/2 to both sides, becomes the equation in n unknowns

$$\frac{1}{2} + \frac{1}{2x_1} + \frac{1}{2x_2} + \cdots + \frac{1}{x_i} + \frac{1}{2x_{i+2}} + \cdots + \frac{1}{2x_n} = 1,$$

where x_n has been replaced by a larger integer. Furthermore, in the solution containing the biggest x_n, not even the last two denominators can be equal *and even,* if $n > 2$. For $2/x_n$ can be replaced by the sum with a larger denominator

$$\frac{1}{\dfrac{x_n}{2} + 1} + \frac{1}{\dfrac{x_n}{2}\left(\dfrac{x_n}{2} + 1\right)}.$$

It seems as if the difference equations (2) and (3) might merit study, not only because they are probably the simplest type of non-linear difference equations, but because of some rather interesting arithmetical properties of their solutions. I will only remark that the second gives a rapidly convergent series for unity. It is

$$1 = \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \cdots, \qquad \text{or} \qquad 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots.$$

Not that it is recommended to use the series for computing the value of its sum approximately, but that Liouville[1] has proved the existence of infinitely many transcendental numbers by setting up rapidly convergent numerical series whose sums are transcendental. The present series does not converge as rapidly as those given by him, but it is more rapidly convergent than the exponential series for $x = 1$, which defines the transcendental number e. In fact v_n lies between $2^{(2^n-2)}$ and $2^{(2^n)}$.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

10. Delambre Aids in Freeing Spencer Stanhope.

Not only was Delambre influential in securing the release of the founder of the Smithsonian Institution from prison in Hamburg, as stated in a preceding article,[2] and thus in an indirect way in the establishing of the Institution itself, but he was of service to British science and art in performing a similar service for Spencer Stanhope, a young member of the famous Stanhope family and one who at that time gave much promise in antiquarian research.

One of Delambre's letters in my collection reads as follows:

The Perpetual Secretary for mathematical sciences, to His Excellency the Minister of War.
Monsieur Le Duc,

A young Englishman, M. Spencer Stanhope, made a prisoner of war through a series of unhappy circumstances surrounding his peaceful voyage to Greece, has begged the Imperial Institute to interest itself in his favor with respect to Your Excellency.

The research which M. Stanhope is undertaking is concerned particularly with antiquities, and this is the reason which has determined the Class of Ancient Languages to recommend to Your Excellency the [favorable consideration of the] petition of this young traveler. He also includes in his plan the rendering of aid to geography, to astronomy, and to all the sciences; he is supplied with telescopic octants and he proposes in his travels to join certain geographers with whom he expects to work in Greece and the adjoining countries. His undertaking will be of service to practically all Classes and sections of the Imperial Institute. He goes out from England in search of new information which he proposes to take back for the common advantage of all civilized nations. The Imperial Institute, in applauding his designs, believes that it participates in the intentions of a generous government in recommending to Your Excellency the [granting of the] request of M. Stanhope, who earnestly desires that he may be permitted to continue the voyage so unhappily interrupted.

I am, with respect, Monsieur Le Duc,
Your Excellency's very humble and
very obedient servant,
Delambre.

The letter bears an official memorandum of reference to the same official as the one relating to James Smithson, and doubtless was equally successful in accomplishing its purpose.

11. Voltaire and Mathematics.

Of those who admire that greatest champion of popular liberty in the eighteenth century, it is not probable that one in a thousand connects Voltaire's

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[1] *Journal de mathématiques pures et appliquées*, series 1, vol. 16, 1851, pp. 137ff.

[2] No. 1 of this series, *1921*, 64-65.

name with the science of mathematics. Perhaps it should be so, since the candle that he placed at the altar of science was dimmed by the great lights which he set before his shrines erected to literature and to political justice. And yet there was no man living in France in his time who did so much to make the philosophy of Newton known to the literati of Paris as this same Voltaire. He knew few of the details of mathematics, but he had read and admired the *Principia*, he had caught the spirit of the work, and he wished to bring this spirit to the attention of the literary world of France. No one could have done this so successfully as he, because, while half the literati hated him, all read his every written word. He first came really to know England when Newton was dying. He visited the niece, Mrs. Conduitt; from her learned the story of Newton and the apple; and twice repeated this story in his works. He was at Newton's funeral in Westminster Abbey, and one of his biographer's remarks, "When Voltaire was very old it is said 'his eye would grow bright and his cheek flush' when he said that he had once lived in a land where 'a professor of mathematics, only because he was great in his vocation,' had been buried 'like a king who had done good to his subjects.'"

He knew Maupertuis, and despised him; knew Koenig, "a very good mathematician and a very dull man," and became his champion; knew D'Alembert, and worked with him on the *Encyclopédie*; and knew—too well—the Marquise du Chastellet and assisted her on her *Principes Mathématiques de la Philosophie Naturelle.*

His own work on Newton's philosophy (*Elémens de la Philosophie de Neuton*) contains but little mathematics. It appeared twice in the same year (1738), in Amsterdam and in London, and made Newton's mathematics known to others if not to its author.

Among my autographs are two letters written by Voltaire. One refers to his *Oreste* which appeared in 1749 and was apparently written about that time. It begins in the style that Houdon had in mind when he chiseled the well-known bust that looks at the visitor when he walks through the foyer of the Théatre Français,—

> The old invalid of Ferney has written a very consoling letter to friend Pankouke, but the true consolation consists in having many purchasers.
>
> The good old man is well aware that there should be a modification of the last scene of *Oreste*,—[and so on, with a closing line]
>
> Ut ut est, mille amitiés.

Voltaire delighted to speak of himself as "the old invalid of Ferney." He was always ill, and always at work. For half a century he was dying, and finally passed away at the age of eighty-four, with intellect clear and with pen as vitriolic as ever.

Pankouke (to take Voltaire's spelling) was his publisher. *Oreste* had not been a financial success. He had changed and changed it during its short life on the stage, until one of the actresses rebelled and declined to receive Voltaire or his letters. The story goes that he finally sent her a pâté of partridges for a dinner she was giving,—and each partridge held in its beak a note containing changes in her rôle!

The second letter was written at Ferney on April 30, 1766, and is addressed to M. le Chevalier de Taulès. Voltaire was then seventy-two years old and his

handwriting had begun to show the advancing years; but his pen had not lost its cunning nor had his sarcasm begun to fail. It is, like others of his earlier years, a plea for human rights,—this time for two men in prison in Geneva.

"I do not know the rubrics of the city of Calvin—and I do not wish to know them! Twenty citizens have come to see me, as once the fishwives of Paris paid me a similar honor. I prepared for the latter a little compliment for the King, which was well received; and I did the same for these citizens, but this has not been received in similar fashion. It appears that certain gentlemen of twenty-five are greater seigneurs than the King. I did not know that fishwives had greater privileges than such citizens; but I ask your protection for these poor devils who know only that they exist. I do not speak of the 'perruques quarrées' but of these native citizens."

This is not mathematical; it is not the letter of a mathematician; but it gives a personal view of a man who did something for mathematics and everything that man could do in the middle of the eighteenth century for human liberty.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTION.

The discussion below by Professor Bradley leads to the request for consideration by readers of the MONTHLY of the following question:

45. Is every non-trivial solution in integers of the equation $t^3 = x^3 + y^3 + 1$ expressible in the form $x = 9r^4 - 3r$, $y = 9r^3 - 1$, $t = 9r^4$? If there are non-trivial solutions not expressible in this form, can a general solution be found?

REPLIES.

30. (*1916*, 88, 354; *1920*, 114, 362; *1921*, 124). A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

REPLY BY T. G. RODGERS, New Mexico Normal University.

It is disappointing that the above question has not called forth a full and varied discussion from which, at least, a certain minimum of essential courses might be agreed upon. Such a discussion would be of great value to a certain class of institutions and the secondary schools with which they are closely associated.

Readers of the MONTHLY may have considered that the replies to Question 31 (*1916*, 395–399), included the above. These excellent articles gave existing conditions in standard colleges and universities but had no reference to normal schools.

Until comparatively recent years no professional training of college rank for the benefit of secondary teachers of mathematics existed in this country and the normal schools confined their efforts to the training of teachers for the grades. At present, as the above replies point out, many colleges, especially the universities with large graduate departments, offer some courses designed primarily for teachers of secondary mathematics. The normal schools which are collegiate institutions, but not graduate institutions, are gradually teaching other college subjects besides their traditional two years of professional training for teachers of the grades.

The colleges and universities, however, though offering courses in the teaching of subjects, put the emphasis upon the development of an atmosphere conducive to the pursuit of scholarship and research rather than good teaching, while the normals lay the emphasis upon the development of enthusiasm for good teaching rather than for research. The universities supply the teachers for the large high schools which can afford to pay for graduate training; the normals supply the

best grade teachers. This leaves the small high schools, many offering less than a four years' course, which abound in the nation, without provision. This class of schools has been forced to recruit its teachers too largely from the grade, teachers who have had no direct training in the subject they are called to teach other than that gained in the secondary school.

With conditions as they are, the normal schools, by extending the scope of the work which they are doing, can best serve this class of high schools and also the rapidly developing junior high schools. For these schools thorough training in the best methods of teaching a subject with a fair amount of scholarship is of much greater value than more extended scholarship and indifferent methods of teaching.

Normal schools cannot offer more than thirty-five or forty semester hours of work in any particular subject and remain distinct institutions with a definite task to perform. Otherwise they lose their first love and become mere duplications of colleges. What should these courses be in mathematics? Who can say? For the purpose of provoking a valuable discussion, the writer ventures to suggest the following:

Plane trigonometry, three hours; algebra, three; analytic geometry, five; differential and integral calculus, six; modern geometry, five; the pedagogy of arithmetic, two; the pedagogy of algebra, two; the pedagogy of plane geometry, two; the pedagogy of solid geometry, two; observation in the practice high school, two; the remaining three hours to be given to a further study of analytic geometry, calculus or modern geometry, as the particular class elects. For the ends sought these introductory courses are better than longer courses in fewer subjects.

The work in trigonometry should emphasize the use of tables, the slide rule, the solution of triangles and a fair amount of theoretical work including inverse functions. The algebra should put the emphasis upon theory and logic rather than technique, and center around the simple progressions, quadratics, combinatory analysis, the theory of equations and the determinant. In analytics the straight line, the circle, functions and graphs, and the transformation of coordinates should receive the emphasis. For the purpose in view a six hour course based on some recent text in calculus of the type of that of March and Wolff will yield richer results than the more formal work in differential and integral calculus of the older type.

A high school teacher who can give but five hours to modern geometry will gain more that is of direct value to him if he divides it into a two hour course in simple theorems pertaining to concurrent lines, collinear points, the homothetic transformation, the transformation by inversion, the radical axis, coaxal circles and possibly polar reciprocal figures and applies them to construction work. This should be followed by a three hour course in synthetic projective geometry.

In each of the pedagogical courses a portion of the time should be given to a serious study of the history and literature of the subject of the course. There is an immense amount of material for this phase of the work. It must be gained from elementary histories of mathematics; books on the teaching of mathematics of such authors as Smith, Young and Stone; government bulletins on the teaching of mathematics, including the reports of the National Committee on Mathematical Requirements and the International Commission on the Teaching of Mathematics; and text books of different types, including some of foreign countries.

In addition to this purely pedagogical phase the prospective teacher needs the special study of the most difficult parts of each subject. In algebra he should take a dip into analysis, studying such topics as the number system, irrational numbers, limits, infinite series and topics that the individual class seems to need. In plane geometry he should get a survey of the foundations of geometry, become acquainted with the famous problems of antiquity and as far as time permits study the subject from the larger view of such works as Hadamard, *Leçons de Géométrie*. In solid geometry this larger view point so essential to a teacher can be gained by the study of selected portions of such works as Hadamard, *Leçons de Géométrie dans l'Espace*.

It is highly desirable that secondary teachers of mathematics become acquainted with the scholarly ideals of some of the best foreign works on elementary algebra and geometry. But for such a brief course the professor will have to make the selections and give them in lecture form to be worked up by the students.

With this as a foundation the students are in fine condition to observe in the practice high school, and meet one hour a week for conference and discussion of what they have observed. For a semester's work of this kind two hours of college credit seem sufficient.

Since this work is taken in a normal school it is understood that these students have also had the regular training in psychology and education.

It is admitted that this is a brief course in mathematics for a secondary teacher of mathematics. But until the standards of preparation for this class of teachers can be raised throughout

the nation, the writer believes this represents a step in the right direction for the preparation of teachers for the junior high school and the small high school.

DISCUSSIONS.

The two short discussions seem to require no comment. We should be glad if Question 45, printed above, which is called forth by Professor Bradley's discussion of the Diophantine equation $t^3 = x^3 + y^3 + 1$, should receive some interesting replies. Mr. Roman's note on a humble phase of the mathematics of investment should be entertaining.

I. On a Diophantine Equation.[1]

By H. C. Bradley, Massachusetts Institute of Technology.

On page 77 of the February number of the Monthly, Professor R. D. Carmichael asks, among other things, for a general solution of the Diophantine equation $t^3 = x^3 + y^3 + 1$.

I wrote to Professor Carmichael, calling attention to the following. In his own *Diophantine Analysis*, p. 65, he gives a general solution of $x^3 + y^3 = u^3 + v^3$ as follows:

$$x = -(a - 3b)(a^2 + 3b^2) + 1, \qquad y = (a + 3b)(a^2 + 3b^2) - 1,$$
$$u = -(a^2 + 3b^2)^2 + (a + 3b), \qquad v = (a^2 + 3b^2)^2 - (a - 3b).$$

Now let $a = 3b$, then let $2b = r$, and change the variables, and we have

$$x = 9r^4 - 3r, \qquad y = 9r^3 - 1, \qquad t = 9r^4;$$

which is a solution of $t^3 = x^3 + y^3 + 1$.

Or, if we assume the possibility of a solution of the form $x = Ar^4 - Br$, $y = Cr^3 - 1$, $t = Ar^4 + Dr$, the coefficients may be determined as above.

This solution gives $9^3 = 6^3 + 8^3 + 1$, $144^3 = 138^3 + 71^3 + 1$, $729^3 = 720^3 + 242^3 + 1$, etc. It fails to include the trivial case $x = y = 0$, $t = 1$.

Professor Carmichael replied that this was interesting, but asked if I could prove that it gave all non-trivial integral solutions of the given equation. This I cannot do. So at his suggestion I am sending the result, thinking that perhaps some other readers of the Monthly might be interested to work on it.

II. A Note on War Savings Stamps.

By Irwin Roman, Northwestern University.

The accompanying table illustrates an interesting property of the War Savings Stamps issued by the government in 1918. A similar table holds for later issues. The third column gives the amount which the post-office will pay for each five dollar stamp during the month. If we consider this amount as reinvested, the fourth column gives the profit accrued at maturity, while the fifth column gives the per cent. this profit is of the amount thus reinvested. The final column shows the annual rate of interest corresponding to the case.

[1] Extract from a letter to the Editor.

Table.

Year	Month	Redemption Value	Profit Actual	Profit Per Cent.	Rate Per Year
1918	Jan.	4.12	0.88	21.3	4.3
1919	Jan.	4.24	0.76	17.9	4.5
1920	Jan.	4.36	0.64	14.6	4.9
1921	Jan.	4.48	0.52	11.6	5.8
1921	April	4.51	0.49	10.9	6.2
1921	July	4.54	0.46	10.1	6.7
1921	Oct.	4.57	0.43	9.4	7.5
1922	Jan.	4.60	0.40	8.7	8.7
1922	April	4.63	0.37	8.0	10.7
1922	July	4.66	0.34	7.2	14.4
1922	Oct.	4.69	0.31	6.6	26.4
1922	Nov.	4.70	0.30	6.4	38.4
1922	Dec.	4.71	0.29	6.2	74.4
1923	Jan.	5.00	Maturity		

While the table speaks for itself, four conclusions may be mentioned:

1. Stamps bought at the redemption value bear a rate of interest, calculated as simple interest, varying from 4.3 to 74.4 per cent. per year.
2. It becomes increasingly desirable to hold the stamps till maturity.
3. When sale is necessary, the post-office becomes an increasingly cheaper buyer than a private individual.
4. A loan with the stamps as collateral becomes increasingly desirable as opposed to sale.

For the sake of simplicity, all calculations are referred to the first day of the month and all interest is assumed to be simple. The writer has never had this aspect of the War Savings Stamps called to his attention, and believes it might be of interest, if not of value, to readers of the MONTHLY.

RECENT PUBLICATIONS.

BOOK REVIEWS.

Les Principes de L'Analyse Mathêmatique. Exposê Historique et Critique. By PIERRE BOUTROUX. 2 volumes, Paris, A. Hermann & Fils, 1914–1919. Royal 8vo. xi + 547 + 512 pages. Price 14 + 20 francs.

In these two volumes we have one of the most suggestive historical and critical surveys of the whole field of elementary mathematics that has appeared in recent times. The purpose of the work may be best stated in the words of the author: "Except for these restrictions [demonstrations of certain theorems and detailed explanations of technique] the present work contains all or nearly all the material contained in the course in 'general mathematics' given in our

Faculties of Science. It moreover goes beyond them notably in that it touches in some places on certain of the highest chapters of modern analysis, and on the other hand it takes up mathematical science at its origin, in its principles, in order to present as nearly as can be done a view of the whole. . . . Its first object is none the less to furnish objective information and to serve as a guide to beginners in mathematics."

If the word "beginners" is properly interpreted—let us say, beginners in the work of our graduate schools—this object has been very successfully attained. The greatest value of the work, however, is for the teacher or advanced student who wishes to gain a comprehensive view of the whole field of mathematics, either as a body of knowledge, or more especially as a mode of thought. And many parts of it will be found valuable for supplementary reading on the part of undergraduates who are interested in getting at the bottom of things.

The care with which the fundamentals are discussed may be inferred from the fact that the first chapter, on (rational) "numbers," covers 61 pages, and the second, on "magnitudes" (including irrational numbers defined both arithmetically and geometrically) covers 118 pages. After passing more rapidly over some of the important topics and methods of elementary geometry, plane trigonometry, and algebra, we reach the study of functions, to which the remainder (239 pages) of the first volume is devoted, the point of view being almost purely arithmetical. The second volume enters upon the discussion of analytic geometry in two and in three dimensions, to which about 125 pages are given; and then follows a chapter (97 pages) on "extension of algebra, and logical constructions," which includes the elementary parts of the algebra of complex numbers, the elementary parts of group theory, with an indication of their applications to the theory of algebraic and differential equations; and a discussion of the foundations of algebra and geometry. Then follow developments in series (36 pages) and infinitesimal calculus (85 pages). The last two chapters, entitled "Analysis of Mathematical Principles" and "Analysis of the Notion of Function" cover 37 and 71 pages respectively. The last chapter includes the elementary properties of functions of a complex variable, such as Cauchy's integral theorems, Taylor's series, singular points, analytic extension, and simply and doubly periodic functions, with mention of elliptic functions and integrals.

As an example of the methods of treatment which are employed, we may take the subject of the conic sections. They are first taken up as sections of a cone, in vol. I, pages 241ff., where the work of the Greeks is cited, without however being described in detail; then the curves are discussed more at length as plane loci, but without any use of coördinates; and only in vol. II, pages 50–65, do we meet with the study of these curves from their equations. At this point it seems regrettable that the author has let slip the opportunity to point out the connection between the locus problem and the equation; there is no intimation that the equation could have been obtained from the definition of the curve as a locus, but only the converse procedure is mentioned, and that, too, starting from the general equation. Another detail of arrangement seems to the re-

viewer defective: that determinants are taken up immediately *after* the study of transformations of coördinates has been completed; had this order been changed, the classification of equations of the second degree could have been stated in a neater and more concise manner, and much of the work in analytic geometry of 3 dimensions could likewise have profited. Indeed no mention is made in connection with determinants that they are used at all in geometry.

The author's view of the nature of mathematical analysis is brought out clearly in the admirable introductory chapter to the third book (volume II, pages 259–267). Analysis, as understood today, may be regarded as a combination and reconciliation of the two tendencies which have been more or less in opposition one to the other throughout the whole history of mathematics: the tendency to look outward upon the world and seek to understand and explain it, and the tendency to look inward to the forms of thought itself, and to develop methods of reasoning for their own sake. Modern analysis is in fact concerned with both these points of view, and Professor Boutroux has summarized the historical developments in such a way as to make the fact stand out strikingly. The same subject is continued and amplified in the penultimate chapter, where the fundamental relations of number and magnitude are subjected to a careful scrutiny, in the light of the results of the infinitesimal calculus. It is brought out clearly that in the modern idea of the irrational number we have the bridge that connects the discrete world of (cardinal) *numbers* with the continuous world of (geometric) *magnitude*. Thus we gain a firm basis for the concept of continuity, and the general notions of the function theory, with which the book concludes.

A few slips occur: thus, vol. I, p. 20, line 4 from the end, *n times unity* should be $(n+1)$ *times unity*, and similar changes should be made three times at the top of p. 21. On p. 38, at the end of § 31, the statement that "the first rigorous and complete exposition of calculation with fractions is found in Stevin's *Arithmétique*" is too sweeping, as (to mention only one name) Leonardo of Pisa[1] gives an account that leaves nothing to be desired—unless probably brevity. On p. 139, the two expressions are equal $(= 4/\pi)$ instead of one being $= 2/\pi$ as stated. On p. 183, Thales is said to have had scarcely any but a practical knowledge of geometry; this seems overdrawn in view of Proclus's ascription to him of "general treatment" (*καθολικώτερον*) of certain problems, along with the general, if less well-authenticated, crediting of the theorem of the right angle inscribed in the semicircle to him. On page 521, Fermat is credited with having stated "very clearly the conception of the derivative and the characteristic [differential] triangle." As to the derivative, this is surely overdrawn, and it seems doubtful if it can be maintained as to the differential triangle. In vol. II, page 58, the word "supplementary" in the next to the last line should be replaced by "one exceeding the other by π."

There is an unfortunately large number of misprints, even after those noted in the "Errata" have been corrected; perhaps those most likely to be confusing are vol. I, page 90, next to last line, read *parallèles* for *perpendiculaires*; page 146,

[1] *Scritti di Leonardo Pisano*, Rome, 1857, vol. I, pp. 47–83.

line 4 from end, interchange *grande* and *petite*; page 169, lines 5 and 6, interchange *sinus* and *cosinus*; page 183, line 4 of the note, for *qualité* read *quantité;* page 266, line 11, for *aa, bb, cc,* read *a, b, c*; page 406, last and 4th from last lines, interchange *maximum* and *minimum;* and similarly on page 522, where in the last line the inequality signs should be reversed; vol. II, page 56, the figure should be numbered 245.

None of these minor blemishes, however, can affect the final judgment of the work, which is that through its clear presentation and discriminating evaluation of so many of the most important topics in the whole field of mathematics, as well as through its stimulation to further thought and investigation along all lines, this book is a really great contribution to mathematical literature, which should be in the possession of every teacher.

R. B. McClenon.

Tables of the Exponential Function and of the Circular Sine and Cosine to Radian Argument. By C. E. Van Orstrand. (*Memoirs of the National Academy of Sciences,* volume 14, fifth memoir.) Washington, D. C., 1921. 4to. 79 pp.

Mr. Van Orstrand, a physical geologist of the U. S. Geological Survey, and a charter member of the Association, has once more rendered fine service to workers in certain fields of applied mathematics. In 1909 he collaborated with G. F. Becker in publishing the 360 page volume, *Hyperbolic Functions* (Smithsonian Mathematical Tables). He has now published fourteen tables concerning which some preliminary publications appeared in *Journal of the Washington Academy of Sciences*, 1912–13. The tables are as follows:

I: Values of the reciprocal of $n!$ to 108 places of decimals at intervals of unity from 1 to 74; II: Values of e^x to 42 significant figures at intervals of unity from 0 to 100; III: Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0; IV: Values of e^x to 62 places of decimals at decimal intervals from 1×10^{-14} to 9×10^{-1}; V: Values of e^{-x} ranging from 52 to 62 places of decimals at intervals of unity from 0 to 100; VI: Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0; VII: Values of e^{-x} to 62 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-1}; VIII: Values of $e^{\pm(n\pi/360)}$ to 23 places of decimals or significant figures at intervals of unity from $n = 0$ to $n = 360$; IX: Values of $e^{\pm n\pi}$ to 25 places of decimals or significant figures for various values of n; X: Values of $\sin x$ and $\cos x$ to 23 places of decimals at intervals of unity from 0 to 100; XI: Values of $\sin x$ and $\cos x$ to 23 places of decimals at intervals of 0.1 from 0.0 to 10.0; XII: Values of $\sin x$ and $\cos x$ to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600; XIII: Values of $\sin x$ and $\cos x$ to 25 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-4}; XIV: Miscellaneous values of e^x, e^{-x}, $\sin x$ and $\cos x$ to a great number of decimals, including Boorman's value of e.

In preparing such tables the author sought not only to obtain "a few high place values at sufficiently small intervals of argument for general use in the evaluation of integrals and other functions," but also to obtain "a basis for

subsequent interpolation to small intervals of argument for use in the construction of complete 10-place tables which are applicable in the various fields of pure and applied mathematics."

The author remarks that the most important tables of extended values of the exponential function in which the exponents are integers or fractions were those constructed by Schulze (1778), Bretschneider (1843), Newman (1883, 1889), Gram (1884), Glaisher (1883), and Burgess (1900). The extent of the contribution of each is indicated. There is no reference to Salomon's *Tafeln*, 1827, where the values of e^n, $e^{\cdot n}$, $e^{\cdot 0n}$, $\cdots$ $e^{\cdot 000000n}$, for $n = 1, 2, 3, \cdots 9$, may be found.

The only previous table of the reciprocal of $n!$ seems to have been the one by Glaisher,[1] in 1877, to twenty-eight figures as far as $n = 50$. Mr. Van Orstrand shows that Galisher's table is in error, by one unit, in the twenty-eighth figure of each of the numbers $n = 20, 27, 41, 50$. An elaborate table of $\log_{10} n!$ from $n = 1$ to $n = 1200$ to eighteen places[2] was given in C. F. Degen, *Tabularum Enneas*, Copenhagen, 1824.

In Table IX the values of $e^{n\pi}$ are given for the following sixty values of n: $\pm 7/6$, $\pm 13/6$, $\pm 19/6$, $\pm 5/4$, $\pm 9/4$, $\pm 13/4$, $\pm 4/3$, $\pm 7/3$, $\pm 10/3$, $\pm 3/2$, $\pm 5/2$, $\pm 7/2$, $\pm 5/3$, $\pm 8/3$, $\pm 11/3$, $\pm 7/4$, $\pm 11/4$, $\pm 15/4$, $\pm 11/6$, $\pm 17/6$, $\pm 23/6$, ± 2, ± 3, ± 4, ± 5, ± 6, ± 7, ± 8, ± 9, ± 10. Three of these, namely for n equal to $-9/4$, -4, and -9, are included in the seventeen values of $e^{n\pi}$ (or $2e^{n\pi}$) given in this MONTHLY, *1921*, 115–120. On page 11 of his introduction Mr. Van Orstrand remarks that "The value of $e^{\pi/2}$ given by Gauss is incorrect in the twenty-third and following decimals"; this error seems to have been first pointed out in this MONTHLY, *1921*, 120.

In table XIV is included (except for one error[3]) the value of e to 346 places of decimals published by J. M. Boorman, "consultative mechanician and attorney at law, Brooklyn, N. Y.," in *Mathematical Magazine*, Washington, August, 1887; it is here pointed out that Shanks's computation of e (1854) to 205 places was incorrect beginning with the 188th decimal. Mr. Van Orstrand states that Tichánek and Minks verified (1892) Boorman's value of e to 223 decimal places, giving as authority *Jahrbuch über die Fortschritte der Mathematik*, volume 23, p. 441 and volume 25, p. 736. A comparison of the numbers shows that they differ in the forty-third decimal place; at that place Boorman gives (correctly) "0" and not "6."

R. C. ARCHIBALD.

April 29, 1921.

The Copernicus of Antiquity (*Aristarchus of Samos*). By T. L. HEATH. (Pioneers of Progress, Men of Science.) London, Society for Promoting Christian Knowledge, 1920. 4 + 59 pages. Price 2 shillings.

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[1] *Cambridge Philosophical Society Transactions*, vol. 13, pp. 246–247.

[2] De Morgan gave a six-place abridgment in his article on "Theory of Probabilities" in *Encyclopedia Metropolitana*, 1837.

[3] Mr. Van Orstrand gives "0" instead of "6" in the thirty-second decimal place. This error may be verified by the computations of Shanks (*l.c.*) and of Glaisher (*l.c.*).

First three paragraphs: "The title-page of this book necessarily bears the name of one man; but the reader will find in its pages the story, or part of the story, of many other Pioneers of Progress. The crowning achievement of anticipating the hypothesis of Copernicus belongs to Aristarchus of Samos alone; but to see it in its proper setting it is necessary to have followed in the footsteps of the earlier pioneers who, by one bold speculation after another, brought the solution of the problem nearer, though no one before Aristarchus actually hit upon the truth. This is why the writer has thought it useful to prefix to his account of Aristarchus a short sketch of the history of the development of astronomy in Greece down to Aristarchus's time, which is indeed the most fascinating portion of the story of Greek astronomy.

"The extraordinary advance in astronomy made by the Greeks in a period of little more than three centuries is a worthy parallel to the rapid development, in their hands, of pure geometry, which, created by them as a theoretical science about the same time, had by the time of Aristarchus covered the ground of the Elements (including solid geometry and the geometry of the sphere), had established the main properties of the three conic sections, had solved problems which were beyond the geometry of the straight line and circle, and finally, before the end of the third century B.C., had been carried to its highest perfection by the genius of Archimedes, who measured the areas of curves and the surfaces and volumes of curved surfaces by geometrical methods practically anticipating the integral calculus.

"To understand how all this was possible we have to remember that the Greeks, pre-eminently among all the nations of the world, possessed just those gifts which are essential to the initiation and development of philosophy and science. They had in the first place a remarkable power of accurate observation; and to this were added clearness of intellect to see things as they are, a passionate love of knowledge for its own sake, and a genius for speculation which stands unrivalled to this day. Nothing that is perceptible to the senses seems to have escaped them; and when the apparent facts had been accurately ascertained, they wanted to know the *why* and the *wherefore*, never resting satisfied until they had given a rational explanation, or what seemed to them to be such, of the phenomena observed. Observation or experiment and theory went hand in hand. So it was that they developed such subjects as medicine and astronomy. In astronomy their guiding principle was, in their own expressive words, to 'save the phenomena.' This meant that, as more and more facts became known, their theories were continually revised to fit them."

Contents—Part I, *Greek Astronomy to Aristarchus*, 1–37: Thales; Anaximander; Anaximenes; Pythagoras; Parmenides; Anaxagoras; Empedocles; The Pythagoreans; Œnopides of Chios; Plato; Eudoxus, Callippus, Aristotle; Heraclides of Pontus. Part II, *Aristarchus of Samos*, 38–56: The heliocentric hypothesis; On the apparent diameter of the sun; On the sizes and distances of the sun and moon; On the year and 'great year'; Later improvements on Aristarchus's figures. Bibliography, 57–58. Chronology, 59.

Problems and Solutions. Associateship Examinations, Parts I and II, 1915–1919. New York, Actuarial Society of America, 1921. 8vo. 133 pages + 46 figures. Price $2.00.

Foreword: "The problems herein set forth with their solutions comprise all of the problems set in the years 1915 to 1919 inclusive, in Parts I and II of the examinations for admission to Associateship in the Actuarial Society of America. These problems and solutions are published primarily for the use of students preparing for these particular examinations; but they will unquestionably be of value to many who are teaching or studying mathematics in high school or college.

"Prior to 1920 these two examinations comprised what was known as *Section A* of the Associateship examination. The nomenclature has been changed so that they are now known simply as Part I and Part II of the Associateship examination.

"In many instances the solutions as set forth are not the only solutions of the given questions and it is not our intention to infer that the published solutions would have been more acceptable than any others to the Examination Committee of the Society. We have simply attempted to present a correct solution to each problem. The major part of the work of editing and arranging the solutions for publication was done by Dr. LESTER R. FORD of The Rice Institute, working in coöperation with the Educational Committee [J. M. Laird, J. F. Little, E. W. Marshall, H. N. Stephenson, and M. A. Linton, chairman] of the Actuarial Society, under whose supervision the book has been published. Charles M. Taylor, a student of the Society, rendered valuable service in reading the proof and making helpful suggestions."

The volume contains 279 problems in all: 9 in arithmetic, 51 in elementary algebra, 16 in plane geometry, 36 in plane trigonometry, 15 in analytical geometry, 9 in bookkeeping, 45 in advanced algebra, 20 in the theory of probabilities, 28 in calculus of finite differences, and 50 in calculus.

NOTES.

The concluding number of *Proceedings of the Royal Society*, London, series A, volume 98, published March 24, 1921, contains a fifty page notice of "John William Strutt, Baron Rayleigh, 1842–1919." There is also a fine frontispiece portrait.

The Mannheim & Polyphase Slide Rules. A Self Teaching Manual with tables of settings, equivalents and gauge points is the title of an 80-page pamphlet mainly by W. E. BRECKENRIDGE, associate in mathematics at Columbia University (New York, Keuffel & Esser, 1920). The supplement, "The slide rule in trigonometry" (pages 63–77), was written by Professor J. M. WILLARD, of the State College of Pennsylvania.

A. E. H. Love's *Theoretical Mechanics: an introductory treatise on the principles of dynamics, with applications and numerous examples* was first published by the Cambridge University Press in 1897. A second edition with few changes appeared ten years later. Of this, an authorized German translation by R. Polster was published in 1920 (Berlin, Springer, 14 + 424 pages. Price 48 marks). For English measures German have been substituted, and an alphabetic subject index has been added.

Various reviewers of Sir Thomas Heath's *Euclid in Greek, Book I, with Introduction and Notes* (Cambridge, 1920) [see, for example, *1920*, 263–266] seem to have overlooked the fact that only four years previously G. C. Sansoni of Florence published the very neat little volume edited by Giovanni Vacca with the following title: *Euclide. Il primo Libro degli Elementi, testo Greco, versione Italiana, Introduzione e note* (1916. 12mo. 20 + 122 pp.). On the last four pages there is a glossary of Greek words with the Italian equivalents.

The concluding number of the *Bulletin of the American Mathematical Society*, volume 27, was for June–July, 1921. It has been decided that in the future the volumes shall begin in January. Hence the first number of volume 28 is to be that for January, 1922.

Between 1913 and 1919 the following volumes of the *Opera Omnia* of Tycho Brahe have been published at Copenhagen (*1920*, 421): I, II, III, IV part 1, and VI. The first part of volume V (*Astronomiae Instauratae Mechanica*, 1598; 213 pages) has recently been published.

In *Il Bollettino di Matematica*, volume 17, 1920, pages 61–77, A. Natucci reviews for the tercentenary of the invention of analytic geometry (*1921*, 179) the edition of the works of Descartes edited by C. Adam and P. Tannery.

The second heft of *Jahrbuch über die Fortschritte der Mathematik,* volume 45, for 1914–1915, was published in July, 1921, nearly two years after the first part was issued (*1920,* 268). The concluding part is promised by November, 1921. The present heft (pages 369–944) contains, apart from the conclusion of combinatory analysis and calculus of probability, series, differential and integral calculus, theory of functions, pure elementary and synthetic geometry, analytic geometry, and three pages on mechanics. There are in the heft about 116 pages more than in the corresponding section of the *Jahrbuch* for the year 1912.

In *Revista Matemática Hispano-Americana,* January–April, 1921, there are a portrait, biographical sketch, and bibliography of the published papers of T. Levi-Civita (pages 1–10, 46–49). The list of published papers contains over one hundred titles. Supplementary to the first five numbers of the *Revista* for 1921 have been published 80 pages of a Spanish translation of *Questioni riguardanti le Matematiche elementari,* edited by Federigo Enriques, volume 1, which is possibly more familiar to American readers in the German translation. The above mentioned 80 pages cover about the same ground as the corresponding number of pages in the German translation.

We are indeed happy to learn that a considerable increase in the number of subscribers to *Journal de Mathématiques Pures et Appliquées* has made the period of suspension of publication a very brief one (compare *1921,* 134). Not only has volume 85, 1920, been published, but also (in May and August, 1921) the the first and second numbers of volume 86. The latter include an address, by Camille Jordan, pages 1–2, at the funeral of Georges Humbert (*1921,* 237), and a paper "Transformations of surfaces applicable to a quadric" by L. P. Eisenhart, 37–66.—The first four numbers of *Annales Scientifiques de l'Ecole Normale Supérieure,* volume 56, and fascicules I–II of *Bulletin de la Société Mathématique de France,* volume 49, have also appeared in 1921.

Two new Italian mathematical periodicals have been started this year by the Circolo Matematico di Catania (each 25 lire a year and printed by V. Giannotta, Catania). Of the one, *Esercitazioni Matematiche,* for the use of university students, Professor Michele Cipolla is the editor. The first number (52 pages) was for January–February. Professor Gaetano Scorza is the editor of the other, *Note e Memorie di Matematica,* fasc. 1 (64 pages).

Revista Matematică din Timişoara is the title of a periodical published by the Scoala Politecnică, for mechanics and mining, recently established (1920) in Timişoara, lately of Hungary but now of Roumania. The first number appeared in March, and the second in April, 1921. These issues contain brief articles, problems proposed and solved, notes and news, examination questions. The periodical is designed to assist those preparing to enter the Scoala.—*Gazeta Matematică,* the other Roumanian mathematical periodical, has been published

for twenty-six years and is the organ of the Society "Gazeta Matematică" with about fifty members desiring to promote the interests of mathematics in the secondary and higher schools of the country.

Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1921, no. V, contains the Festvortrag, "Geometrie und Erfahrung" (pages 123–130), delivered by Einstein January 27, 1921, and an account (pages 116–123, 138–139) of the new edition of the complete works of Leibnitz. It was planned that this edition should occupy about 60 quarto volumes. By using smaller type and cutting down the editorial remarks, the commission of the Akademie now plans to reduce this estimate to 39 octavo volumes. The collected papers will be published in four series: philosophical (6 volumes), mathematical, in the field of natural science, and politico-historical; the correspondence in three series is to be classified: philosophical (6 volumes), in the fields of mathematics and natural science, and politico-historical and general (10 volumes). The 22 volumes of philosophical writings, and correspondence dealing with philosophy, politico-historical matters, and generalities are to be prepared first. One of these volumes is ready for publication.

The University of Illinois published, "November 22, 1920," an eight page illustrated pamphlet, whose contents are dated "January, 1921," entitled *Mathematical Models.* The author is Professor ARNOLD EMCH. He describes 18 models "designed and constructed in the mathematical department of the University of Illinois, . . . the first results of an effort to represent certain desirable features of mathematical instruction and research by adequate models, mechanisms, or graphs when they are not available otherwise. . . . For those interested . . . it will be possible to make arrangements with private firms for the reproduction and sale of the . . . models at prices which will be quoted upon application." The models include: (*a*) a variable string model of the hyperbolic paraboloid; (*b*) string models of certain cubic, quartic, quintic and sextic ruled surfaces; (*c*) tangent surfaces of a rectilinear (2, 2)-congruence; (*d*) models of projective generation of surfaces; (*e*) plane sections of a torus; (*f*) cellular division of space; and (*g*) three linkages—(α) Hebbert's cardioidograph, described in this MONTHLY, *1915*, 12–13; (β) a mechanism illustrating the description of certain ruled surfaces (Hebbert); and (γ) a cinematographic film of a Poncelet "polygon," *i.e.*, "a triangle remaining inscribed and circumscribed to two fixed circles respectively."

Since our last report (*1921*, 177), six more parts have been issued in *Publications of the Massachusetts Institute of Technology* "Contributions from the department of mathematics," series (since no. 7 changed from "serial") 2. These parts are number 17–22, February–July, 1921, and contain articles, by F. L. Hitchcock, Joseph Lipka, C. L. E. Moore, L. H. Rice, and Norbert Wiener (2), (see *1921*, 177, 178), reprinted from *American Journal of Mathematics, Bulletin of the American Mathematical Society, Comptes Rendus du Congrès International*

des Mathématiques, and *Proceedings of the American Academy of Arts and Sciences.* The first article in the *Proceedings*, volume 56, was "Motion on a surface for any positional field of force" by J. Lipka, in no. 4, March, 1921, pp. 157–182; the second article, "The axes of a quadratic vector" by F. L. Hitchcock, appeared in no. 9, June, 1921, pp. 331–351. The articles by N. Weiner were respectively three and four pages in length, being the papers he read at the so-called international congress of mathematicians (*1920*, 440).

Attention is directed to the interesting and admirably edited first three numbers (January–March, 1921, pp. 1–232) of *Periodico di Matematiche* (*Storia, Didattica, Filosofia*), organ of the Società Italiana di Matematiche "Mathesis," edited by F. Enriques and G. Lazzeri. Although these numbers[1] are marked series IV, vol. 1, nos. 1–3, the periodical is really an entirely new one having very little in common with its immediate predecessors *Periodico di Matematiche* and *Bollettino della Società Italiana di Matematiche "Mathesis."* The second series of the *Periodico*[2], 1899–1903, edited by A. Lugli, was formerly the official organ[3] of the Society of "Mathesis" which was founded[4] in 1895 by a group of teachers, in secondary schools, interested in the discussion of didactic questions.

For several years prior to 1908 interest in this society was decidedly on the wane, but in that year it was reorganized, in collaboration with university professors, under the name Società Italiana di Matematiche "Mathesis."

We are indebted to Professor Enriques, of the University of Bologna, for many of the facts contained in this note.

Of the ten mathematical periodicals started since January, 1919, none are of such notable importance for mathematical research as *Fundamenta Mathematicae* of which two volumes have been published; the first (224 pages) in 1920, the second (287 pages) in 1921, before May 1. The periodical is confined to the publication of memoirs, notes, and problems dealing with the theory of aggregates and related questions (immediate applications of the theory of aggregates,

[1] Among the articles in them are: "The teaching of dynamics" by F. Enriques; "The theory of irrational numbers in antiquity" by T. Bonnesen; "Paradoxes of infinity" by G. Vivanti; "Reform in the teaching of mathematics in the United States of America" by D. E. Smith; and "On the construction of a triangle given the lengths of its angular bisectors" by O. Chisne—a problem discussed by R. P. Baker in his doctor's dissertation at the University of Chicago, published in 1911 (100 pages, 4to).

[2] *Periodico di Matematica per l'insegnament secondario* was founded by David Besso in 1886. The first series contained 13 volumes, published 1886–1898. The third series, edited by G. Lazzeri, was published at Leghorn (Livorno), 1904–1920. The first volume of the fourth series is being published by Zanichelli at Bologna (20 francs).

[3] The first official organ of the society was *Bollettino della Mathesis* of which three volumes were published at Rome and Turin, 1896–1898. Under the title *Bollettino della Società Italiana di Matematiche "Mathesis"* it was published with Severi as editor 1909–1910; with Castelnuovo as editor 1911–1914; with Berzolari as editor 1915–1918; with Enriques as editor 1919–1920. It was only since about 1910 that the *Bollettino* commenced to contain scientific articles apart from reports of transactions of various sections of the society.

[4] Under the society's auspices national conferences were held at Turin (1898), Leghorn (1901), Naples (1903), Florence (1908), Padua (1909), Genoa (1912), Trieste (1919) and Naples (1921).

analysis situs, mathematical logic, research regarding axioms). Most of the articles are in French, and the use of languages other than English, French, Italian and German is disallowed.

The first volume opens with a portrait frontispiece, and a brief sketch, of ZYGMUNT JANISZEWSKI, the founder of the journal, and a member of the editorial board. He was born at Warsaw July 12, 1888, became doctor de l'Université de Paris in 1911, maître de conférence in mathematics at the University of Leopold in 1913, and professor at the University of Warsaw in 1919. He died January 3, 1920. A list of his publications is given. The volume contains also 24 articles or notes by 8 authors, and 10 proposed problems.

In the second volume there are 30 articles or notes by 13 authors, and 6 more problems. The last article, 256–285, "Sur les correspondances entre les points de deux espaces," is by H. Lebesgue.

The chief editors of the volumes are STEFAN MAZURKIEWICZ and WACLAW SIERPINSKI, professors of mathematics at the University of Warsaw. The periodical is excellently printed on good paper, royal octavo size. Subscriptions to the volumes (15 French francs) may be sent to *Fundamenta Mathematicae*, Mathematical Seminary, University of Warsaw, Warsaw, Poland.

In 1912 it was arranged that the collected papers of Sophus Lie should be assembled under the general direction of Friedrich Engel, and published by Teubner in about seven large volumes. On account of the great increase in cost of publication, the undertaking would have fallen through had not the Norwegian mathematical union (Norsk Matematisk Forening) laid the matter before the committee on the research fund of three million crowns voted by the Norwegian Storting in 1919. As Engel writes, in the last number of the *Jahresbericht der deutschen Mathematiker Vereinigung*, Bäcklund of Lund, Bianchi of Pisa, Hjelmslev of Copenhagen, Klein of Göttingen, Study of Bonn, Veblen of Princeton, and Vessiot of Paris, united in stating that the publication of the collected papers of Sophus Lie was not only highly desirable but also necessary. As a result, the committee of the Storting fund voted 5000 crowns a year for four years, beginning with 1921, to assist in carrying the undertaking to a successful conclusion, which is now assured. Professor Poul Heegaard, of the University of Christiania, is to be one of the editors. It is expected that the first published volume, the third of the set, will be issued during 1921. The tentative title page of the edition is: SOPHUS LIE, *Gesammelte Abhandlungen, im Auftrage des Norwegischen Mathematischen Vereins und mit Unterstützung der Akademien zu Kristiania und Leipzig herausgegeben von Friedrich Engel und Poul Heegaard.*

ARTICLES IN CURRENT PERIODICALS.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 5, February, 1921: "The thirteenth regular meeting of the southwestern section" by L. Ingold, 197–200; "A remark on skew parabolas" by G. Loria, 201; "The pseudo-derivative of a summable function" by W. L. Hart, 202–211; "Note on minimal varieties in hyperspace" by C. L. E. Moore, 211–216; "Notes on electrical theory" by H. Bateman, 217–225; Review by D. N. Lehmer of E.

Lebon's *Table de Charactéristiques de Base 30030, donnant en un seul coup d'œil les facteurs premiers des nombres premiers avec 30030 et inférieurs à 901,800,900.* Tome I, Premier Fascicule (Paris, 1920), 225–227; Reviews by D. E. Smith, of G. Loria's *Newton* (Rome, 1920), 228–230, and of T. L. Heath's *Archimedes* (London, 1920), 232–234; Reviews by R. D. Carmichael of W. Ahrens's *Mathematiker Anekdoten* (2d edition, Leipzig and Berlin, 1920), 230–231, and of G. Giraud's *Leçons sur les Fonctions automorphes* (Paris, 1920), 231–232; Review by V. Snyder of J. L. S. Hatton's *The Theory of the Imaginary in Geometry together with the Trigonometry of the Imaginary* (Cambridge, 1920), 234–236; "Comment on a previous review" [of Eddington's *Space, Time, and Gravitation,* by E. B. Wilson] by C. N. Moore, 236; Notes, 237–241; New Publications, 242–244—No. 6, March: "The twenty-seventh annual meeting of the American Mathematical Society" by R. G. D. Richardson, 245–265; "Reciprocal subgroups of an abelian group" by G. A. Miller, 266–272; "Proof of an arithmetic theorem due to Liouville" by E. T. Bell, 273–275; "A sequence of polynomials connected with the nth roots of unity" by T. H. Gronwall, 275–279; "The minimum area between a curve and its caustic" by P. R. Rider, 279–284; Review by L. W. Dowling of R. Marcolongo's *Il Problema dei Tre Corpi da Newton ai Nostri Giorni* (Milan, 1919), 284–285; Review by V. Snyder of T. Schmid's *Darstellende Geometrie,* volume 1, second edition (Berlin and Leipzig, 1919), 285; Notes, 286–289; New Publications, 289–292—No. 7, April: "The Chicago meeting of the American Mathematical Society" by A. Dresden, 293–308; "Pleasant questions and wonderful effects" [Presidential address delivered before the American Mathematical Society, December 28, 1920] by F. Morley, 309–312; "Fallacies and misconceptions in Diophantine analysis" by L. E. Dickson, 312–319; "On the Fourier coefficients of a continuous function" by T. H. Gronwall, 320–321; "Extension of an existence theorem for a non-self-adjoint linear system" by H. J. Ettlinger, 322–325; "On the Cauchy-Goursat theorem" by R. L. Borger, 325–329; "On a general arithmetic formula of Liouville" by E. T. Bell, 330–332; Review by L. W. Dowling of O. S. Adams's *General Theory of Polyconic Projections* (Washington, United States Coast and Geodetic Survey, 1919, special publication no. 57), 332–333; Review by G. A. Pfeiffer of A. Fraenkel's *Einleitung in die Mengenlehre* (Berlin, 1919), 333–334; Review by E. B. Cowley of E. Kenison and H. C. Bradley's *Descriptive Geometry* (New York, 1917), 334–335; Notes, 336–337, New Publications, 337–340.

L'Education Mathématique, volume 18, January, 1921: "Sur la relation d'Euler" by J. Coissard, 57–58—February: "Sur les cercles d'Apollonius d'un triangle" by A. Julson, 65–66, 73–74.

Financier, New York, volume 116, March 15, 1921: "Is banking a science?" by C. C. Grove, 10–11, 71–73.

Isis, volume 3, no. 1, January, 1920: "The purpose of Zeno's arguments on motion" by F. Cajori, 7–20—No. 2, September: "Did Fermat have a solution of the so-called Pellian equation?" by J. M. Child, 255–262 [Last sentence: "Hence I conclude that Fermat did have a general proof, and that it was substantially what I have given."].

Journal of the Indian Mathematical Society, volume 12, October, 1920: "An algebra of arithmetical functions" (continued) by F. Hallberg, 161–169; "Double points and lines" (concluded) by B. Rao, 170–174; "Invariants of a conic" by G. A. Srinivasan, 175–177; Problems and Solutions, 178–200—December: "An algebra of arithmetical functions" (continued) by F. Hallberg, 201–216; "Singularities of pedal curves" by A. Narasinga Rao, 217–222; Problems and Solutions, 223–240.

Nature, volume 106, February 3, 1921: "Mathematical text-books," 722–723 [review of H. T. H. Piaggio's *An Elementary Treatise on Differential Equations and their Applications* (London. 1920), of C. V. Durell and G. W. Palmer's *Elementary Algebra, Part i* (London, 1920), of R. E. Moritz's *A Short Course in College Mathematics* (New York and London, 1919) and of F. W. Dobbs and H. K. Marsden's *Arithmetic, Part ii* (London, 1920)]—Volume 107, March 3: "Mathematical papers of Huygens" by J. L. E. D[reyer], 4–5 [review of *Œuvres Complètes de Christian Huygens,* vol. 14]—March 10: "Relativity and the velocity of light" by C. O. Bartrum and J. H. Jeans, 42–43; "Relativity and the deviation of spectral lines" by H. J. Priestley, 43—March 24: Review of A. L. Bowley's *Elements of Statistics* (4th edition, London, 1920), 102–103.

Philosophical Magazine, sixth series, volume 41, February, 1921: "Notes on times of descent under gravity, suggested by a proposition of Galileo's" by W. B. Morton and T. C. Tobin, 225–239 [Proposition 36 of the *Dialogues:* A particle which slides to the lowest point O of a vertical circle starting from rest at any point B of the circumference below the level of the center, will make the journey in a shorter time if it moves along two successive chords BA, AO of the circle than if it goes directly along the one chord BO].

Printing Art, volume 37, no. 2, April, 1921: "Golden proportions in design" by F. T Singleton, 133–144.

Revue de l'Enseignement des Sciences, volume 14, November–December, 1920: "Formules relatives à l'ellipse et à l'hyperbole" by G. Fontené, 193–196; "Sur la composition des vitesses et la composition des accélérations" by P. Montel, 196–199; "Sur deux théorèmes relatifs aux courbes planes algébriques et aux surfaces algébriques" by R. Malloizel, 199–203; "Sur le calcul de π au moyen d'expressions analogues à $4 \text{ arctg } \frac{1}{5} - \text{arctg } \frac{1}{239}$" by A. Lévy, 203–210; "Sur les sommes des puissance semblables des n premiers nombres entiers" by H. Girard, 210–212; "Intersection d'une droite et d'une hyperbole" by J. Angelloz-Pessey, 213–214; "Sur un procédé de Fermat" by J. Angelloz-Pessey, 214–215; "Polaire d'un point par rapport à un cercle" by J. Lemaire, 215; "Note de géometrie descriptive" by J. Juhel-Renoy, 216; "Sur le trapèze harmonique" by C. Michel, 217–218; "Sur l'équation différentielle linéaire du second ordre à coefficients constants" by C. Michel, 218–221; "Examens et concours de 1920," 221–233; Index, 235–236.

Revue de Mathématiques Spéciales, volume 13, April, 1921: "Sur lés hélices cylindriques" by M. Anzemberger, 441–444.

Revue Générale des Sciences, volume 32, February 28, 1921: "Georges Humbert" by R. d'Adhémar, 97–98 [First paragraph: "Georges Humbert vient de mourir et tous ceux qui l'ont connu regrettent, en lui, le gallant homme, amiable, serviable, au regard vif et bienveillant, et le géomètre éminent, dont l'œuvre est d'une clarté éblouissante et bien française." (See also *1921*, 237)].

Revue Scientifique, volume 59, February 12, 1921: "Le mathématicien Georges Humbert" by G. Lemoine, 88–89.

School Science and Mathematics, volume 21, March, 1921: "Mathematical ability as related to general intelligence" by B. R. Buckingham, 205–215; Problems and solutions, 280–284.

Science, new series, volume 53, April 1, 1921: "National temperament in scientific investigations" by R. D. Carmichael, 298–301 [Fourth paragraph from the end: "Of the three countries which have led in scientific development it seems to be the impartial verdict of history that we owe to France the largest number of works perfect in form and substance and classical for all time; that the greatest bulk of scientific work, at least in more recent decades, has been produced in Germany; but that the new ideas which have fructified science, in earlier times and also in the nineteenth century, have arisen more frequently in Great Britain than in any other country."]; "The American Association for the Advancement of Science, Section L—History of Science sessions" by F. E. Brasch, 315–318—April 22: "Sherburne Wesley Burnham, 1838–1921" by E. B. Frost, 373–377 [see this Monthly, *1921*, 236–237].

Scientia, volume 29, March, 1921: "Les contributions des différents peuples au développement des mathématiques. Ière partie: Evènements mémorables et hommes représentatifs dans l'histoire des mathématiques" by G. Loria, 169–184; Review by G. Scorza of L. D. Weld's *Theory of Errors* (New York, 1916) and J. Hadamard's *Four Lectures on Mathematics delivered at Columbia University* (New York, 1915), 220–221.

Scientific American, volume 124, February 19, 1921: "The new concepts of time and space. A brief account of what Einstein has done to these fundamental notions" by M. Francis, 146–147, 155, 157—March 12, 1921: "The Einstein prize winner," 207 [Lyndon Bolton, autobiographical sketch and portrait; "I was born in Dublin in 1860, but I have lived in England since 1869. My family belonged to the landed gentry class, but I owe nothing to wealth or position. . . . After taking my degree (at Cambridge) in 1883 as a wrangler, I taught science and mathematics at Wellington College but I was attracted by what I had heard of the Patent Office and I entered it through open competition in 1885" (compare *1921*, 191)].

Scientific American Monthly, volume 3, March, 1921: "Leonardo da Vinci as an inventor. Remarkable achievements in sciences and invention of the great Italian artist" by A. A. Hopkins, 263–265.

Scientific Monthly, volume 12, no. 4, April, 1921: "The history of physics" by H. A. Bumstead, 289–309 [First paragraph: "The beginnings of anything like a connected history of the science which is now called physics may be placed with considerable definiteness about the beginning of the 17th century and associated with the great name of Galileo. It is of course

true that innumerable isolated facts had been known for many centuries which are now included among the data of this science; and many tools and simple machines which are now regarded as applications of physical principles had been devised and used. Even prehistoric man knew some of these—to his very great advantage. But, with one important exception which will be mentioned later, there was, in the ancient world, no connected body of knowledge in this field which can properly be called scientific. In this respect physics differs radically from mathematics, or astronomy, natural history, or medicine, each of which began its modern career with a store of scientific knowledge that had been obtained and put in order before the Renaissance."]; "Perfect and amicable numbers" by L. E. Dickson, 349–354 [First paragraph: "The two types of numbers in our title have had a continuous history extending from the early Greeks to date, and may be justly called the most human of all numbers. To them were early attributed certain social qualities, and later also ethical import, while mystics of the middle ages believed that they possessed special powers as talismans. Continuously for twenty centuries a wide-spread interest has been taken in the purely numerical questions and puzzle problems which arose in the study of these remarkable numbers. We shall present here the more essential facts and fancies in the quaint history of these most human of all numbers."].

Sphinx-Œdipe, volume 15, October, 1920: "Sur le folium double" by G. Loria, 1–4—November: "Sur les nombres qui sont, de n façons, la somme de trois bicarrés" by R. Goormaghtigh, 161–162; "Une illusion de Fermat" by L. Aubry, 162–163—December: "Résumé de la conférence générale de M. L. E. Dickson," 177–178; "Errata dans les tables de facteurs linéaires des formes $x^2 \pm Dy^2$" by P. Poulet, 180–181—Volume 16, February, 1921: "Sur les carrés d'Euler" by A. Margossian, 17–21; "Deux nombres curieux" by V. Thebault, 21 [$567^2 = 321489$ and $854^2 = 729316$]; "Au sujet d'une description du folium double due à J. B. Suardi et retrouvée par M. Gino Loria" by R. Goormaghtigh, 22–23—March: "Notice sur Charles Ange Laisant" (à suivre), 33–36; Review by H. Brocard of R. C. Archibald's "Notes on the logarithmic spiral, golden sections and the Fibonacci series" (New Haven, Conn., 1920).

The Times Literary Supplement, London, volume 19, December 30, 1920: "Correspondence—First use of the decimal point" by J. D. White, 891 [The letter: "Sir,—With reference to my letter on 'The First Use of the Decimal Point,' in *The Times Literary Supplement* of September 9, [cf. this Monthly, *1921*, 83], Professor Florian Cajori, of Colorado College, has kindly written to me, quoting from the Second Edition of his *History of Mathematics* (1919) this passage:

"'Historians of Mathematics do not yet agree to whom the first introduction of the decimla point or comma should be ascribed. Among the candidates for the honour are Pellos (1492), Bürgi (1592), Pitiscus (1608, 1612), Kepler (1616), Napier (1616, 1617). This divergence of opinion is due mainly to different standards of judgment. If the requirement made of candidates is not only that the decimal point or comma was actually used by them, but that they must give evidence that the numbers were actually decimal fractions, that the point or comma was with them not merely a general symbol to indicate a separation, that they must actually use the decimal point in operations including multiplication or division of decimal fractions, then it would seem that the honour falls to John Napier, who exhibits such use in his 'Rabdologiæ,' 1617. Perhaps Napier received the suggestion for this notation from Pitiscus, who, according to Eneström, uses the point in his 'Trigonometria' of 1608 and in 1612, not as a regular decimal point, but as a more general sign of separation.'

"Francis Pellos, the first of the writers thus mentioned, was the author of a work in the dialect of Nice, entitled 'Sen Segue de la Art de Arithmeticha,' which was published at Turin in 1492—the year that Columbus discovered America, and within fifty years of the first printing from movable types. An examination of the copy of this rare work in the British Museum Library shows that on pp. 10–12 Pellos gives the following, among other examples, of the method of division 'per una figura' and 'per desenals he centenals.' In the first case the divisor, dividend, and quotient are arranged as here shown, and I have set out the others in the same form, the figures, of course, being as in the original work:

$$5)\underline{5789657} \qquad 30)\underline{583604.3} \qquad 400)\underline{78965.73}$$
$$1157931\tfrac{2}{5} \qquad 194534\tfrac{23}{30} \qquad 19741\tfrac{173}{400}$$

"It will be seen that Pellos—as he explains in the text—uses the point for marking off from the dividend as many figures as there are 0's at the end of the divisor, so as to assimilate, for instance, division by 400 to division by 4; but he can hardly be said to use it as a true decimal point for separating the integers from the decimals in the expression of a quantity, and he states the remainder as a fraction of the original divisor.

"Stevinus introduced the operative use of decimals in 1585, and they were also used operatively by Bürgi in 1592, and Beyer in 1603. Both Stevinus and Beyer indicated the last integer and the first, second, &c., decimal figures by placing after or above them as exponents the numbers 0, 1, 2, &c. (a notation probably suggested by that which had long been used for sexagesimal fractions and survives in the marks for minutes and seconds), and even Bürgi, who thought it sufficient to place an 0 underneath the last integer, failed to attain 'the elegant simplicity' of the decimal point.

"Pitiscus, as shown in my previous letter, appears to have been the first to use the decimal point for separating the integers from the decimals in expressions of quantity. But he does not appear to have used it operatively; the evidence indeed is rather the other way. In the tables to both the second and the third editions of his 'Trigonometria' (1608, 1612) he gives, for instance, the sine of 30° 20′ to radius 100,000 as 50502.98; but in the example (at the beginning of book 3 in both these editions) of ascertaining the length of the sine of this angle for a radius of 24 ft., he takes the sine as 5050298 to radius 10,000,000, and states the result as $12\,\frac{1207152}{10000000}$.

"The decimal point was not used in the original Latin edition of Napier's 'Descriptio' of 1614, but it was used in the English edition of 1616, as described in my previous letter. In neither edition, however, is there an example of its operative use. But in Napier's 'Rabdologiæ,' Edinburgh, 1617, bk. 1. ch. 4, the description of the use of the numbered rods, commonly called Napier's Bones, in compound division is followed by an 'Admonitio pro Decimali Arithmetica,' in which Napier refers to Stevinus's Decimal Arithmetic, and says—to quote the translation in Mr. Macdonald's book mentioned below—

"'Since there is the same facility in working with these fractions as with the whole numbers, you will be able, after completing the ordinary division, and adding a period or comma, as in the margin, to add to the dividend or to the remainder one cypher to obtain tenths, two for hundredths, three for thousandths, or more afterwards as required; and with these you will be able to proceed with the working as above' . . .

"In the margin is the example of dividing 1180 by 432 (sic), with the result stated as 1993,273 which, as he explains, is equivalent to $1993\frac{273}{1000}$ or 1993, 2′ 7″ 3‴. This is probably the earliest printed instance of the operative use of decimals with the decimal point, or its equivalent, the decimal comma.

"In Napier's 'Constructio,' with Notes by Briggs, which was printed in 1619 after Napier's death, decimals are both described and used operatively with the decimal point, strictly so called. Mr. W. R. Macdonald, in his translation with notes (1889) of the 'Constructio,' makes special reference to this feature on pp. 88, 89, setting out the above quoted passage from 'Rabdologiæ' (1617), and observing that it is interesting as Napier's first published reference to decimal arithmetic, though sections 4, 5, and 47 of the 'Constructio' which deal with it 'must have been written long before that date.'"]

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. Dodd, 3012 West Ave., Austin, Texas

CLUB ACTIVITIES.

The Mathematics Club of Albion College, Albion, Michigan.
[*1918*, 354; *1919*, 409.]

Officers, elected each semester in 1919–20 and 1920–21, were as follows: Presidents, Esther Pearl '20, Joyce Hadaway '20, Elizabeth Gordon '21, Gertrude Pratt '21; vice-presidents, Joyce Hadaway '20, Harland Hatch '20, Gertrude Pratt '21, Mary Hutchins '21; secretary-treasurers, Almira Priest '20, Almira Priest '20, Mary Hutchins '21, Harold Black '23; third member of program committee, Elizabeth Gordon '21, Clark Dean '21, Christine Niemann '21, Marie de Vries '21.

The programs, which are listed below, usually included a general consideration of some topic at roll call, followed by one or more papers or talks.

October 28, 1919: "Our great inheritance—Mathematics" by Esther Pearl '20.

November 4: "Graphical algebra" by Carl Anderson '21; "Mathematical lists" by Elizabeth Gordon '21.

December 2: The work of great mathematicians; "The Flatlanders" by Gertrude Pratt '21.

December 9: Myths of constellations; talk by Mary Hutchins '21.

December 16: Applications of mathematics; "Geometrical representations of intermediate forms" by Donald Herrick '21.

January 14, 1920: Women in mathematics; "Hyperbolic functions" by Esther Pearl '20.

January 20: Letters from alumnæ read by Almira Priest '20.

January 27: Mathematical publications; "Mathematical fallacies" by Carl Anderson '21; "History of mathematics in the United States during the nineteenth century" by Christine Niemann '21.

March 2: Historical theorems; "The Leibnitz-Newton controversy" by Charlotte LaMastus '21 and Gertrude Pratt '21.

March 16: Present day mathematicians; "Graphs in ninth-grade algebra" by Elizabeth Gordon '21; "Eulerian integrals and Gamma functions" by Clark Dean '21.

April 13: Difficult solutions in algebra; "The junior high school" by Joyce Hadaway '20; "Historical development of the number system" by Clark Dean '21.

April 27: Trigonometric functions and geometrical constructions; "Elementary algebra and geometry" by Professor Head.

May 11: "Harmonics" by Joyce Hadaway '20; "Anharmonics" by Miss Hubert.

November 2: Facts about mathematicians; "A 'flu' dream in mathematics" by Gertrude Pratt '21; "The project method in teaching" by Mary Hutchins '21.

November 16: Mathematical jokes; "Applications of the forms of zero and unity" by Marie Plumb '22; "The circles of Appollonius" by Christine Niemann '21.

November 30: Constellations and myths about them; "Calculation by geometry of astronomical distances" by Elizabeth Gorden '21; "Mathematics, the farmer, and the weather" by Charlotte LaMastus '21.

December 21: Trigonometric functions and their development; "The math quest" by Harold Black '23; "Hyperbolic functions" by Gertrude Pratt '21.

January 18, 1921: Mathematical current events; "The Courtis intelligence test" by Rex Miller '21; "Problems in junior high school mathematics" by Lona Stockmeyer '21.

January 25: "Minimum requirements for high school mathematics" by Marie deVries '21; "The monkey and cocoanut problem" by Professor E. R. Sleight.

(Report by Mr. Black.)

MU THETA EPSILON, ALPHA CHAPTER, University of California, Berkeley, Cal.

A mathematics club was organized at the University of California under the name of Mu Theta Epsilon in April, 1920, with three aims: (*a*) The stimulation of interest in mathematics and the inducement of better work therein; (*b*) The discussion of mathematical topics; (*c*) The promotion of congeniality among students and closer coöperation between students and professors.

Membership is limited to women students, and includes juniors, seniors, and graduates. Twenty-two active members have participated during 1920–21, and regular monthly meetings have been held.

The officers for the current year are: President, Mamie Cohen '20; vice-president, Evelyn Aylesworth '20; secretary, Helene Clark, '21; treasurer, Nellie Bartlett '20; active faculty member, Dr. Pauline Sperry.

The following programs have been given:

September 1, 1920: "A special quartic curve" by Elsie McFarland '17; "Devices in mathematics used by peasants of Russia" by Professor B. A. Bernstein.

October 6: "Mathematics clubs in high schools" by Constance Kendall '20; "Geometry of the circle or compass" by Gladys-Mary Campbell '18.

November 3: "Mathematical theory of the satine arrangement" by Mildred Hurd '21; "The fourth dimension" by Ruth Brant '21.

December 1: "A discussion of hyperspace" by Nina Alderton '14; "Mathematical fallacies" by Nellie Bartlett '20.

February 1, 1921: "Concrete multipliers" by Lora Lind '21.

March 2: "Lambert's solution of the equation by infinite series" by Mamie Cohen '20; "D. M. D. Method of computing areas" by Thelma Hansen '21.

The Beta Chapter of Mu Theta Epsilon was established at the University of Southern California in Los Angeles in December, 1920. Plans for the organization of chapters at Leland Stanford University at Palo Alto, and Pomona College at Claremont, are under way. It is hoped to make Mu Theta Epsilon a national mathematics honor society in the near future.

(Report by Miss Clarke.)

THE JUNIOR MATHEMATICAL CLUB OF THE UNIVERSITY OF CHICAGO, Chicago, Ill. [*1918*, 34, 448.]

In the autumn of 1918, owing to the great confusion in connection with S. A. T. C., the club was not organized, but in the autumn of 1919 its work was again resumed.

The officers for 1919–20 were as follows: President, Ernest Zeisler Gr.; secretary and treasurer, Lila Nelson Gr.; program committee, Mrs. Mayme I. Logsdon Gr., Gladys Freeman Gr., and J. W. Lasley Gr. The number of members for the year was forty-two, and the average attendance, thirty-one.

The programs for 1919–20 were as follows:

October 22, 1919: "History and purposes of the Club" by Professor H. E. Slaught.

November 5: "Methods of counting" by Mrs. Mayme I. Logsdon Gr.

November 26: "Isosceles triangles" by Isaac Schour '21; "Magic squares" by Leah Libman '21; "Game of kim" by Lila Nelson Gr.

December 3: "The isosceles triangle" by C. C. MacDuffee Gr.; "Zeno's paradoxes of motion" by V. D. Gokhale Gr.

December 17: "Non-euclidean geometry" by Roy Haskell Gr.

January 14, 1920: "Arithmetical properties of numbers" by Gladys Freeman Gr.

February 11: "Some simple examples of covariants and invariants" by J. W. Lasley Gr.

February 25: "Concerning two particular matrices" by Professor E. H. Moore.

March 10: "Hyperbolic functions" by E. T. Browne Gr.

April 7: "Solution of quadratics" by J. C. Kamplain Gr.; "Interpolations" by S. J. Jacobson Gr.

April 21: "The theory of the complex number system" by C. C. MacDuffee Gr.

May 5: "The fundamental theorem of algebra" by Gladys Freeman Gr.

May 19: "The philosophical conception of the universe" by Dr. Edward Lasker, efficiency engineer, Chicago, Ill.

June 9: A beach party was given for the Graduate Mathematical Club and the Faculty of the department.

The officers for 1920–21 were as follows: President, E. T. Browne Gr.; vice-president, E. B. Miller Gr.; secretary and treasurer, Agnes Jones Gr.; program committee, Claribel Kendall Gr., E. T. Browne Gr., Myrtle Collier Gr. The number of members for the year was twenty-five, and the average attendance, twenty-one.

The programs for 1920–21 were as follows:

November 10, 1920: "History, purposes and aim of the Club" by Professor Slaught.

November 24: "Hyperbolic functions" by E. B. Miller Gr.

December 8: "The history of the parallel postulate" by Claribel Kendall Gr.

January 5, 1921: "Exterior ballistics" by Major H. E. Miner, U. S. Army.

January 19: "Linear algebras" by C. C. MacDuffee Gr.

February 2: "Zermelo's theorem: 'Every class can be well ordered'" by V. D. Gokhale Gr.

February 16: "Two famous geometry problems of antiquity" by Roy Haskell '21.

(Report by Miss Jones.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, or sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible

sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2908. Proposed by L. E. DICKSON, University of Chicago.

If f is a homogeneous polynomial in n variables and H is its Hessian determinant, prove that the Hessian of f^2 is cHf^n, where c is a constant.

2909. Proposed by J. S. STROMS, University of Maine.

A pinochle pack contains 48 cards, eight each of aces, kings, queens, jacks, tens, and nines. When three play, they are distributed by giving fifteen to each player and leaving three in the "kitty." When one holds two jacks of diamonds and two queens of spades in the same hand he has what is called double pinochle. When he holds eight aces he wins the game with 1000 points. What are the chances (*a*) of getting double pinochle; (*b*) of getting eight aces?

2910. Proposed by DANIEL KRETH, Wellman, Iowa.

The segments formed on the base of a triangle by the perpendicular from the opposite vertex are m and n. The product of the other two sides is p. Compute the two unknown sides and give a simple construction for the triangle.

2911. Proposed by D. A. ABRAMS, Chicago, Ill..

In the analysis of stresses in a flat slab of reinforced concrete, the following integral arises: $\int_0^x\int_0^y\int_0^z dxdydz$, where the upper limits for x, y, z are b, a, and $wb^3y^3(a-y)^3/[a^3x^3(b-x)^3+b^3y^3(a-y)^3]$. Evaluate this integral.

2912. Proposed by T. W. JACKSON, Jamestown College, N. D.

Given c, the chord of a circle, determine r, the radius, so that $3c$ is equal to the major arc of the circle.

2913. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given

$$S(z) \equiv nz\left[1+\sum_1^\infty(-1)^p\frac{(n^2-1)(n^2-3^2)\cdots\{n^2-(2p-1)^2\}}{(2p+1)!}z^{2p}\right]$$

$$C(z) \equiv 1+\sum_1^\infty(-1)^p\frac{n^2(n^2-2^2)(n^2-4^2)\cdots\{n^2-(2p-2)^2\}}{(2p)!}z^{2p}$$

show that, for any value of n, $|\cos x|<1$, $|\sin x|<1$:

$$\sin(nx) \equiv S(\sin x) \equiv \sin\left(\frac{n\pi}{2}\right)C(\cos x)-\cos\left(\frac{n\pi}{2}\right)S(\cos x);$$

$$\cos(nx) = C(\sin x) \equiv \cos\left(\frac{n\pi}{2}\right)C(\cos x)+\sin\left(\frac{n\pi}{2}\right)S(\cos x).$$

2914. Proposed by HARRIS HANCOCK, University of Cincinnati.

If $a_1, a_2, \cdots, a_n$ are n positive integers, a_{ij} the greatest common divisor of a_i and a_j, d_m the greatest common divisor of all products of every m of these numbers $(m=1, 2, \cdots, n-1)$, then is

$$\prod_{ij} a_{ij} = d_1d_2\cdots d_{n-1};\quad (j>i;\ i=1, 2, \cdots, n;\ j=2, 3, \cdots, n).$$

In general show that this theorem is true if $A_1, A_2, \cdots, A_n$ are any functions integral in any number of variables, with rational integral coefficients, or with algebraic integral coefficients. [Remark: This is a generalized statement in positive rational integers of the following theorem of much importance in the theory of algebraic numbers and due to Dedekind (Dirichlet, *Zahlentheorie*, supplement XI): Let **A**, **B**, **C** be three moduls (Dedekind). Denote the greatest common divisor of **A** and **B** by **A** + **B** and their product by **A**·**B**. Dedekind proves that

$$(\mathbf{A}+\mathbf{B})(\mathbf{B}+\mathbf{C})(\mathbf{C}+\mathbf{A}) = (\mathbf{A}+\mathbf{B}+\mathbf{C})(\mathbf{AB}+\mathbf{BC}+\mathbf{CA}).$$

Denote the greatest common divisor of two moduls $\mathbf{A}_i$ and $\mathbf{A}_j$, that is, $\mathbf{A}_i + \mathbf{A}_j$ by $\mathbf{A}_{ij}$; and write

$$\mathbf{D}_1 = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n, \qquad \mathbf{D}_2 = \mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_1\mathbf{A}_3 + \cdots + \mathbf{A}_{n-1}\mathbf{A}_n,$$
$$\mathbf{D}_3 = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_1\mathbf{A}_2\mathbf{A}_4 + \cdots + \mathbf{A}_{n-2}\mathbf{A}_{n-1}\mathbf{A}_n, \cdots$$
$$\mathbf{D}_{n-1} = \mathbf{A}_2\mathbf{A}_3 \cdots \mathbf{A}_n + \mathbf{A}_1\mathbf{A}_3 \cdots \mathbf{A}_n + \cdots + \mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_{n-1}.$$

Show that

$$\mathbf{A}_{12}\,\mathbf{A}_{13} \cdots \mathbf{A}_{n-1,\,n} = \mathbf{D}_1\mathbf{D}_2 \cdots \mathbf{D}_{n-1}.]$$

2915. Proposed by HARRIS HANCOCK, University of Cincinnati.

Determine x so as to satisfy the two congruences $3x^2 \equiv 0\ (3N)$, $x^3 + a \equiv 0\ (9N^2)$, where $a = N^2 \cdot n$, and the two integers N, n have no common factor, and neither contains a squared factor.

2916. Proposed by HARRIS HANCOCK, University of Cincinnati.

If p is any rational prime integer, and if $\alpha(\neq 1)$ is any root of $x^p - 1 = 0$, show that $p = \mathbf{P}_1 \cdot \mathbf{P}_2 \cdots \mathbf{P}_{p-1}$, where $\mathbf{P}_i$ $(i = 1, 2, \cdots, p - 1)$ are the ideals $(p, 1 - \alpha^i)$, which in turn may be reduced to principal ideals. [Remark: This is rather a good elementary example to show that an integer prime in one realm is factorable in a more extended realm.]

2917.

A parabola is rolled upon a fixed right line. Find the locus of (a) its vertex; and (b) its focus.

2918. Proposed by NATHAN ALTSHILLER-COURT, University of Okla.

Find planes which cut four given lines in four concyclic points.

2919. Proposed by V. M. SPUNAR, Chicago, Ill.

An equilateral hyperbola which touches a conic and is concentric with it is called a hyperbolic tangent to the conic. Being given two hyperbolic tangents to a conic, the arc of any third hyperbolic tangent which is intercepted by the first two subtends a constant angle at either focus of the given conic.

NOTES.

19. A Curve of Pursuit. Apropos of Note **10** (*1921*, 184; cf. *1921*, 278, 281), referring to a problem proposed by Lucas in 1877, a very similar problem is given as a worked out exercise in Bateman, *Differential Equations*, 1918, pp. 8–10. It is as follows: "Three boys running at the same speed u chase one another. A pursues B, B pursues C and C pursues A. Find a differential equation which will indicate the way in which the ratios of the sides of the triangle ABC vary." A footnote states that this problem is due to Professor Frank Morley. His differential equation is discussed in a paper by F. E. Hackett, "A numerical solution of the triangular problem of pursuit," in *The Johns Hopkins University Circular*, July, 1908, pp. 135–137.

A. H. WILSON.

20. A Problem in Investment. There has been inquiry concerning the following problem given in a less general form, with a reference to *Engineering News*, volume 48, 1902, pp. 362–363, in E. B. Skinner, *The Mathematical Theory of Investment*, Boston, 1913, p. 140.

Bonds are issued, N in number, of a face value of A each, bearing interest at rate r. At the end of a years and at the end of each year thereafter a certain number of these bonds is to be redeemed at a price which bears the ratio R to the face value A. How many bonds must be redeemed each year in order that the whole issue shall be paid for at the end of n years, and that the sum of the interest

on the unpaid bonds and the amount paid to redeem the bonds shall be the same for each of the $n - a + 1$ years?

This problem is completely solved in *Engineering News* (l.c.).

21. Involutes of a circle and a pasturage problem. On page 128 of *Problems and Solutions. Associateship Examinations, Parts I and II, 1915–1919* (New York, Actuarial Society of America, 1921) is a brief solution of the following problem in the examinations set for 1918: "A circular wall of radius a stands in the middle of a large field. A horse is tethered to the outside of this wall by a rope the length of which is equal to half the circumference of the wall. Show that the area over which the horse can graze is $(5/6)\pi^3a^2$." The area is evidently composed of a semicircle of radius πa and two areas with arcs of involutes of the given circle as outer boundaries.

A somewhat more complicated problem was proposed over one hundred and seventy years ago, in *The Ladies Diary or the Woman's Almanack*, 1748. The problem was as follows (page 41): "Observing a *Horse* tied to feed in a Gentleman's *Park*, with one End of a Rope to his *Fore-foot*, and the other end to one of the Circular *Iron-Rails*, inclosing a Pond, the Circumference of which Rails being 160 yards, equal to the Length of the Rope, what Quantity of Ground, at most could the Horse feed?" A solution was given on pages 25–26 of the *Diary* for 1749, and the answer found was[1] "76257.86 sq. yards = $15A.2R.12P$."

Involutes of a circle may be connected with many other curves. For example: If an involute of a circle, of radius a, rolls on a straight line, the locus of the center of the circle is a parabola whose parameter is $2a$ (J. Clerk Maxwell,[2] 1849)—The locus of the center of the circle (radius a) of an involute, rolling on an orthogonal tractory of the catenary, whose equation is

$$y = \frac{x}{2a}\sqrt{x^2 - a^2} + \frac{a}{2}\log\left(\sqrt{\frac{x^2}{a^2} - 1} + \frac{x}{a}\right),$$

is the axis of y (Maxwell, 1849)—If an involute of a circle rolls on an equal involute with corresponding points in contact, the center of the circle traces a spiral of Archimedes (Maxwell, 1849)—The involute of a circle is the locus of the pole of a logarithmic spiral rolling on a concentric circle (Maxwell, 1849; often attributed to Haton de la Goupillière)—The pedal of an involute with respect to the center of its circle is a spiral of Archimedes (Practically the same as the third of the results by Maxwell; see also Mannheim,[3] 1858)—The caustic by reflection of an involute for rays emanating from the center of its circle is an evolute of a spiral of Archimedes (Haton de la Goupillière,[4] 1863)—The inverse of

[1] See also *The Mathematical Questions proposed in the Ladies' Diary*, edited by T. Leybourn, volume 2, London, 1817, pp. 6–7; *The Diarian Miscellany* by C. Hutton, volume 2, London, 1775, p. 269; *The Diarian Repository*, London, 1774, pp. 507–508.

[2] This and the next three results are taken from the remarkable memoir on "The theory of rolling curves" presented to the Royal Society of Edinburgh when Maxwell was 18 years of age. See the *Scientific Papers of James Clerk Maxwell*, volume 1, 1890, pp. xi, 22, 26, 28.

[3] *Nouvelles Annales de Mathématiques*, 1858, pp. 186–187 and 1860, 186–187.

[4] *Nouvelles Annales de Mathématiques*, 1863, pp. 336, 494–500, 548–550.

an involute with respect to the center of its circle is a spiral tractrix, that is, a curve which, in polar coördinates, has a tangent of constant length (Haton de la Goupillière,[4] 1863)—The centers of curvature for the points of contact of an involute rolling on a straight line is a parabola (Cesàro,[1] 1884)—The points of contact of tangents drawn to an involute from any point of its plane lie on a limaçon of Pascal (Fouret,[2] 1888).

The involute of a circle seems to have been first conceived in 1693 when Huygens was considering clocks without pendulums which might be of service on sea going vessels.[3] In this connection he originated an apparatus in which the involute of a circle plays an essential rôle.

In 1891 it became desirable to install in the Royal Observatory at Greenwich a larger telescope. This necessitated that a larger dome, 36 feet in diameter, be built upon a circular wall 31 feet 4 inches in diameter. The form adopted was that of a "surface generated by the revolution of an involute of a circle, 7 feet in diameter, with its center in the plane of the rail, and 5 feet from the axis, the curve being completed near the apex by an arc of a circle (of 13 feet 3 inches radius) so that it cuts the axis at right angles. The diameter of the dome is 36 feet at a height of 7 feet above the rail."[4]

ARC.

SOLUTIONS.

2809 [1920, 80]. Proposed by the late L. G. WELD.

Find the nth term of the series defined by the relation, $u_{i+2} = u_i + u_{i+1}$, in which $u_1 = u_2 = 1$.

SOLUTION BY HENRI SEBBAN, Boufarik, Algeria.

Consider the sequence

$$1, 1, 2, 3, 5, 8, \cdots$$

in which each term beginning with the third is the sum of the two terms immediately preceding. In order to express a term as a function of u_1, u_2 and i, let us set $u_i = Ax_1^i + Bx_2^i$ and endeavor to determine A, B, x_1 and x_2 so as to satisfy the law of the sequence. Then

$$\begin{aligned} Ax_1^i + Bx_2^i &= Ax_1^{i-1} + Bx_2^{i-1} + Ax_1^{i-2} + Bx_2^{i-2} \\ &= Ax_1^{i-2}(x_1 + 1) + Bx_2^{i-2}(x_2 + 1), \end{aligned}$$

and from this it follows that $x_1 + 1 = x_1^2$, $x_2 + 1 = x_2^2$. Hence x_1 and x_2 are the two roots of the equation $x^2 - x - 1 = 0$, for we cannot have $x_1 = x_2$ unless the sequence reduces to a geometric progression, $u_i = Kx^i$ of ratio x which is not the case when $u_1 = u_2 = 1$. It will be seen that in the general case A and B are determined by the initial conditions and are different from zero. Hence we may set

$$x_1 = \frac{1 + \sqrt{5}}{2}, \qquad x_2 = \frac{1 - \sqrt{5}}{2}, \qquad \text{and} \qquad u_i = A\left(\frac{1 + \sqrt{5}}{2}\right)^i + B\left(\frac{1 - \sqrt{5}}{2}\right)^i,$$

where A and B are to be determined from the conditions $Ax_1 + Bx_2 = 1$, $Ax_1^2 + Bx_2^2 = 1$. Hence, we have after certain reductions

$$A = -B = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad u_i = \frac{(1 + \sqrt{5})^i - (1 - \sqrt{5})^i}{2^i\sqrt{5}}.$$

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[1] *Mathesis*, 1884, pp. 233–235.

[2] *Journal de Mathématiques Spéciales*, 1888, p. 261.

[3] *Œuvres Complètes de Christian Huygens*, volume 10, 1905, pp. 514–515.

[4] *Monthly Notices of the Royal Astronomical Society*, volume 51, May 8, 1891, p. 436.

It will be found that for this series the following relations are true:

$$u_{i+2}u_{i-1} - u_iu_{i+1} = \pm 1, \qquad u_i^4 - u_{i-2}u_{i-1}u_{i+1}u_{i+2} = 1, \qquad \text{and} \qquad S_i = u_{i+2} - u_2,$$

where S_i is the sum of the first i terms of the series. This latter series has received the names Fibonacci's and Lamé's, and is the subject of several interesting studies. In the *Journal de Mathématiques Elémentaires*(Vuibert), 44e année, p. 11, it is shown that the sum of the squares of two consecutive terms of Fibonacci's series is a term of the series. In the *Intermédiaire des Mathématiciens*, 1916, pages 64 and 599, it is proved that if p is a prime of the form $10n \pm 3$ it is a divisor of u_{p+1}; if of the form $10n \pm 1$ it is a divisor of u_{p-1}. In *Nouvelles Annales de Mathématiques*, 1884, p. 533, is a proof of the theorem that $(2u + 1)^i - u^{3i} = 1$ provided that the exponents in the expansion are replaced by indices and the resulting terms u_n are terms of Fibonacci's series.[1]

Also solved by NORMAN ANNING, J. P. BALLANTINE, E. B. ESCOTT, H. L. OLSON, E. J. OGLESBY, and LOUIS WEISNER.

2811 [1920, 81]. Proposed by J. L. RILEY, Stephenville, Texas.

Given cube roots of 60, 61, 63 and 64, to find the cube root of 62 by the method of differences.

SOLUTION AND REMARKS BY E. B. ESCOTT, Oak Park, Illinois.

Let u_1, u_2, u_3, u_4, u_5 be a series of functions with equidistant arguments. By the method of differences, the first term of the fourth order of difference of u_1 is

$$u_5 - 4u_4 + 6u_3 - 4u_2 + u_1.$$

Let this equal zero and solve for u_3, whence

$$u_3 = \tfrac{1}{6}[4(u_2 + u_4) - (u_1 + u_5)].$$

If

$$u_1 = \sqrt[3]{60} = 3.91486\ 76412 -, \qquad u_2 = \sqrt[3]{61} = 3.93649\ 71831, \qquad u_3 = \sqrt[3]{62},$$

$$u_4 = \sqrt[3]{63} = 3.97905\ 72079 - \qquad \text{and} \qquad u_5 = \sqrt[3]{64} = 4,$$

we have by above formula, $\sqrt[3]{62} = 3.95789\ 16538$. This is true to 7 decimals, the value to 10 decimals being $3.95789\ 16097 -$.

A more general method is to use Newton's Interpolation Formula, where the intervals are not necessarily equal, *i.e.*,

$$u_x = u_a + (x - a)\cdot\delta'(a, b) + (x - a)(x - b)\cdot\delta''(a, b, c) + \cdots, \tag{1}$$

where

$$\delta'(a, b) = \frac{u_a - u_b}{a - b}, \qquad \delta''(a, b, c) = \frac{\delta'(a, b) - \delta'(b, c)}{a - c}, \qquad \cdots.$$

See Thiele, *Interpolationsrechnung*, page 5, or Boole, *Finite Differences*, Chapter 3, Exercise 19. In this problem, $u_{62} = \sqrt[3]{62}$, $u_a = u_{60} = \sqrt[3]{60}$, $u_b = u_{61} = \sqrt[3]{61}$, $u_c = u_{63} = \sqrt[3]{63}$, $u_d = u_{64} = \sqrt[3]{64}$.

Then

$$\delta'(61, 60) = \frac{\sqrt[3]{61} - \sqrt[3]{60}}{61 - 60} = .02162\ 95419,$$

$$\delta'(63, 61) = \frac{\sqrt[3]{63} - \sqrt[3]{61}}{63 - 61} = .02128\ 00124,$$

$$\delta'(64, 63) = \frac{\sqrt[3]{64} - \sqrt[3]{63}}{64 - 63} = .02094\ 27921,$$

$$\delta''(63, 61, 60) = \frac{\delta'(63, 61) - \delta'(61, 60)}{63 - 60} = -.00011\ 65098,$$

[1] The literature of this series has been discussed already in this MONTHLY, *1918*, 234–238, 462, by R. C. Archibald; reference to a recent extended elaboration of these notes was made *1920*, 314.—EDITORS.

$$\delta''(64, 63, 61) = \frac{\delta'(64, 63) - \delta'(63, 61)}{64 - 61} = -.00011\ 24068,$$

and

$$\delta'''(64, 63, 61, 60) = \frac{\delta''(64, 63, 61) - \delta''(63, 61, 60)}{64 - 60} = .00000\ 10258.$$

Substituting in Newton's Formula, we have

$$\sqrt[3]{62} = \sqrt[3]{60} + 2 \times .02162\ 95419 + 2 \times 1 \times (-.00011\ 65098) + 2 \times 1 \times (-1) \times .00000\ 10258 \quad (2)$$

$= 3.95789\ 16538$ true to 7 decimals, as in the other solution.

In the above formula, the arguments may be any number, and, in particular, we may have any number of them equal. In this case the "divided differences" become

$$\delta'(a, a) = \lim_{b \doteq a} \frac{u_a - u_b}{a - b} = \frac{du_b}{db}, \qquad \delta''(a, a, a) = \lim_{b \doteq a} \frac{\delta''(a, a) - \delta''(b, b)}{a - b} = \frac{1}{2}\frac{d^2u_b}{db^2},$$

$$\delta'''(a, a, a, a) = \frac{1}{2 \cdot 3}\frac{d^3u_b}{db^3} \cdots.$$

Newton's Interpolation formula for the given functions $u_a, u_b, u_c, u_d, u_d, u_d, \cdots$, where the last function may be used an indefinite number of times, becomes

$$u_x = u_a + (x - a)\delta'(a, b) + (x - a)(x - b)\delta''(a, b, c) + (x - a)(x - b)(x - c)\cdot\delta'''(a, b, c, d) + (x - a)(x - b)(x - c)(x - d)\cdot\delta^{\text{iv}}(a, b, c, d, d) + \cdots.$$

In this form, it is a combination of Newton's Interpolation Formula and Taylor's Formula. In the foregoing problem, using $u_{64} = \sqrt[3]{64}$ three times, we have

$$\delta'(64, 64) = \left(\frac{d\sqrt[3]{x}}{dx}\right)_{x=64}\left(\frac{1}{3}x^{-2/3}\right)_{x=64} = .02083\ 33333,$$

$$\delta''(64, 64, 63) = \frac{\delta'(64, 64) - \delta'(64, 63)}{64 - 63} = -.00010\ 94588,$$

$$\delta'''(64, 64, 63, 61) = \frac{\delta''(64, 64, 63) - \delta''(64, 63, 61)}{64 - 61} = .00000\ 09827,$$

$$\delta^{\text{iv}}(64, 64, 63, 61, 60) = \frac{\delta'''(64, 64, 63, 61) - \delta'''(64, 63, 61, 60)}{64 - 60} = -.00000\ 00108,$$

and

$\delta^{\text{v}}(64, 64, 64, 63, 61, 60) = .00000\ 00001.$

This gives us two more terms in the series (2), viz.,

$$+ 2 \times 1 \times (-1)(-2)(-.00000\ 00108) + 2(1)(-1)(-2)(-2)(.00000\ 00001) = -.00000\ 00440.$$

Then $\sqrt[3]{62} = 3.95789\ 16538 - .00000\ 00440$
$= 3.95789\ 16098$, true to 9 decimals.[1]

[1] The same result may be obtained as follows: In the first part of the discussion above,

$$\Delta_4 = u_5 - 4u_4 + 6u_3 - 4u_2 + u_1$$

was assumed zero, but a correction may be obtained by applying to the terms of Δ_4 the development of u_x in powers of $x - 64$, x being 64 for u_5, 63 for u_4, etc. Then

$$\Delta_4 = u_5^{\text{iv}} - 2u_5^{\text{v}} + \frac{13}{6}u_5^{\text{vi}} - \cdots,$$

where

$$u_5^{(n)} = \frac{d^n\sqrt[3]{x}}{dx^n}\bigg]_{x=64}.$$

Hence

$$\sqrt{62} = u_3 = \frac{1}{6}[4(u_2 + u_4) - (u_1 + u_5)] + \frac{\Delta_4}{6}.$$

Note. Newton's Interpolation Formula is equivalent to the well-known Euler-Lagrange Interpolation Formula, but is frequently more convenient to use. As shown in the above example, if the interpolation is not sufficiently accurate, we need only add more terms to Newton's Formula, while the Euler-Lagrange Formula will need to be entirely recomputed.

Also solved by NORMAN ANNING, E. J. OGLESBY, ARTHUR PELLETIER, J. B. REYNOLDS, C. C. WYLIE, and the Proposer.

NOTES AND NEWS.

It is to be hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

Associate Professor E. J. MOULTON, of Northwestern University, has been promoted to a full professorship of mathematics.

Professor E. G. BILL, of Dartmouth College, has been appointed to the newly created office, dean of freshmen. He will give up all teaching.

At their Alma Mater, Wellesley College, Miss RACHEL BLODGETT, recently of Radcliffe College, and Miss RUBY WILLIS, have been appointed instructors of mathematics.

At the University of Pennsylvania, Assistant Professors G. G. CHAMBERS and H. H. MITCHELL have been promoted to full professorships.

Mrs. THERON CLARK, of Brown University, has been appointed instructor of mathematics at Bucknell University, Lewisburg, Pa.

Dr. F. D. MURNAGHAN, of The Johns Hopkins University, has been promoted to be an associate professor of applied mathematics.

Associate Professor W. A. WILSON, of Yale University, has been promoted to a full professorship.

Instructors C. S. DOAN and F. H. HODGE, of Purdue University, Lafayette, Ind., have been promoted to assistant professorships.

Mr. L. E. WARD, of Harvard University, has been appointed instructor of mathematics at the Rice Institute, Texas.

Miss MAY B. CARTER, of Brown University, has been appointed instructor of mathematics in The Western College for Women, Oxford, O.

Professor E. R. SMITH, of Pennsylvania State College, has been appointed head of the department of mathematics at Iowa State College, Ames, Ia.

Dr. J. R. RITT, of Columbia University, has been promoted to an assistant professorship of mathematics.

At Princeton University, Mr. PHILIP FRANKLIN, who received his doctorate there in June, has been appointed instructor of mathematics.

Dr. J. E. ROWE, recently of the Aberdeen Proving Ground, has been appointed head of the department of mathematics at College of William and Mary, Williamsburg, Va. Compare *1920*, 280.

In this case the correction $\Delta_4/6 = -.00000\ 00441 +$ and the value of $\sqrt[3]{62}$ is given to 9 places of decimal as above.—EDITOR.

At Lehigh University, Assistant Professors J. E. STOCKER and J. B. REYNOLDS have been promoted to associate professorships of mathematics and astronomy.

At Hobart College, Geneva, N. Y., where W. P. DURFEE, professor of mathematics, is also dean of the college, his son W. H. DURFEE has been appointed professor of freshman mathematics.

Dr. E. L. POST, of Columbia University, for the past year a fellow at Princeton University, has been appointed instructor of mathematics at Columbia University for the current academic year.

Mr. E. T. BROWNE, instructor of mathematics at the University of Virginia, but more recently a graduate student at the University of Chicago, has been appointed assistant professor of mathematics at Trinity College, Hartford, Conn.

DR. C. N. REYNOLDS, of Dartmouth College, (*1920*, 238), has been appointed assistant professor of mathematics at the University of West Virginia.

Assistant Professor E. S. ALLEN, of the University of West Virginia (*1919*, 419) has been appointed associate professor of mathematics in Iowa State College.

Associate Professor M. O. TRIPP, of the University of Maine, has been appointed head of the department of mathematics at Wittenberg College, Springfield, Ohio.

MR. E. L. MACKIE, instructor of mathematics at Harvard University during 1920–1921 (*1920*, 382), has been appointed assistant professor of mathematics at his Alma Mater, the University of North Carolina.

At Clark University, Dr. W. E. STORY has been made professor emeritus since the age limit for teachers there is seventy years. Dr. HENRY TABER has had to retire on account of ill health, and has also been made professor of mathematics, emeritus.

At Colgate University, Assistant Professor E. P. SISSON has retired from active teaching, Mr. TORWALD FREDERICKSEN (see *1921*, 235) has been given permanent appointment as instructor of mathematics, and Mr. H. A. DOBELL has also been made an instructor.

At Oberlin College, Mr. F. E. CARR has been promoted to an assistant professorship of mathematics, and Dr. C. H. YEATON, of Milwaukee College of Engineering, has been appointed assistant professor of mathematics.

At Knox College, Galesburg, Ill., Miss HELEN CALKINS, of Columbia University, has been appointed instructor of mathematics. Professor G. T. SELLEW, for twenty-two years head of the department of mathematics, has been granted a year's leave of absence for study and travel abroad. For a time he expects to study statistics with Karl Pearson in London.

Professor KATHARINE S. ARNOLD, who has been in the department of mathematics at Milwaukee-Downer College, Milwaukee, Wis., since 1912, has been appointed head of the department of mathematics at the Constantinople Woman's College, formerly known as the American College for Girls at Constantinople. She sailed in August to take up her new work.

Assistant Professor W. C. GRAUSTEIN, of Harvard University, and Assistant Professor MARY F. CURTIS, of Wellesley College, were married on June 10, 1921, and sailed for England on the following day. Professor Curtis had received leave of absence from Wellesley for 1921–1922.

One of the three exchange professors whom Harvard sends to France this year is one of our charter members, Professor A. E. KENNELLY, who goes under the auspices of a committee of American Universities interested in exchanging with France professors of engineering and applied science.

Dean T. F. HOLGATE, of Northwestern University, has been invited by the University of Nanking, China, to spend his sabbatic year at that institution, lecturing on mathematical subjects and assisting in the general organization of the university. He sailed for China on August 18.

At Harvard University, Dr. J. L. WALSH, Sheldon Fellow at Paris during the past year, has been appointed permanent instructor of mathematics (*1920*, 109, 440). The appointment as instructors of mathematics for 1921–1922 are as follows: Dr. C. E. HILLE as Benjamin Peirce instructor (see *1920*, 438); C. A. GARABEDIAN, R. E. LANGER and HARRY LEVY, who have been instructors during the past year (*1920*, 382); LINCOLN LA PAZ, L. L. SMITH (*1920*, 322) and R. M. FOSTER (*1921*, 106), of Harvard University; and H. W. BRINKMAN, of Stanford University (*1920*, 241). On August 9, 1921, LARNED LINN SMITH was drowned. He was a young mathematician of exceptional promise, aged twenty-two years. Dr. H. T. STETSON has been promoted to be an assistant professor of astronomy (*1921*, 285).

C. H. PEABODY, professor of naval architecture and marine engineering at the Massachusetts Institute of Technology since 1893, was made professor emeritus in June, 1920, after thirty-seven years of service in connection with the Institute. He was professor of mathematics and engineering at the Imperial Agricultural College in Sapporo, Japan, 1878–1880. His first work was *Thermodynamics of the Steam Engine*, 1889, which reappeared in several revised and enlarged editions, finally including the theory of turbines and internal combustion engines. His well known book on *Naval Architecture* was first published in 1904, and has gone through several editions.

Professor HORACE LAMB has been made professor of mathematics emeritus, at the University of Manchester, which, on May 7, conferred on him the degree D. Sc. He has been appointed to the honorary (Rayleigh) University lectureship at Cambridge.

WILLIAM ROBERT BROOKS, director of the Smith Observatory since 1888, and professor of astronomy at Hobart College since 1900, died at his home, Geneva, N. Y., on May 3, 1921. He was born in Kent, England, June 11, 1844, and came to this country in 1857. He founded the Red House Observatory, Phelps, N. Y., in 1874, and since that time discovered twenty-seven comets, more than any living astronomer has to his credit.

NATHANIEL FRENCH DAVIS, professor of pure mathematics, emeritus, in Brown University since 1915, died May 17, 1921. He was born at Lake Village (now Laconia), N. H., June 11, 1847, and graduated B. A. from Brown University in 1870. Returning to his Alma Mater in 1874 as instructor of mathematics, he was made assistant professor in 1879, associate professor in 1889, and professor of pure mathematics in 1890. We have recently referred (*1921*, 97) to the Fund of ten thousand dollars founded in honor of Professor Davis. His son, H. N. DAVIS, is professor of mechanical engineering at Harvard University (*1919*, 275).

Professor PAUL ARNOLD, head of the department of mathematics in University of Southern California since 1901, died February 24, 1921. He was born in Columbia, Missouri, December 16, 1870, but went to California with his father at an early age. Graduating from the University of Southern California, Ph.B., 1890, Ph.M., 1893, he taught in the department of mathematics 1890–1893. During 1894–1896 he was a graduate student in mathematics at Cornell University and held a fellowship from this University while studying at Berlin and Leipzig, 1896–1897. He remained abroad till 1899 studying with Lie at the University of Christiania.

Professor ALFRED MONROE KENYON, head of the department of mathematics at Purdue University, Indiana, since 1908, died while traveling on an interurban train after attending his mother's funeral, July 27, 1921. Born at Medina, O., December 10, 1869, he graduated from Hiram College and took graduate work at Western Reserve University, at the Case School of Applied Science, and at Harvard University (A.M., 1898). Appointed instructor of mathematics at Purdue University in 1898, he became an associate professor in 1901, and a professor in 1903. He was joint-author, with W. V. Lovitt, of *Mathematics for Collegiate Students of Agriculture and General Science*, New York, 1917; and with Louis Ingold, of (*a*) [*Plane and Spherical*] *Trigonometry*, New York, 1913; (*b*) *Elements of Plane Trigonometry*, New York, 1919.

The death is announced of M. FRANCOIS-JOSEPH PICAVET, secrêtaire du Collêge de France, directeur â l'Ecole Pratique des Hautes Etudes, chargê de cours â la Facultê des Lettres de l'Universitê de Paris, and chevalier de la lêgion d'honneur. M. Picavet died on May 19, 1921, at his residence in the Collêge de France, at the age of seventy years, since he was born May 17, 1851. He is known in the history of mathematics for his great work on Gerbert,—*Gerbert, un pape philosophe, d'apres l'histoire et d'aprês la lêgende*, Paris, 1897. Few men of his time have done more to maintain high standards in the field of scholarly research.

ALBRECHT WILHELM ADOLF THAER, director of the Oberrealschule at Hamburg, died March 1, 1921. He was born January 13, 1855, and was granted his doctorate at Giessen in 1878. Apart from his dissertation, published in *Mathematische Annalen*, volume 14, his publications related to matters connected with secondary schools. He collaborated with others in the preparation of a series of elementary mathematical texts, 1911–1915, and of a report to the International Commission on Mathematical Instruction, 1911.

JOHANNES KARL THOMAE, ordinary professor of mathematics in Jena from 1879 to his retirement in 1914, died April 1, 1921. Born December 11, 1840, he received his doctorate at Göttingen in 1864. During the next fifty years he was a prolific writer of articles (especially in the *Berichte* of the Academy of Sciences at Leipzig), pamphlets, and books. Among his books were the following: (*a*) *Abriss einer Theorie der complexen Functionen und der Thetafunctionen einer Veränderlichen,* Halle, 1870; third edition, 1890; (*b*) *Elementare Theorie der analytischen Functionen einer complexen Veränderlichen,* Halle, 1880; second edition, 1898; (*c*) *Die Kegelschnitte in rein projectivischer Behandlung,* Halle, 1894; (*d*) *Grundriss einer analytischen Geometrie der Ebene,* Leipzig, 1906.

Mr. ALFRED DAVIS, of the Soldan High School, St. Louis, is president of the Mathematics Club of St. Louis (see *Mathematics Teacher,* 1921, page 156).

Professor W. C. EELLS, of Whitman College, Wash., is chairman of the mathematics section of the Inland Empire Teachers Association.

On June 10, Professor A. G. WEBSTER, of Clark University, delivered before the Royal Institution of Great Britain a lecture on "Researches on sound."

The inauguration of Dr. E. F. NICHOLS as president of Massachusetts Institute of Technology occurred June 8; his address was published in *Science,* June 10.

Dr. J. S. TAYLOR, of Massachusetts Institute of Technology, has been granted a year's leave of absence which he is spending at the University of Louvain as a Belgian-American fellow.

Professor R. C. ARCHIBALD, of Brown University, was on June 20, 1921, elected an honorary member of the Harvard Chapter of Phi Beta Kappa. Brown University has granted him leave of absence for the second half of the academic year, 1921–1922. He expects to spend it in visiting mathematicians at universities of Italy, France, Belgium, Holland, Scandinavia, and Great Britain.

Miss ANNIE J. CANNON, who has been connected with the Harvard Observatory since 1897, and curator of the astronomical photographs there since 1911, has received from Groningen University, Holland, an honorary doctor's degree in mathematics and astronomy, in acknowledgment of her work in the study of stellar spectra which will fill nine quarto volumes of the *Annals of the Harvard Observatory;* three of these are now published.

The Committee on Algebraic Numbers, formed in 1920 by the Division of Physical Sciences of the National Research Council, is composed of Professors H. H. MITCHELL, H. S. VANDIVER, G. E. WAHLIN, and L. E. DICKSON, chairman. At its meeting in Philadelphia, April 20–22, 1921, final plans were formulated for the preparation of a report on the literature of algebraic numbers subsequent to Hilbert's report, "Die Theorie der algebraischen Zahlkörper," in *Jahresbericht der deutschen Mathematiker-Vereinigung,* 1897.

At the inauguration of L. T. COFFMAN (joint author, with J. C. Brown, of *How to Teach Arithmetic,* Chicago, 1914; and, with W. A. Jessup, of *The Supervision of Arithmetic,* New York, 1916) as president of the University of Minnesota,

Professor W. H. BUSSEY, of the University of Minnesota, was the delegate from the American Mathematical Society, and Professor H. E. SLAUGHT from the Mathematical Association of America.

Among the delegates at the inauguration on June 22, 1921, of Dr. J. R. Angell, president of Yale, were the following: Dr. E. G. BILL, Dartmouth College; Acting Dean JAMES HARKNESS, McGill University; President W. A. GRANVILLE, Pennsylvania College; Professor A. S. GALE, University of Rochester; Professor JOSEPH BOWDEN, Adelphi College; President E. O. LOVETT, Rice Institute; Professor R. G. D. RICHARDSON, American Mathematical Society; Professor E. W. BROWN, American Philosophical Society.

At the meeting of the American Academy of Arts and Sciences in May, 1921, the following fellows were elected in Section I—Mathematics and Astronomy: C. G. ABBOT, astrophysicist, Smithsonian Institution; FLORIAN CAJORI, historian of mathematics, University of California; O. D. KELLOGG, mathematican, Harvard University; H. N. RUSSELL, astronomer, Princeton University; FRANK SCHLESINGER, astronomer, Yale University; JOEL STEBBINS, astronomer, University of Illinois. G. H. HARDY, mathematician, of Oxford University, was elected a foreign honorary fellow. Among officers of the Academy elected for the ensuing year were H. W. TYLER, corresponding secretary, A. G. WEBSTER, librarian, and E. B. WILSON, a member of the "committee on meetings."

At the meeting on April 11 of the Academy of Sciences of the Institute of France (see *1920*, 384), EMILE BOREL, professor of the calculus of probabilities and of mathematical physics at the University of Paris, was elected to succeed HUMBERT (see *1921*, 237) as one of the six members in the section of geometry. There are in the Academy, apart from its two permanent secretaries one in each of the departments, physics and mathematics, 66 regular members, six in each of the eleven sections: geometry, mechanics, astronomy, geography and navigation, general physics, chemistry, mineralogy, botany, agriculture, anatomy and zoology, and medicine and surgery.

At the meeting of the Southwestern Michigan Association of Teachers of Mathematics and Science held in Kalamazoo on April 22, 1921, H. E. SLAUGHT spoke on behalf of the National Committee on Mathematical Requirements concerning the history and work of this committee.

At a meeting of the Twin Cities Mathematical Club held at the University of Minnesota on May 14, 1921, H. E. SLAUGHT addressed the high school teachers of St. Paul and Minneapolis on "Certain marvelous magnifications of mathematics." The club of some sixty members was organized two years ago by W. D. REEVE, head of the department of mathematics in the University of Minnesota High School.

At the fourteenth regular meeting of the Association of Mathematics Teachers of New Jersey, at Princeton, N. J., May 21, 1921, the following papers were read: "A high school test in arithmetic" by A. W. BELCHER; "Some geometry propositions proven by elementary mechanics" (presidential address) by C. R. MAC-

INNES; "Recent developments in connection with the work of the National Committee" by H. E. WEBB; "A geometric introduction to the theory of relativity" by L. P. EISENHART.

The twenty-fourth regular meeting of the Rochester section of the Association of Teachers of Mathematics in the Middle States and Maryland was held at the University of Rochester on May 7, and the following papers were read: "The place of the formula in elementary algebra" by ELIZABETH M. LOETZER; "Experiences based on the report of the National Committee on Mathematical Requirements" by WILLIAM BETZ; "Discussion of Einstein's theory of relativity" by A. S. GALE.

In our report (*1921*, 239) of the meeting of the Mathematics Section of the New York State Teachers Association at Rochester, November 23, 1920, record of G. R. MIRICK's paper, "The proper correlation of intermediate algebra, trigonometry, and solid geometry, in the senior high school," was unintentionally omitted. A similar omission was made (*1921*, 240) in the case of M. M. S. MORIARTY's paper, "Some geometric notions," read at the meeting of the Connecticut Valley Section of the Association of Teachers of Mathematics in New England, November 6, 1920.

At the meetings of the American Mathematical Society at Wellesley College, September 8–9, 36 papers were presented by the following mathematicians: J. W. ALEXANDER (2), G. D. BIRKHOFF (2), B. H. CAMP, W. L. CRUM (2), E. L. DODD, JOHN EIESLAND, L. P. EISENHART, C. A. FISCHER, R. E. GILMAN, O. E. GLENN, T. H. GRONWALL, C. F. GUMMER, EUGENIE C. HAUSLE, E. V. HUNTINGTON, EDWARD KASNER, J. R. KLINE (2), G. A. MILLER, C. N. MOORE, R. L. MOORE, H. M. MORSE, J. R. MUSSELMAN, M. B. PORTER, J. E. ROWE, I. J. SCHWATT (3), J. L. SYNGE, J. L. WALSH, A. G. WEBSTER (2), N. WIENER, S. D. ZELDIN. Details concerning the papers may be found in *Bulletin of the American Mathematical Society*, January, 1922.

At Göttingen on January 6, 1921, was organized the Reichsverband deutscher mathematischer Gesellschaften und Vereine (National Association of German Mathematical Societies and Clubs). The members of the Association are: the Deutsche Mathematiker-Vereinigung, the Deutscher Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterricht, the Mathematisch-Naturwissenschaftlicher Verein in Württemberg and the mathematical societies in Berlin, Vienna, Hamburg, Göttingen and Aix-la-Chapelle. It is desired to unite in this Association all mathematical societies and clubs of German speaking countries, whether their aim is primarily educational or scientific. In this way it is expected that the ideals of all will be more effectively promoted, that the position of mathematics in practical life will in every direction be represented, and that especially, the interests of mathematical instruction of every grade will be guarded.

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RELIGIO MATHEMATICI.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE MATHEMATICAL ASSOCIATION OF AMERICA, SEPTEMBER 7, 1921.

By DAVID EUGENE SMITH, Columbia University.

Apologia. Well aware that I, today, break all precedents in the nature of a presidential address, it is proper that I offer due apology and confess my unconventionality. It is proper, too, that I should frankly say that I am aware that my message is wanting in the rigor of demonstration to which our science accustoms us; that it involves no mathematics beyond the merest elements; that it will be listened to with regret by some and with disapproval by others; and that it is likely to carry conviction to only a limited number.

Why, then, should I choose such a topic? Why should I force my friends to apologize for my thoughts, and why should I place others in a favorable position to condemn my action? The answer is a simple one:—I believe it to be a duty laid upon us, that one of our number should, on some such occasion as this, say what I shall say. I regret that another,—one with more faith, one better endowed with power of expression, one whose words would carry greater weight than any I can utter,—might not have had the opportunity that is mine to deliver the message. And finally, by way of apology, I say with perfect candor that I speak not at all to the audience here present,—for what I shall say will convey nothing that is new to any one of you; I speak rather to those who are not present, those who are not of our special guild, and those to whom the teaching of mathematics may possibly, by this means, appear in a somewhat different, and I hope in a somewhat more favorable light.

The General Message. It is nearly three centuries ago that Sir Thomas Browne wrote, for private circulation among a few of his intimate friends, his *Religio Medici.* He did not call it *Religio Medicorum,* for he could not assume to speak for others. Indeed, as he confessed, the men who practiced the healing art in his time were generally, like loud-mouthed boys in their teens, boastful of their atheism; but for himself, a humble practitioner in an ancient town of England, he could testify; and he, the giver of balm for the body, could unobtrusively pass the message to others, as a balm for the spirit.

And so, today, I speak not of the *Religio Mathematicorum,* for I have not the authority; I feel, indeed, that *Religio Mathematici* is a misnomer; but I also feel that it is proper to speak of the relation of mathematics to a religious attitude of mind, and I know of no better title for my purpose than the one I have chosen.

Do not, however, feel that the message seeks to change the faith or lack of faith of any man; do not feel that it contains a plea for any creed or for any sect; do not feel that it sets forth anything that is new; but rather feel that it represents the mere commonplace knowledge that most of us have, and the mere basis of belief that mathematics may possibly foster. It goes no farther, it seeks only a

single corner of the foundation walls upon which we may build, it lacks warmth, it lacks beauty, it lacks the fervor of religious faith, and it is hard of texture; but the foundation stones of palaces, and of temples, and of homes, all lie below the surface; they, too, are cold, but they sustain structures which give to the world pleasure and protection and repose. The message is transmitted by one who has joined in serious conference with those who hold the Brahmin faith; by one who has learned much from the high priests in the temples of Buddha, who has sat at the feet of the followers of Zoroaster, and who has communed with those who wear the green turban that marks the seed of the Prophet,—one who believes that

> "Creed and rite perchance may differ,
> Yet our faith and hope be one,"

and who, with the author of the *Religio Medici* itself, is able to say:

"For my Religion, though there be several Circumstances that might perswade the World I have none at all, . . . yet, in despight hereof, I dare without usurpation assume the honourable Stile of a Christian."

Specifically, I wish to consider, even though totally unable to answer satisfactorily, these questions: What bond of concord, if any, is there between mathematical knowledge and religious faith? What influence can an exact, abstract, reputedly frigid science like ours have upon the religious nature of man? What, in fact, is the soul of mathematics, and to what wave lengths must our own souls be tuned to catch its message?

Such, you will say, are the imaginations of a dreamer; not the serious thoughts of a mathematician. So be it. Were it necessary to make the choice, I would rather be a dreamer without mathematics, than a mathematician without dreams or a teacher without imagination. What I wish to show to those who are not of our calling, however, is that there is no other science that leads so directly to a recognition of the reasonableness of a broad religious faith, and none that parallels so completely the broader tenets of the Fathers.

What is Religion? In the domain of mathematics we find it necessary at certain times and convenient at certain other times to define our terms; but unless we need a particular term in a proof we do not feel bound to define it with precision. For example, in elementary mathematics we do not deem it necessary to define space, number, or straight line, the terms being basic and, for the beginner, indefinable.

So if you ask me to define religion before we consider its relation to mathematics, I must simply refer you to the theologians, wishing you good fortune in your quest. To many people the Buddhist is not religious, but I have often found him intensely and beautifully so. To many religiously minded persons the Parsee is a heathen, but I have seen as great faith and as pronounced goodness of work among the Parsees as I have found among many Christians with whom I worship. For our purposes, therefore, I prefer to think of religion with respect to certain general characteristics, one of which is faith; faith, as a very intellectual and catholic-minded writer has remarked, in "The Eternal, not ourselves,

that makes for righteousness"; faith that this life does not end all; faith that our lives today prepare for our lives somewhere beyond.

Tangible Immortality. Out of the manifold characteristics of the mind of youth, one of the most interesting is best described by the word "opinionatedness"—a word that I grant should be in the *Index expurgatorius*. All normal individuals reach this milestone, most of them pass it, but some find their development arrested at this point, and here they bind themselves for life. In particular, they absorb (I will not say develop) the idea that immortality is an idle word of an idle faith, a part of the "opiate of the proletariat," as our most modern autocracy has phrased it.

One thing that mathematics early imparts, unless hindered from so doing, is the idea that here, at last, is an immortality that is seemingly tangible,—the immortality of a mathematical law. The student of algebra, for example, may well question the use of the traditional curriculum, but when he finds the value of $(a + b)^2$ he has come in contact with an eternal law. The laws of the Medes and Persians, unchangeable though they were thought to be, have all perished; the canons that bound Egyptian activities for thousands of years exist only in the ancient records, preserved in our museums of antiquity; the laws of Rome, which at one time dominated the legal world, have given place to modern codes; and the laws that we make today are certain to be changed tomorrow. But in the midst of all these changes it has ever been true, it is true today, it shall be true in all the future of this earth, and it is equally true throughout the universe whether in the algebra of Flatland or in that of the space in which we live, that $(a + b)^2 = a^2 + 2ab + b^2$.

We may change the symbols,—they are temporary expedients to convey the idea; we may speak in different tongues,—they are local expedients to convey thought; but it is inconceivable to us that the relation which the formula expresses should not be true always and everywhere,—a tangible symbol of the immortality of law.

What I learned in chemistry, as a boy, seemed true at the time, but much of it today is known to be false. What I learned of molecular physics seems at the present time like children's stories, interesting but puerile. What we learn in history may be true in some degree, but is certain to be false in many particulars. So we may run the gamut of learning, and nowhere, save in mathematics alone, do we find that which stands as a tangible symbol of the immortality of law, true "yesterday, today, and forever."

But does the teacher make this known to the student? Does the student come to feel the significance of this fact,—this fact so full of awe to the normal mind, this evidence of immortality that never comes to his consciousness until he meets it in mathematics? I do not know, nor do I know how much else that is great, that has tremendous significance, is taught or is not taught in science, in letters, in history, or in art. I only know that mathematics can do this thing; that it can (and it should) give, to the degree that the pupil is able to receive it, the idea that before the world was created, before our solar system was formed,

and after our system shall cease to be, the every-day laws of mathematics stood and shall stand for immortal truth,—for laws that are divine in their infinite endurance. It is not necessary, it is not desirable, that we should preach these things in the halls of learning; but it is essential that we should feel their significance. This done and all the rest "shall be added unto you." Stated in another way, the immortality of law means that we come in touch with the invariant. The tyro in mathematics comes early upon the invariant properties of a figure as seen in the theory of elementary projection. In a wider sense, however, all geometry is a science of invariance. We prove a law for a general plane triangle and it never varies, whatever we do to the figure. If we prove that $a^2 = b^2 + c^2 - 2bc \cos A$, then, however A may change, the law itself will never vary. In it the pupil comes into touch with the unchangeable, with the absolute.

It is the same with all other laws of geometry. In any convex polyhedron, whatever its shape, the law remains that the number of faces plus the number of vertices is equal to the number of edges increased by two.

"Change and decay in all around I see," but the established properties of a general geometric figure, in our space, are as unchangeable in that space as divinity itself. Stated in still another way, the immortality of law and the invariability of mathematical principles mean the eternity of mathematics. To come into relation with a science which was illustrated by the spiral nebulae before our solar system was formed, which only now reveals to us those laws of crystals which were in operation long before life appeared upon the earth, and which is also entirely independent of matter, so that if we could imagine the universe destroyed absolutely, the laws would still be true,—to come into relation with such a science makes real to us, as no other discipline in our curriculum can possibly do, the ideal of truth eternal.

Our Infinitesimal Nature. I know of nothing which acts as such a powerful antidote to that which I ventured to call "opinionatedness," as a study of mathematics. To know that the light from solar systems far larger than our own has been thousands of years in reaching us, gives us an idea of our infinitesimal nature, in comparison with space about us, that can come only with a study of the science that it is ours to teach. A bacillus in our veins, so small as to be invisible through a powerful microscope, is a giant compared with ourselves in our relation to this space in which we live. Our doubts, our beliefs, our hopes, our fears are all so trivial, so infinitesimal, so like a lost electron in our solar system, as compared with our relative importance in the universe as revealed to us by the calculations which mathematics brings to bear upon the great problem! Cowper wrote well when he put in verse the words,

> "God never meant that man should scale the heavens
> By strides of human wisdom."

and even the mathematics of youth confirms the thought.

With our feeble voices we cry out that we are certain that this life ends our existence, but mathematics shows us the truth of Voltaire's words, in speaking of human opinion of the significance of life,—

"Doubt is not a pleasant condition, but certainty is an absurd one."

Our feeble voices join in a protest that we cannot understand God. Again the "old man of Ferney," condemned by so many as an atheist, speaks out and says,

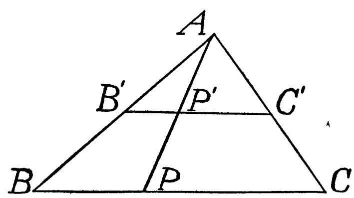
"My reason tells me that God exists; but it also tells me that I cannot know what He is."

The infinitesimal still cries out that it cannot understand what our soul is, and again the sage speaks:

"Is it indeed likely that we should know what our soul is, when we could form no idea of light if we had the misfortune to be born blind?"

We agree that spaces of higher dimensions than the one in which we think we live can easily be conceived by analogy, and we agree without question to the paradoxes which we meet in the study of infinity, and yet we feel that it shows our great wisdom, or perhaps our boldness, if we deny the soul an existence. Strange, that in algebra we accept without the slightest question the idea of the permanence of law, but that our little natures should so often boast that we deny the permanence of the soul!

Contact with the Infinite. One of the impressive experiences which comes to the devotees of our science is the continual contact with the infinite,—an experience which is inspiring beyond words to express,—and sometimes as discouraging. To take an illustration that is a mere commonplace to those who have even visited the borderlands of our science, suppose that a line segment $B'C'$ joins the mid-points of the sides AB and AC of the triangle ABC, and that the straight line $AP'P$ connects the vertex A with points P' and P on $B'C'$ and BC respectively.



It is then evident that, in this construction, to any point P on BC there is one point and only one point P' on $B'C'$ that corresponds to it, and conversely. Like it or not as we please, there is only one possible conclusion, namely, that there is a one-to-one correspondence between the points on $B'C'$ and the points on BC; or, taking the usual definition of equality, that there are precisely as many points on $B'C'$ as there are on BC, which is twice as long. But similarly, the number of points on $B'C'$ can be shown to be the same as the number on any part of BC, and so a part (not of a line, but of a group of points) is equal to the whole. To adopt a paradox of old Tertullian, "Certum est quia impossibile est."

Against such a conclusion our little minds revolt; we have been taught, or we have empirically learned, that the part is less than the whole,—as it really is in dealing with the finite. But when we come to deal with the infinite, our little, narrow, finite laws break down; they are even more feeble than the beliefs and experiences of the child, when viewed by the eyes of a mature man. We must face the inevitable, that in the domain of the infinite, the part may indeed be equal to the whole, in spite of the childish beliefs of our finite minds. The mathe-

matician continually meets this necessary conclusion; it is a commonplace; to him the laws of the finite give way without question when he enters the domain of the infinite. In all the finite, indeed, he sees the infinite. He knows that there is a one-to-one correspondence between the points in a sphere and those in the entire universe about him, and between the number of vibrating points in his brain and the number in all space. In other words, within the brain of each of us, there is a point that corresponds to any given point in the universe, say one on the surface of Neptune. If we take another point, say the center of a fixed star, there is one special point, and only one, within each brain that also corresponds to that. If we move the point in the fixed star ever so slightly, we move the point in the brain so that it shall continue to correspond to it. Does all this signify anything to us? I certainly do not know. Does it mean that the planets have their influence upon us? I do not know, but with my finite mind I am led to say I do not think so. Does this mathematico-physical relation of our brains to all the universe about us have any deeper significance? I do not know. But one thing I do know, that thoughts like these give new meaning to the words, "For, behold, the Kingdom of God is within you."

Do such ideas signify that it is ours to preach? Shall mathematics become a medium for religious instruction? Do we not lower religious belief when we link it to the certainties of cold mathematics? With respect to each of these questions I would reply, "I do not think so." With respect to the last one, I see no difference in the sanctity of truth, whether the truth be taught in the books of Euclid, in the holy books of the East, or in the Christian Church;—in the *significance* of truth,—yes; in the *sanctity* of truth,—no.

Our Impotence in Relation to the Eternal. I know of no other branch of learning that makes so clear to us our impotence in relation to the Eternal. We fail today in a problem in chemistry, but we feel that we may succeed tomorrow; we are at a loss to know how to overcome a certain difficulty in physics, but we have confidence that someone, sometime, somehow, will overcome it; we do not attain to the success we had hoped for in the painting of a certain picture, but we "carry on" in the hope of bettering our work from day to day; and so in mathematics, we fail but we persevere. But there come times in mathematics when we fail and know that we must fail, because we come in conflict with the Eternal. At first sight we say that we can construct a seven-edged polyhedron. We fail, we seek out the reason, and we find that we are combatting the everlasting truth of which I have spoken, the truth that $f + v = e + 2$. Protest as we will, we are powerless when we combat the Eternal.

Physical Permanence. Our mathematics also comes to the aid of science and assists in the proof that force is not lost, and shows us that even a thought generates a wave which has its eternal influence. I like the ancient theory, and I know of no reason why it should not be a fact, that the very walls and arches of the venerable cathedrals of the old world, which have heard the daily chants of the priests for many centuries, have come to vibrate in unison therewith, so

that the same singers, if transported to another spot, fail to produce the same sonorous, musical effect. Why, then, should not the thoughts of others influence us, as the telepathists affirm, and why should not our body of thought be permanent in space, even from a materialistic point of view?

Mathematics is such a science of harmony! The universe is such a science of harmony! The "music of the spheres" meant precisely this in the ancient philosophy;—and how out of all harmony with what mathematics reveals to us is the theory of annihilation, even, as I have said, from the materialistic standpoint!

To quote again from Voltaire, whose name is anathema to so many ignorant religionists:

"All nature cries aloud that He *does* exist; that there *is* a supreme intelligence, an immense power, an admirable order, and that everything teaches us our own dependence upon it."

Spoken like a philosopher, but also like the mathematician that he was; for Voltaire was a great student of Newtonian philosophy and did more than any man in his century to make the teachings of Newton known to the general intellectual element of France. That Voltaire, the bitter antagonist of fraud and sham, in the Church as well as in the State, should honestly confess his faith in this manner need not surprise us. It shows his greatness. Cicero told the world two thousand years ago that the greatest thinkers always have such faith:—

"There is, I know not how, in the minds of men a certain presage, as it were, of a future existence; and this takes the deepest root and is most discoverable in the greatest geniuses and the most exalted souls."

The Drama of Space. What a science is ours that raises the curtain on the drama of space! That shows us a finite space in which all bodies are at rest until acted upon by some external force; and then the space of the infinitesimal, in which a radically different code of laws obtains,—where everything (molecules, atoms, electrons, and very likely sub-electrons) is an automaton, self moved and never resting. John Burroughs has expressed this idea, adding:

"When we reach the astronomic world, or the sidereal universe, we find the same condition that prevails in the world of the infinitely little: perpetual motion goes on, friction is abolished, and nothing is at rest. . . . Height and depth, upper and under, east and west, north and south, weight and inertia, as we experience them, have vanished. There are no boundaries, no ending and no beginning, no center and no circumference; the infinite cannot have any of these."

What a science is ours, moreover, that reveals to the youth the secret of indirect measurement! That shows him how the distances to the stars are found! That opens to him the moving picture of the universe! That reveals the world of the electronic bodies as the reciprocal of the world of the sidereal bodies! And who can measure the influence of this revelation upon the soul of one who is standing upon the threshold of young manhood or young womanhood?

Scientific Religion. Religion is generally felt to be unscientific. One of the world's great authorities, Professor Harald Höffding of Copenhagen, in his *Philosophy of Religion,* speaks of it as that state of mind

"in which feeling and need, fear and hope, enthusiasm and surrender play a greater part than do meditation and inquiry, and in which intuition and imagination have the mastery over investigation and reflection."

I cannot believe that this should be the case with respect to the great basal facts of religion, although it is so traditionally, at least in our sectarian doctrines. We lay down certain postulates in geometry; they may, as in the case of parallels, be true or false; mathematics simply says, "If A, then B,"—if these postulates are sound, then these conclusions are true. I have often wondered mildly why religion did not do the same, postulating certain statements and then proceeding precisely as in mathematics,—"If A, then B." The postulates, like those of Euclid, might be true or false, but the deductions would be absolute. I have no idea as to what postulates would be assumed; I simply know that it seems entirely scientific to assume at least a few that are reasonable. We do not, in elementary mathematics, feel that our postulates must be independent or that they must cover all possible needs; we simply assume what seems to be necessary for the mind of youth, and on that we build.

When I think, in a kind of indefinite fashion, of what my mind postulates with respect to some of the larger features of mathematics, my thought runs to something like this:

1. The Infinite exists.
2. Immortal laws exist.
3. The laws relating to finite magnitudes do not hold respecting the infinitely large or the infinitely small.
4. The existence of hyperspace is entirely reasonable.
5. No factor is ever lost.
6. Time may be a closed curve.

Such a list of postulates might easily be put into theological language as well, and might be extended when necessary. For example, the theologian might phrase these same postulates like this:

1. God exists.
2. God's laws exist.
3. God's laws are entirely different from ours.
4. There are spaces beyond ours.
5. The soul exists and is eternal.
6. God looks at time as a whole.

After making out his list of postulates, the theologian might formulate his definitions. It would be as easy, I should think, to make an attempt to define God, heaven, angel, and miracle as it is to make an attempt to define infinity, hyperspace, straight line, solid angle, and dozens of other terms that we use in mathematics; or, if precise definition were found unnecessary, as is the case with these mathematical terms, at least some reasonable limitations would be in order.

Given the postulates and the definitions, I see no reason why a perfectly rigorous set of propositions should not be erected and religion put on a cold, scientific foundation. I should not wish to see this done; I think it would be

about as sensible as to build up a scientific, deductive system of love or beauty; but what I mean to say is that if religion is unscientific, it is partly because the world wishes it to be so. Love would not be any more potent, or more real, or more beautiful if we formulated a set of postulates and deduced a series of propositions relating to it; and the same may be said of religion.

Duality of Mathematics and Religion. Schopenhauer's duality between Time and Space is well known. Time is homogeneous, for example; it is a continuum, and no part is separated from any other part by something which is not time. The dual proposition for Euclidean space is simply formed by a substitution of "space" for "time."

We need not be surprised at this dualism. Time has often been called a fourth dimension, and it harmonizes with our three dimensional, Euclidean space to think of time alone as a space of one dimension. The two united gave to Sir William Rowan Hamilton the science of quaternions, and now they unite again to give birth to one feature of the Minkowski-Einstein hypothesis.

How much of Schopenhauer's duality is real we cannot say, because we cannot say how much of time and space are real. If we leave the domain of Euclidean space and think of ourselves as living in a space that curves through a fourth dimension, as a spherical two-dimensional space curves through a third dimension, then this space becomes limited, as Einstein suggests, and we are led to consider other three-dimensional spaces like ours,—all quasi-finite, and the Schopenhauer dualism automatically changes.

Similarly, when we consider the duality between mathematics and religion, we have nothing positive. Change the space in which we live, make time a fourth dimension, let our new universe curve through a fifth dimension, and the details of the parallelism would necessarily change. Grant, however, certain postulates, derived empirically like the postulates of mathematics, or, as we say, derived from common sense,—grant these, and some such parallelism as the following is suggested to the mathematical mind:

Mathematics.	Religion.
1. The infinite exists.	1. God exists.
2. Eternal laws exist.	2. Eternal laws exist.
3. The laws relating to finite magnitudes do not hold respecting the infinitely large or the infinitely small.	3. God's laws are so different from ours as to be absolutely non-understandable by us.
4. The existence of hyperspace is entirely reasonable.	4. The existence of a heaven, with gradations, is entirely reasonable.
5. No factor is ever lost.	5. The soul is eternal.
6. Time may be a closed curve.	6. God looks at time as a whole.
7. Time may be a fourth dimension.	7. In the next world, the direction of time may actually be seen.

8. Positive infinity may physically coincide with negative infinity, if lines curve through four-space.
9. A Flatlander has enough of the third dimension in his being to give him some feeling of that dimension; and so this may explain the fact that we have some feeling of the fourth dimension.
10. Mathematics is a vast storehouse of the discoveries of the human intellect. We cannot afford to discard this material.
11. It is not necessary that the solution of a problem, by limited means,—say the trisection of an angle,—should be found in order that we may feel certain that the problem can be solved by *some* means.
12. Every term in an infinite sequence is in a small way a part of infinity.

8. In God's sight the infinite past and the infinite future are the same.
9. The human soul has enough of the divine within it to have some feeling of the reality of divinity and of the world beyond.
10. Religion is a vast storehouse of the discoveries of the human spirit. We cannot afford to discard this material.
11. It is not necessary that the solution of the problem of religion, by our limited human means, should be found in order that we may feel certain that the problem can be solved by *some* means.
12. Lucretius spoke wisely when he said, "Everyone is in a small way the image of God."

Conclusion. And what is the conclusion? Does mathematics make a man religious? Does it give him a basis for ethics? Will the individual love his fellow man more certainly because of the square on the hypotenuse?—Such questions are trivial; they are food for the youthful paragrapher. Mathematics makes no such claim. What we may safely assert, however, is this,—that mathematics increases the faith of a man who has faith; that it shows him his finite nature with respect to the Infinite; that it puts him in touch with immortality in the form of mathematical laws that are eternal; and that it shows him the futility of setting up his childish arrogance of disbelief in that which he cannot see.

And if this be the case, then what is the duty of teachers of our science? To preach?—that should be the last thought. The greatest sermons are preached in silence. The most ancient religions that we have, if there be more than one fundamental religion, have always recognized this fact. And so it must be with us,—that we should teach "the science venerable" not merely for its technique; not solely for this little group of laws or that; not only for a body of unrelated propositions or for some examination set by the schools; but that we should teach it primarily for the beauty of the discipline, for "the music of the spheres," and for the faith that it gives in truth, in eternal law, in the Infinite, and in the reality of the imaginary; and for the feeling of humility that results from our comparison of the laws within our reach and those which obtain in the transfinite

domain. With such a spirit to guide us, what teachers we would be!—whether of those who are standing on the threshold, of those who are passing through the realms of mystery that lead to manhood and womanhood, of those of mature years, or of those who, as the ancients were wont to say, "number their years upon their right hand." For then, unconsciously but none the less surely, would we prove the words of a seer among poets:

"And Reason now, through number,
time, and space
Darts the keen lustre of her serious eye;
And learns, from facts compared,
the laws to trace
Whose long procession leads to Deity."

MATHEMATICAL ASSOCIATION OF AMERICA.

The sixth annual meeting of the Association is to be held at Toronto December 29–30, 1921. The American Association for the Advancement of Science and the American Mathematical Society are to hold their annual meetings there during the same week.

The following 49 persons and 1 institution, on applications duly certified, have been elected to membership.

G. H. ALBRIGHT, A.M. (Harvard). Prof., Colorado College, Colorado Springs, Colo.
THURMAN ANDREW, B.S. (W. Va. Wesleyan). Teacher, Wesleyan College, Buckhannon, W. Va.
W. F. C. ARNDT, Ph.D. (Göttingen). Prof., Grey Univ. Coll., Bloemfontein, South Africa.
E. A. BAILEY, A.B. (Georgia). Instr., U. S. Naval Academy, Annapolis, Md.
W. W. R. BALL, M.A. Trinity College, Cambridge, Eng.
E. B. BEATTY, A.M. (California). Asso. prof., Oregon Agric. Coll., Corvallis, Ore.
B. C. BELLAMY, B.S.C.E. (Wyoming). Civil Engineer, Laramie, Wyo.
O. J. BOND, LL.D. (South Carolina). Supt., The Citadel, Milit. Coll. of South Carolina, Charleston, S. C.
B. H. CAMP, Ph.D. (Yale). Prof., Wesleyan Univ., Middletown, Conn.
C. H. CHEPMELL, Major, Sussex, Eng.
CARRIE A. DAY (Fredonia State Normal). Teacher, H. S., Smethport, Pa.
L. G. DUPASQUIER, Ph.D. (Zurich). Prof. ord., Univ. Neuchatel, Switzerland.
H. W. ENGLISH. Master in Chancery, Jacksonville, Ill.
HENRI FEHR, Docteur ès sciences (Geneva). Prof., Univ. of Geneva, Switzerland.
PHILIP FITCH, A.B. (Colorado Coll.). Instr., physics, N. Side H. S., Denver, Colo.
A. R. FORSYTH, Sc.D., LL.D., Math.D. Chief prof. of math., Imperial Coll. of Sc. and Tech., S. Kensington, London, Eng.
TORWALD FREDERICKSEN, Ph.B. (Chicago). Instr., Colgate Univ., Hamilton, N. Y.
GUIDO FUBINI. Prof., Univ. and Polytech. School of Turin, Turin, Italy.
J. C. FUNK, A.M. (Columbia). Head of dept. of math., H. S. and Junior Coll., Santa Maria, Calif.
C. L. GARNER, B.E. (N. C. State Coll.). Hydrogr. and geod. engr., U. S. Coast and Geod. Survey, Washington, D. C.
R. A. P. GÉRARDIN, Editor *Sphinx-Œdipe*, Nancy, France.
J. HADAMARD, Prof., Ecole Polytech. and Coll. de France, Paris, France.
G. H. HARDY, Savilian Prof. of Geom., Oxford Univ., Oxford, Eng.
R. R. HAUN, A.M. (Vanderbilt). Prof., Ashland Coll., Ashland, Ohio.
I. O. HORSFALL, A.B. (Utah). Head of dept. of math., Latter Day Saints H. S., Salt Lake City, Utah.
L. A. HOWLAND, Ph.D. (Munich). Prof., Wesleyan Univ., Middletown, Conn.

H. M. JEFFERS, Ph.D. (California). Instr. in astr., Iowa State Univ., Iowa City, Iowa.
R. P. JOHNSON, A.M. (Virginia; Harvard). Instr., U. S. Naval Academy, Annapolis, Md.
H. V. KNORR, A.B. (Susquehanna Univ.). Head of dept. of math., Central Wesleyan College, Warrenton, Mo.
ANNA D. LEWIS, Ph.D. (Carleton). Prof. of math. and physics, Lake Erie College, Painesville, Ohio.
C. F. LEWIS, A.B. (Denver). Instr., Kansas State Agric. Coll., Manhattan, Kansas.
FLORENCE LONG, M.S. (Illinois). Asst. prof., Earlham College, Earlham, Ind.
E. S. MANSON, Jr., M.S. (Mass. Inst. of Tech.). Asso. prof. astr., Ohio State University, Columbus, Ohio.
W. J. MARTYN, M.A. (New Zealand). Math. Master, Otago Boys H. S., Dunedin, N. Z.
NINA M. MCLATCHEY, A.B. (Washburn). Instr., Univ. of Kansas, Lawrence, Kans.
A. A. MCSWEENY, A.M. (Montana). Asst. prof., math. and astr., Albion Coll., Albion, Mich.
YOSHIO MIKAMI. Research assoc., Imperial Acad., Tokyo, Japan.
A. L. MILLER, Ph.D. (Harvard). Brookline, Mass.
N. B. MITRA, M.A. (Calcutta). Prof., Ewing Christian Coll., Allahabad City, India.
W. B. PIETENPOL, Ph.D. (Wisconsin). Asso. prof., physics, Univ. of Colorado, Boulder, Colo.
C. J. REFS, A.B. (Franklin and Marshall). Asst. prof., Delaware College, Newark, Del.
L. P. RIPPY, Ph.B. (Elon College). Elon College, N. C.
JOSEPH SEIDLIN, A.M. (Columbia). Prof., physics, Alfred Univ., Alfred, N. Y.
C. J. STOWELL, Ph.D. in Econ. (Illinois). Prof., math. and physics, McKendree Coll., Lebanon, Ill.
MARY S. TAYLOR, A.B. (Cornell). Teacher, N. Y. Training School for Teachers, New York, N. Y.
W. A. TITSWORTH, A.M. (Alfred), M.S. (Wisconsin). Prof., Alfred Coll., Alfred, N. Y.
H. G. TITT, A.M. (Michigan). Dean and prof. of math., Fairmount Coll., Wichita, Kans.
E. B. VAN VLECK, Ph.D. (Göttingen). Prof., Univ. of Wisconsin, Madison, Wis.
B. F. WALTON. Instr., Randolph-Macon Coll., Ashland, Va.

HARVARD COLLEGE, Cambridge, Mass., to institutional membership.

W. D. CAIRNS,
Secretary-Treasurer.

June, 1921.

PROPOSED AMENDMENT TO THE BY-LAWS OF THE ASSOCIATION.

At the Wellesley meeting of the Board of Trustees of the Association, the following amendment to the By-Laws was recommended to the Association for adoption at the next annual meeting, embodying the report of a committee appointed by the Trustees from among the actuarial members of the Association. The committee (Professors E. L. Dodd, chairman, J. W. Glover, and H. L. Rietz) express their judgment that this is an actuarial equivalent for the four dollar membership.

"On the payment of $100 - x$ dollars in one payment, where x is the number of years in the member's age, a member may become a life member and shall thereafter be exempt from all annual dues. A separate account shall be kept of life membership dues, this account to be credited at the end of each interest period with interest earned, and to be charged at the beginning of each year with $3.75[1] for each surviving life member."

This amendment, if adopted by the Association, will appear as By-Law 6 of

[1]Annual dues less cost of collection.

Article VII; it is proposed for action at the annual meeting[1] under the provisions of By-Law 1 of Article VIII.

W. D. CAIRNS, *Secretary-Treasurer.*

THE SIXTH SUMMER MEETING OF THE ASSOCIATION.

The sixth summer meeting of the Mathematical Association of America was held by invitation at Wellesley College on Tuesday, Wednesday and Thursday, September 6–8, 1921, in conjunction with, and immediately preceding, the summer meeting of the American Mathematical Society. 146 were present at the meeting, including the following 106 members of the Association:

C. R. ADAMS, Harvard University.
R. C. ARCHIBALD, Brown University.
CLARA L. BACON, Goucher College.
I. A. BARNETT, University of Saskatchewan.
IDA BARNEY, New Haven, Conn.
SUZAN R. BENEDICT, Smith College.
A. A. BENNETT, University of Texas.
E. G. BILL, Dartmouth College.
G. D. BIRKHOFF, Harvard University.
VEVIA BLAIR, Horace Mann School.
G. A. BLISS, University of Chicago.
L. A. BRIGHAM, Boston University.
B. H. BROWN, Harvard University.
R. E. BRUCE, Boston University.
W. D. CAIRNS, Oberlin College.
B. H. CAMP, Wesleyan University.
F. E. CARR, Oberlin College.
A. B. CHACE, Brown University.
J. W. CLAWSON, Ursinus College.
G. M. CONWELL, N. Y. State College for Teachers.
J. L. COOLIDGE, Harvard University.
LENNIE P. COPELAND, Wellesley College.
C. H. CURRIER, Brown University.
C. E. DIMICK, U. S. Coast Guard Academy.
ELEANOR C. DOAK, Mount Holyoke College .
W. P. DURFEE, Hobart College.
J. A. EIESLAND, West Virginia University.
L. P. EISENHART, Princeton University.
H. J. ETTLINGER, University of Texas.
J. A. FOBERG, Penna. Department of Public Instruction.
T. C. FRY, Western Electric Company.
C. A. GARABEDIAN, Harvard University.
H. D. GAYLORD, Browne and Nichols School.
D. C. GILLESPIE, Cornell University.
R. E. GILMAN, Brown University.
O. E. GLENN, University of Pennsylvania.
C. F. GUMMER, Queen's University.
E. R. HEDRICK, University of Missouri.
ARCHIBALD HENDERSON, University of North Carolina.
GOLDIE P. HORTON, University of Texas.
L. A. HOWLAND, Wesleyan University.
L. S. HULBURT, Johns Hopkins University.
E. V. HUNTINGTON, Harvard University.
W. A. HURWITZ, Cornell University.
NELLE L. INGELS, Washington, D. C.
DUNHAM JACKSON, University of Minnesota.
O. D. KELLOGG, Harvard University.
J. P. KELLY, Boston College.
W. D. LAMBERT, Coast and Geodetic Survey.
MARCIA L. LATHAM, Hunter College.
ANNA D. LEWIS, Lake Erie College.
FLORENCE P. LEWIS, Goucher College.
W. R. LONGLEY, Yale University.
A. C. LUNN, University of Chicago.
E. S. MANSON, Ohio State University.
R. M. MATHEWS, Wesleyan University.
NANNIE J. MCKNIGHT, Western High School, Washington, D. C.
T. E. MERGENDAHL, Tufts College.
HELEN A. MERRILL, Wellesley College.
A. L. MILLER, Brookline, Mass.
G. A. MILLER, University of Illinois.
E. B. MODE, Boston University.
C. N. MOORE, University of Cincinnati.
R. K. MORLEY, Worcester Polytechnic Institute.
H. C. M. MORSE, Cornell University.
J. R. MUSSELMAN, Johns Hopkins University.
C. E. NORWOOD, U. S. Naval Academy.
W. F. OSGOOD, Harvard University.
ANNA H. PALMIÉ, Western Reserve University.
ANNA J. PELL, Bryn Mawr College.
L. R. PERKINS, Middlebury College.
E. C. PHILLIPS, Woodstock College.
SUSAN M. RAMBO, Smith College.
W. R. RANSOM, Tufts College.
C. N. REYNOLDS, West Virginia University.

[1] The unavoidable delay in the appearance of the October issue made it impossible to give the necessary notice, in the MONTHLY, one month before the annual meeting; hence it was not possible to adopt this amendment at the annual meeting in December, 1921.

HARRIS RICE, Worcester Polytechnic Institute.
L. H. RICE, Mass. Institute of Technology.
R. G. D. RICHARDSON, Brown University.
E. D. ROE, Jr., Syracuse University.
GEORGE RUTLEDGE, Mass. Institute of Technology.
F. H. SAFFORD, University of Pennsylvania.
HAZEL E. SCHOONMAKER, Gulf Park College.
W. G. SIMON, Western Reserve University.
H. L. SLOBIN, New Hampshire State College.
CLARA E. SMITH, Wellesley College.
D. E. SMITH, Columbia University.
P. F. SMITH, Yale University.
MAY J. SPERRY, Knox College.
J. A. TOBIN, Boston College.
M. O. TRIPP, Wittenberg College.
H. W. TYLER, Mass. Institute of Technology.
ROXANA H. VIVIAN, Wellesley College.
J. L. WALSH, Harvard University.
A. G. WEBSTER, Clark University.
J. K. WHITTEMORE, Yale University.
RUBY WILLIS, Wellesley College.
EUPHEMIA R. WORTHINGTON, Wellesley, Mass.
RUTH G. WOOD, Smith College.
F. S. WOODS, Mass. Institute of Technology.
FRANCES W. WRIGHT, Elmira College.
C. H. YEATON, Oberlin College.
JESSICA M. YOUNG, Washington University.
J. W. YOUNG, Dartmouth College.
MABEL M. YOUNG, Wellesley College.
S. D. ZELDIN, Mass. Institute of Technology.
NOTRE DAME UNIVERSITY, Joseph Donahue, official representative.

Among those who came from more distant parts of the country there were in attendance three from Canada, three from Texas, two from Missouri and one each from Minnesota, Mississippi, and North Carolina.

The committee on local arrangements planned most effectively for the comfort of guests and this was recognized in a motion presented by Professor C. N. Moore and heartily adopted in the Thursday morning session. Well-appointed quarters were provided in Tower Court, one of the magnificent dormitories, the members using this building also for their meals, for committee meetings and for frequent social gatherings. On Tuesday afternoon an opportunity was afforded to visit the private gardens of the beautiful Hunnewell estate; in the evening an organ recital was given by Professor Macdougall of Wellesley College, and parties were afterward taken on a visit to Whitin Observatory. On Thursday afternoon a visit was made to the unusual collection of butterflies belonging to the Denton brothers in Wellesley. After the close of the Society program Friday noon, automobiles were placed at the disposal of all who wished to remain, for a ride to the historic spots in Lexington and Concord.

The joint banquet of the Society and the Association was attended by about 120 members and visitors. President Ellen F. Pendleton gave a gracious welcome to Wellesley's guests, and under the toastmastership of President Bliss of the Society speeches were made by President Miller of the Association, Professors Huntington, Jackson, Hedrick, Lewis, Eisenhart and Webster.

GIFT TO THE ASSOCIATION FOR THE PUBLICATION OF MATHEMATICAL MONOGRAPHS.

The outstanding feature of the Wellesley meeting was the announcement of a notable gift made to the Association. The nature and the terms of the gift cannot be presented to our members and to the mathematical fraternity more clearly than in the admirable letter of transmission from the donor:

La Salle, Ill., August 24, 1921.

Professor H. E. Slaught,
Chicago, Illinois.
Dear Professor Slaught:

Confirming our recent conferences regarding the publication of a series of monographs on mathematical subjects, to be prepared and published by The Mathematical Association of America, I as trustee for the Edward C. Hegeler Trust Fund hereby offer a yearly contribution to the Association of twelve hundred dollars ($1200) for a period of five years payable on January first of each year and beginning in January 1922, these funds to be under the complete control of The Mathematical Association of America. It is understood that the distribution of these monographs to the members of the Association will naturally be conducted through its own channels, but for their distribution to the general public the Open Court Publishing Company gladly offers its services.

If at the end of five years this project shall have proved successful it is my intention to then give to the Association a permanent endowment fund, and I will so direct my legal representatives, which will yield at least twelve hundred dollars annually, hoping that this will be sufficient income for the publication of two or three monographs a year, as long as this form of publication appears desirable and best suited to the diffusion of mathematical and formal thought as contributory to exact knowledge and clear thinking, not only for mathematicians and teachers of mathematics but also for other scientists and the public at large. But nothing is hereby intended to restrain the Association from the use of this income for the promotion of other mathematical publications if at some future time it should be deemed best to cease the publication of such monographs.

The Open Court Publishing Company has also been working for the diffusion of mathematical knowledge through a related channel, namely, by making certain mathematical classics available to the public through reprints, and in the furtherance of this ideal it is hoped that its coöperation with the Association will prove desirable and mutually helpful, but it is not intended to bind either organization.

It is my hope that The Mathematical Association of America will be successful in editing such a series and that it will set a standard of simplicity and clearness not before attained by mathematical writers in English.

Trusting that this donation may be of general benefit, I am

Yours very truly,
(Signed) Mary Hegeler Carus,
Trustee Edward C. Hegeler Trust Fund.

Appropriate action has been taken by the Trustees in formally accepting this gift and expressing their grateful recognition of this notable opportunity for an advance in American mathematics. Further, by a motion adopted unanimously in the morning session of the Association, Professor Smith and the Secretary were instructed to send a letter to Mrs. Carus embodying the Association's keen appreciation of her generous gift. This letter follows.

Dear Mrs. Carus:

At a meeting of the Mathematical Association of America, held at Wellesley College September 7, 1921, there was read to the members your letter of August 24, 1921, announcing the notable gift made by you for the purpose of enabling the Association to publish a series of mathematical monographs. Although it was further announced that the Trustees had formally accepted the trust, together with the obligations attendant thereon, and that they had expressed to you their appreciation of your generosity, the Association, desiring also to express its own sentiments upon this memorable occasion, passed the following resolution and delegated the undersigned to convey the same by letter to you:

Resolved that the Mathematical Association of America, having been advised of the munificent gift of Mrs. Mary Hegeler Carus as Trustee of the Edward C. Hegeler Trust Fund, desires to express its sincere appreciation of her action and to assure her of its determination to make every effort to carry out the trust in accord with the desires which she has so clearly set forth. The Association joins with her in the expression of the belief that it is both possible and feasible to

present to a large body of intelligent readers a clear and satisfying view of the leading branches of modern mathematics which, under present conditions, they are unable to obtain. To this end the Association pledges its efforts to secure an efficient board of editors who will carry out the project, mindful of its responsibilities and determined to make of the series the success for which we all confidently hope.

Yours very truly,

DAVID EUGENE SMITH,
W. D. CAIRNS.

President Miller presided at the sessions of the Association, save that Vice-President Archibald replaced him for a part of the Wednesday morning session. Presidents Bliss and Miller occupied the chair at the joint sessions of Wednesday afternoon and Thursday morning respectively. The program was formulated under the chairmanship of Professor W. R. Longley and the committee's able rendering of its duties was commended in Professor Moore's motion. When it was announced at the final session that Professor A. A. Bennett had been appointed Editor-in-Chief of the AMERICAN MATHEMATICAL MONTHLY to succeed Professor Archibald, whose resignation had been accepted by the Trustees to take effect at the close of the present volume, the following resolution was offered by Professor Jackson and heartily adopted by the Association:

"Voted, That the Association place on record its appreciation of the faithful, enthusiastic, and scholarly services of Professor Archibald as Editor-in-Chief of the MONTHLY, constituting a contribution to its history which no other man could have made and a significant stage in its advancement toward a permanent place among the most important mathematical journals."

The following papers were read, the last two being given in the joint session of the Society and the Association.

(1) "Freshman Mathematics" by Professor W. F. OSGOOD, Harvard University.

General discussion, led by Dean E. G. BILL, Dartmouth College, and Professor CLARA L. BACON, Goucher College.

(2) Report of progress of the National Committee on Mathematical Requirements, by Professor J. W. YOUNG, Dartmouth College.

(3) "Religio mathematici"—retiring presidential address by Professor D. E. SMITH, Teachers College, Columbia University.

(4) "Synthetic projective methods of generating cubic and quartic curves" by Professor HELEN A. MERRILL, Wellesley College.

(5) "A few questionable points in the history of mathematics" by Professor G. A. MILLER, University of Illinois.

(6) "A simple form of Duhamel's theorem and some new applications" by Professor H. J. ETTLINGER, University of Texas.

(7) "Some mathematical aspects of the theory of relativity" by Professor JAMES PIERPONT, Yale University.

(8) "The place of the Einstein theory in theoretical physics" by Professor A. C. LUNN, University of Chicago.

Abstracts of the papers and discussions follow below, the numbers corresponding to the numbers in the list of titles.

1. Calculus is now rather generally accepted as an essential part of the curriculum of the freshman year, the more readily for the fact that the influential class of technical schools require trigonometry and solid geometry for entrance. A generation ago a leisurely treatment of calculus was given in the sophomore year, largely formal as represented in the texts of that period, the applications being for the most part geometrical and very closely following the prototype. Next followed what may be called the Perry movement. Professor Osgood feels it imperative to counsel a middle course. Numerous important principles were presented in his mapping out of the freshman course. The applications of the differential calculus to maxima and minima and allied topics should be included in a first course, and here as in later courses mathematicians must lay hold of the definite integral as the limit of a sum as an illuminating feature of the calculus. Formal work is to be carried to the point that for the easier applications the student need not be held up by the necessary differentiation or integration.

As to the amount of work that can be included, the speaker noted that in S. A. T. C. days he found that in one third of a year, in about thirty periods, it was possible to take up the vital parts of analytic geometry, including some applications, for example, the geometrical determination of points on an ellipse, etc. It was not a course for *seeing* analytic geometry; nevertheless it transformed the freshman from a high school boy into a college student. Then it was possible to begin the calculus.

Early in this course it is necessary for the pupil to see the derivative at work; the question of what the derivative is can be gained by seeing what it is for. The place for introducing the infinitesimal depends on the individual, Professor Osgood finding it convenient to put it at the end of the first group of differentiations before transcendental functions, this with full recognition of the fact that college teachers should not feel forced to turn out a finished product in their classes. For example, briefly stated, we may begin with the familiar definition $\lim\limits_{\Delta x \to 0} \Delta y/\Delta x = D_x y$, and since $\Delta y/\Delta x = D_x y + \epsilon$, ϵ being a correction term, we have as a consequence $\Delta y = D_x y \cdot \Delta x + \epsilon \Delta x$, whence we get $dy = D_x y \cdot \Delta x$. Then we prove that $dy = D_x y \cdot dx$, etc.

Next comes the differentiation of trigonometric functions, with the addition of which there is ample material for drill in the formal side of differentiation. We may now introduce additional problems in maxima and minima involving trigonometric functions; here is a natural place to treat the analytical part of trigonometry.

The student should now be given the great advantage which the calculus affords in curve tracing. The differentiation of logarithmic functions comes next. Inverse functions need not enter until the work on integration.

But here the new mathematics diverges sharply from the old. We must now get over the subject of indefinite integrals, treating the necessary definitions and

formulas, for example, the clear realization of the fact that these equations all have the same meaning:

$$U = \int u dx; \qquad D_x U = u; \qquad dU = u dx.$$

The several elementary principles of differentiation (derivative of a sum, etc.) must needs be given in this course as much as in the older course. The number of formulas can however be limited to less than twelve, the others being deduced by several methods, the most important of which are *gumption* and substitution.

Professor Osgood then introduces simple problems in physics, explaining the method of defining elementary work for a variable force, in extension of the pupil's high school principle of work, and thus leading up to the use of Duhamel's theorem. He closed his address by saying that there never has been a time when teachers were more at a premium than today; that teachers need continually to adapt their methods to the various conditions, for example, that women students are not as a class vitally interested in the problems of physics; that there should be inculcated in our pupils as fully as possible: (1) a broad and deep interest in mathematical analysis for its own sake, (2) a sympathetic attitude toward the outstanding problems of physics, (3) judgment on the scientific side of choosing that which is genuine and fundamental and of avoiding that which is impossible of attainment, and also, judgment as to the proper presentation of this material.

Dean Bill stated that the calculus is best adapted to present those methods that are fundamental. The pupil should (1) acquire the ability to analyze a problem, (2) have proper tools, (3) seek the one prime requisite of accuracy, the average teacher being apt to pass a student on the ground that he has the right idea, even though he be woefully deficient in expressing this idea accurately. To accomplish this, (1) we must start the student in a correct manner from the outset, it being a psychological error to begin with mathematical reviews rather than to stimulate the student by taking up topics that are at least ostensibly new; (2) we must recognize that the pupil, like ourselves, cannot see many steps in advance and that it is by far better to master one subject fully than to cover many subjects and so to contribute to the great American trait of superficiality. It may be on this last ground that one may criticize Professor Osgood's course for trying to cover too much in the first short calculus course; it may be possible however to cut down the material in trigonometry and in conic sections sufficiently to gain time for the calculus.

Miss Bacon remarked that Professor Osgood's course as outlined was stimulating for those who have presented advanced entrance credits or are specially gifted in mathematics, but that it is difficult to do justice to trigonometry and analytic geometry and to bring all of this material into the second semester, particularly in a three hour course. She advocated having two freshman courses, one similar to that suggested by Professor Osgood, consisting of analytic geometry in the first semester and calculus in the second, the other for the ordinary freshman, less mature mathematically, who offers only plane trigonometry and elementary algebra, the course to consist of trigonometry for the first semester and analytic geometry for the second, leaving the calculus for the sophomore year.

Professor Longley spoke of the great variation in the notion of freshman mathematics, as indicated by the results of a questionnaire sent to about thirty colleges and universities from Yale; for example, the amount of time devoted to advanced algebra varied from zero to fifty-five hours, trigonometry varied greatly, conics varied from zero to twenty-four lessons. Only a few of these taught as much as a half year of freshman calculus. The reports differed so much from each other that no general conclusions could be derived from the data.

Professor Glenn expressed his pleasure at hearing of freshman teaching based on genuine theory rather than pedagogical convenience and observed that something may still perhaps be said as to giving a freshman more in some parts of his first course than he can stand.

In reference to students who had not had trigonometry, Professor Osgood said that while the numerical portion of trigonometry should not be curtailed, the analytical portion can almost all be treated in connection with calculus; that, for example, trigonometry might be taught for five weeks, such topics as the addition theorem being touched upon in the first treatment, the remainder of the first half year reserved for analytic geometry, thus using the second half year for calculus with a fuller treatment of some topics of the first half year.

Professor Ransom, in commenting on Professor Longley's remarks, felt sure that the general tendency of colleges over the country is toward two types: one the old-fashioned type consisting of extracts from college algebra, solid geometry and analytic geometry, the other, now emerging, including parts of trigonometry, analytic geometry and calculus.

2. The National Committee held its last meeting under its present form of organization on September fifth. One phase, at least, of its work has come to an end. In view of several previous reports made to the Association, it hardly seems necessary at this time to review in detail the work of the Committee. The present report may well be confined to a statement regarding present conditions and plans for the immediate future. The manuscript of a summary of the final report of the Committee has been sent to the U. S. Bureau of Education for publication. This summary, which will constitute a bulletin of some eighty pages, virtually presents the first part of the complete report. It contains the following chapters:

I. A Brief Survey of the Report
II. Aims of Mathematical Instruction—General Principles
III. Mathematics for Years Seven, Eight and Nine
IV. Mathematics for Years Ten, Eleven and Twelve
V. College Entrance Requirements in Mathematics
VI. List of Propositions in Plane and Solid Geometry
VII. The Function Concept in Elementary Mathematics
VIII. Terms and Symbols in Elementary Mathematics,

and also a brief synopsis of the remaining chapters of the complete report. It is expected that this summary will appear late in November or early in December.

It was the original intention of the Committee to publish its complete report

also through the U. S. Bureau of Education. It was found, however, that this would involve a delay of two or three years in view of the fact that it would have been necessary for the Bureau of Education to issue the report in parts extending over a considerable period of time. It is hoped at present that sufficient funds will be obtainable to publish the report during the winter and to distribute it free of charge to all who are sufficiently interested to ask for it. The complete report will constitute a volume of about five thousand pages. In addition to the chapters listed in the summary, it will contain an account of a number of investigations instituted by the Committee. Among these may be mentioned:

The Present Status of Disciplinary Values in Education
A Critical Study of the Correlation Method Applied to Grades
Mathematical Curricula in Foreign Countries
Mathematics in Experimental Schools
The Use of Mental Tests in the Teaching of Mathematics
The Training of Teachers of Mathematics.

There will also be included an extensive bibliography on the teaching of mathematics.

In closing this phase of its work, the Committee desires to extend its most cordial thanks to all the individuals and organizations that have helped. The response secured by the Committee to its appeal for assistance in solving the many problems facing it has been extremely enthusiastic and gratifying. This leads the Committee to look forward to the future optimistically. The real work for which the Committee was appointed may be said to begin with the publication of its report rather than to end with it. Continued enthusiastic activity on the part of all individuals and organizations concerned with the teaching of mathematics is needed over a period of many years to put the recommendations of the Committee into effect, to test their validity and to modify them in ways that experience shows to be desirable. In order to be of assistance in this direction, the Committee hopes to be able to maintain an office with a certain amount of clerical help during the next few years so that it may continue to act as a clearing house for ideas and to stimulate the discussion of problems relating to the teaching of mathematics among the nearly one hundred organizations that have in the past been actively coöperating with the Committee. Members of this Association are strongly urged to continue their support and activity in connection with the cause for which the National Committee has been laboring.

Following Professor Young's report the session adopted by a hearty vote a motion of Professor Coolidge recognizing the great importance of the series of reports, their potential influence for good in the cause of mathematics if these reports come fully to the attention of American teachers, and commending the committee members and those collaborating with them for the high order of ability and thoroughness shown in these reports.

3. President Smith's able retiring address is published elsewhere in this issue of the MONTHLY.

4. The methods ordinarily given in the texts for the generation of cubics by two flat pencils of the first and second order respectively can be greatly shortened by substituting for one or both of these pencils a flat pencil of the first order in involution. Schröter has proved that the general cubic may be generated by two such pencils related projectively and having a common corresponding ray, while, if the common ray is not corresponding, a quartic is generated. Unicursal cubics may be generated by relating projectively two flat pencils of the first order, one of which is in involution, and then special cubic curves may be obtained very readily from systems of coaxial circles.

Miss Merrill's paper, embodying a method which has proved especially interesting even to first year students in two or three colleges, and which gives points on the curves very expeditiously, since these ordinarily are given in sets of four, may be expected in an early number of the MONTHLY.

5. Professor Miller's paper, in effect, appeared in *Scientific Monthly*, September, 1921.

6. Professor Ettlinger presented a very simple form of Duhamel's theorem by means of the following geometric lemma: Let the interval I: $a \leqq x \leqq b$ be divided into n equal subdivisions, I_{in}, of length, $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots, n)$. Upon I_{in} as base, draw a rectangle of area, R_{in}, and height h_{in}, such that $|h_{in}| \leqq M$ for all values of n. If $\overline{P}$ is any fixed point of I, there is for each value of n, at least, one rectangle whose base contains $\overline{P}$. If for each fixed $\overline{P}$, the altitude of this rectangle, h_P, approaches zero as $n \to \infty$, then $\lim_{n \to \infty} \sum_{i=1}^{n} R_{in} = 0$.

The advantages of this form are fourfold: (1) it is simple enough for a first course in the calculus, (2) it avoids the question of double limits and uniform convergence, (3) it is more general than earlier contributions, (4) it is easily applied.

A simple proof of the integrability of a continuous function is given by means of this formulation. Finally, new applications are made to the transformation of a double integral and to the solution of an integral equation.

The discussion of this paper turned rather upon the wisdom of using Duhamel's theorem or a substitute; it was observed however that Professor Ettlinger apparently avoided the use of uniform convergence, but that logically this principle must enter into any proof of Duhamel's theorem or its equivalent.

In his inimitable manner Professor Huntington laid, as he said, a wreath on the tomb of Duhamel's theorem for the very good reason that he believed it to be dead, had great respect for it and had a belief in its immortality. He illustrated how one can set up a definite integral with perfect rigor without the use of Duhamel's theorem, his full presentation having been given in this MONTHLY, *1917*, 271–275.

Professor Bliss discussed the problem of the total attraction due to a thin rod, using the substitute for Duhamel's theorem which he published in the *Annals of Mathematics* for December, 1914.

Professor Birkhoff mentioned his increasing attachment to Duhamel's theorem. He has found such an illustration as this illuminating and easily understood: If I am calculating the cost of building from the estimates of various sub-contractors, each being correct to within 10 per cent., I am certain that my whole estimate is correct to within 10 per cent. He considers it wise to use, not the more difficult problem of attraction, but a simpler application, the length of an arc. After simple examples he tells his students that it is well for them to read Duhamel's theorem, his experience being that a gratifying number of students will work their problems by actually using Duhamel's theorem.

7. Professor Pierpont's purpose, achieved in a wonderfully clear manner, was to present in simple form the main mathematical facts on which Einstein's theory rests. As an extension of two- and three-dimensional concepts he described an n-way space or manifold, a hypersurface, distance and its connection with the theory of bilinear forms, angle, volume, geodetic curve, and the term "curvature of space," which is the generalization of the Gaussian curvature of a surface with its invariant $1/(R_1R_2)$. After he explained the mathematical notion involved in tensors and their properties, he showed how by generalizing the vector treatment of Maxwell's equations and similar equations Einstein derived his fundamental equations for a universe in which the geometry of a space is determined not merely through the conventions as to length and time, but by all the material that exists in this space. He finally sketched the still more generalized theory of Weyl, wherein any existing amounts of electricity and magnetism serve also to determine the geometry.

It will be of great service to all interested in the Einstein theory if, as is expected, Professor Pierpont's very helpful introduction to the difficulties of the mathematics of relativity may be made available through an American journal.

8. Because of the nature of the subject it is difficult to outline Professor Lunn's presentation of the place of the Einstein theory in theoretical physics. He sought in the main to give the physical interpretation of the mathematical notions and principles involved in Einstein's theory, indicating for example how the notion of n dimensions may be realized experimentally. A considerable part of the address had to do with a description of the vector notation and expressions used in the development of the theory.

Meeting of the Board of Trustees of the Association.

Ten members of the Board were present at this meeting.

(*a*) The Trustees received the formal letter from Mrs. Mary Hegeler Carus already presented in this report. The Board voted formally to accept the gift and to instruct the Secretary to send to Mrs. Carus the acceptance and the expression of gratefulness with which the Trustees regard this gift. It was voted to express the obligation of the Board to Professor Slaught for his large share in inspiring this gift and for his continued services to the Association in

this great enterprise. A committee, consisting of Professors Slaught, Veblen and Cairns, was appointed to nominate to the Trustees an editorial board on monographs and to formulate a statement of the powers of this board.

(*b*) In view of the resignation of Professor Archibald as Editor-in-Chief of the MONTHLY, to take effect with the completion of the present volume for 1921, Professor A. A. BENNETT of the University of Texas was appointed as his successor.

(*c*) The following seventy persons and five institutions, on applications duly certified, were elected to membership:

To individual membership.

L. C. BAGBY, A.B. (Washburn). Head of dept. of math., Ottawa Univ., Ottawa, Kans.
A. J. BARRETT, A.B. (Arkansas). Instr., U. S. Naval Acad., Annapolis, Md.
E. T. BROWNE, A.M. (Virginia). Asst. prof., Trinity Coll., Hartford, Conn.
JULIA A. BURRELL, A.B. (Rice Inst.). Galveston, Tex.
W. R. BURROWS, Medical student, Univ. of Chicago. Halstead, Kans.
MILDRED E. CARLEN, Student, Brown Univ., Secretary to the Editor-in-Chief of the MONTHLY, Providence, R. I.
E. L. CARR, A.B. (Ewing Coll.). Prof., Shurtleff Coll., Alton, Ill.
MARGARET F. CHAPMAN, A.B. (Wisconsin). Instr., State Normal School, Milwaukee, Wis.
E. F. CHURCH, C.E. (Syracuse). Prof., Penna. Military Coll., Chester, Pa.
LENA R. COLE, A.M. (Missouri). Head of dept. of math., Central Coll., Lexington, Mo.
ALBERTUS DARNELL, Ph.B. (Michigan). Head of dept. of math., Detroit Junior Coll., Detroit, Mich.
H. A. DAVIS, Asst., W. Va. Univ., Morgantown, W. Va.
A. K. DENNY, M.S. (Lincoln). Prof. and registrar, Lincoln Coll., Lincoln, Ill.
L. E. DONNELLY, Student, Univ. of Chicago, Chicago, Ill.
B. F. DOSTAL, A.M. (Indiana). Asst. prof., Univ. of Denver, Denver, Col.
FRANC C. EARHART, B.S. (Lenox). Dean of Women, Lenox Coll., Hopkinton, Iowa.
LOUIS FELDMAN, B.S. in E.E. (Cooper Union). Brooklyn, N. Y.
J. C. FIELDS, Ph.D. (Johns Hopkins). Prof., Univ. of Toronto, Toronto, Canada.
GLADYS H. FREEMAN, A.M. (Chicago). Instr., State Coll. of Washington, Pullman, Wash.
T. C. FRY, Ph.D. (Wisconsin). Mathematician, Research Labs., Amer. Tel. and Tel. Co., and Western Electric Co., New York, N. Y.
LESLIE J. GAYLORD, A.B. (Lake Erie). Instr., Agnes Scott Coll., Decatur, Ga.
GLADYS E. C. GIBBENS, Ph.D. (Chicago). Instr., Univ. of Minn., Minneapolis, Minn.
G. W. GORRELL, A.M. (Ohio State). Prof., Colo. School of Mines, Golden, Colo.
R. F. GRAESSER, A.B. (Illinois). Asst., Univ. of Illinois, Urbana, Ill.
LILIAN HACKNEY, A.B. (W. Va. Univ.). Head of dept. of math., Marshall Coll., Huntington, W. Va.
W. R. HALE, A.M. (Alabama). Asst. prof., Colo. School of Mines, Golden, Colo.
CLARA L. HANCOCK, A.M. (Iowa). Teacher, Junior Coll., Virginia, Minn.
D. C. HARKINS, A.B. (W. Va. Univ.). Asst., W. Va. Univ., Morgantown, W. Va.
C. E. HARRINGTON, M.E. (Cornell). Instr., Univ. of Buffalo, Buffalo, N. Y.
MARY G. HASEMAN, Ph.D. (Bryn Mawr). Instr., Univ. of Illinois, Urbana, Ill.
B. A. HAZELTINE, B.S. (Tufts). Instr., Tufts Coll., Tufts College, Mass.
ARCHIBALD HENDERSON, Ph.D. (North Carolina; Chicago), D.C.L. (Univ. of the South). Head of dept. of math., Univ. of North Carolina, Chapel Hill, N. C.
M. DET. HIGH, A.B. (Franklin & Marshall). Head of dept. of math. and dean of men, Central State Normal School, Lock Haven, Pa.
T. R. HOLLCROFT, Ph.D. (Cornell). Prof., Wells Coll., Aurora, N. Y.
F. F. HOOPER, A.M. (Wisconsin). Dean, Univ. of Chattanooga, Chattanooga, Tenn.
E. M. HORSBURGH, D.Sc. (Edinburgh). Lecturer on technical math., Univ. of Edinburgh, Edinburgh, Scotland.
MILDRED HUNT, A.M. (Chicago). Head of dept. of math., Bessie Tift Coll., Forsyth, Ga.
K. P. JOHNSTON, B.A., B.S. (Queens). Asst. prof., Queens Univ., Kingston, Ont., Canada.

AGNES A. JONES, A.B. (Hood Coll.). Grad. student, Univ. of Chicago. Larimer, Pa.
E. E. JORDAN, M.A. (Dalhousie). Asst. prof., Univ. of Brit. Columbia, Point Grey, B. C., Canada.
M. H. JÛRDAK, M.A. (Beirut). Prof., Amer. Univ. of Beirut, Beirut, Syria.
J. P. KELLY, A.M. (Woodstock). Prof., Boston Coll., Chestnut Hill, Mass.
G. E. KING, M.S. (Wisconsin). Prof., Iowa Wesleyan Coll., Mt. Pleasant, Iowa.
C. G. LATIMER, B.S. (Chicago). Instr., Swarthmore Coll., Swarthmore, Pa.
W. H. LYONS, A.M. (Denver). Muskegon Heights, Mich.
C. C. MACDUFFEE, Ph.D. (Chicago). Instr., Princeton Univ., Princeton, N. J.
H. B. MARSH, A.M. (Amherst). Head of dept. of math., Tech. High School and Junior Coll., Springfield, Mass.
E. J. MAURUS, M.S. (Notre Dame). Prof., Univ. of Notre Dame, Notre Dame, Ind.
W. C. MCCOY, A.B. (Ohio State). Columbus, Ohio.
NANNIE J. MCKNIGHT, A.M. (George Washington). Teacher, Western High School, Washington, D. C.
J. J. MILLER, B.L. (Ouachita). Head of dept. of math., Southeastern State Normal School, Durant, Okla.
Mrs. DECIMA E. MITCHELL, M.S. (Alberta). Instr., Univ. of Alberta, Edmonton, Alb., Canada.
R. L. MODESITT, A.M. (Indiana). Teacher, Eastern Ill. State Teachers Coll., Charleston, Ill.
W. A. MOORE, A.M. (Chicago). Prof., Birmingham-Southern Coll., Birmingham, Ala.
C. M. NOLAND, A.B. (Missouri). Prof., Howard Payne Coll., Brownwood, Tex.
J. O. OSBORN, A.B. (Berea). Instr., Cornell Univ., Ithaca, N. Y.
XAVIER PRUM, A.M. (Louvain). Prof., Columbia Coll., Dubuque, Iowa.
P. L. REA, A.B. (Oberlin). Instr., Marietta Coll., Marietta, Ohio.
C. A. REAGAN, B.S. (Moores Hill). Prof., Hanover Coll., Hanover, Ind.
Sister MARY RESIGNATA, A.M. (Catholic Univ.). Head of dept. of math., Mount St. Joseph Coll., Dubuque, Iowa.
ARVID REUTERDAHL. Dean, Dept. of engg. and archit., Coll. of St. Thomas, St. Paul, Minn.
HAZEL E. SCHOONMAKER, A.M. (Radcliffe). In charge of math., Gulf-Park Coll., Gulfport, Miss.
P. K. SMITH, A.M. (South Carolina). Adj. prof., Univ. of South Carolina, Columbia, S. C.
R. W. THOMAS, B.S. (Washington & Jefferson). Instr., Washington & Jefferson Coll., Washington, Pa.
TELESFORO TIENZO, A.B. (Philippines), B.S. (Chicago). Instr., Univ. of the Philippines, Manila, P. I.
B. L. WAITS, A.M. (Clark). Prof. of math. and physics, Fla. A. & M. Coll., Tallahassee, Fla.
W. W. WEBER, A.M. (Georgia). Prof. and dean, Southern Coll., Clearwater, Fla.
R. L. WILDER, M.S. (Brown). Instr., Univ. of Texas, Austin, Tex.
P. D. WILKINS, A.B. (Bowdoin). Instr., Tufts Coll., Tufts College, Mass.
ARTHUR WOODS, M.A. (Queens). Instr., Western Univ., London, Ont., Canada.

To institutional membership.

UNIVERSITY OF NOTRE DAME, Notre Dame, Ind.
ANTIOCH COLLEGE, Yellow Springs, Ohio.
SOUTHEASTERN STATE NORMAL SCHOOL, Durant, Okla.
VIRGINIA POLYTECHNIC INSTITUTE, Blacksburg, Va.
COLLEGE OF WILLIAM AND MARY, Williamsburg, Va.

(*d*) The President was empowered to appoint a representative of the Association on the Council of the American Association for the Advancement of Science, in addition to the Secretary-Treasurer, who is already a member; President Miller has appointed to this position Professor H. L. RIETZ of the University of Iowa.

(*e*) Professors A. A. BENNETT, H. P. MANNING and H. E. SLAUGHT were appointed as the Committee on Publications beginning with the volume of the MONTHLY for 1922.

(*f*) A committee consisting of three Association members of the National Committee on Mathematical Requirements, Professors J. W. YOUNG, C. N.

MOORE and H. W. TYLER, was appointed for the purpose of coöperating with a similar committee, from the Society for the Promotion of Engineering Education, in an investigation of the teaching of mathematics in engineering schools.

(*g*) The Association received an invitation from the department of mathematics of Cornell University to hold its summer meeting at Ithaca in either of the years 1923 or 1924, in conjunction with the American Mathematical Society.

W. D. CAIRNS, *Secretary-Treasurer.*

MATHEMATICAL PROBLEMS IN THE WORK OF THE UNITED STATES COAST AND GEODETIC SURVEY.[1]

By WALTER D. LAMBERT and OSCAR S. ADAMS,

Division of Geodesy, U. S. Coast and Geodetic Survey.

In the office of the United States Coast and Geodetic Survey there are three divisions that employ mathematics to a considerable extent in the routine work which they perform. These are the Division of Geodesy, the Division of Tides and Currents, and the Division of Terrestrial Magnetism. We shall aim to give some account of the nature and extent of the mathematics required as a tool in each of these three divisions.

The work of the Division of Geodesy is almost entirely mathematical and the problems that are presented for consideration are many and varied. The routine work consists of the computation and adjustment of triangulation, the computation of astronomical observations, the computation of gravity observations, and the computation and adjustment of observations in precise leveling. This is a bill of fare that should be sufficiently comprehensive to suit the taste of any one who is mathematically inclined (or should we say, in accordance with popular belief, mathematically deranged?)

The computation of triangulation and of astronomical observations requires a thorough training in trigonometry with especial attention to the use of logarithmic tables. This training should be directed toward the attainment of both speed and accuracy, with a considerable premium placed on accuracy. It may be thought that common sense would suffice for the use of tables, and so it would seem; but, if so, good common sense is oftentimes sadly lacking in men when they graduate from college. In the examinations given by the Coast and Geodetic Survey, mistakes are made by candidates that are so utterly absurd that one can hardly explain the psychology of the errors.

The adjustment of observations by the method of least squares forms the greater part of the purely mathematical work in the general routine of the geodetic division. This adjustment finds its most extensive field of application when applied to triangulation. The regular method employed in the Survey for this purpose is that of adjustment with condition equations as distinguished from

[1] Read at the meeting of the Ohio Section of the Mathematical Association of America, March 25, 1921.

observation equations. The development of such condition equations for closures in length, azimuth, latitude, and longitude, requires considerable skill in the application of analysis to practical problems. These equations must be rigid enough to fulfil their purpose, but at the same time they must be as simple as possible so as to obviate the performance of any undue labor in the practical formation of the equations for any specific piece of triangulation.

The use of observation equations in the adjustment of triangulation, tried by other countries, has been found to have certain advantages. This Survey has worked out the necessary formulas and published them with examples for their use, but has not so far applied the method extensively to its own work. For the adjustment of differences of elevation, however, whether based on spirit leveling or trigonometric leveling, it is the practice of the Survey to use the method of observation equations, though it is quite practicable to use the method of condition equations. In the treatment of gravity observations and astronomical observations, observation equations are more frequently used than equations of condition.

The two largest adjustments that have ever been made in the Survey consisted of 264 and 284 equations. The amount of labor involved in the formation and solution of such a set of equations is enormous. In addition, one small error may require a month or two of hard work to make the necessary correction. It has been proposed to include all of the primary triangulation in the United States in one piece for scientific purposes, but since this would involve an adjustment consisting of between three and four thousand equations it has not as yet been attempted.

Whatever may be our opinion of the theory of least squares as founded upon probability, it must be recognized that it gives an exceedingly simple method for reconciling discrepancies in observations which, though small and practically insignificant, are yet troublesome because of the inconsistencies that they introduce. When one considers carefully, it is astonishing what degree of accuracy can be attained by careful work. The requirement for precise triangulation is that the average closing error of the triangles should be one second or less. Now on the largest circle used in horizontal angle measures (the 12 inch circle) one second represents about 58 one-millionths of an inch or about one and one half microns. With this fact in mind it is truly remarkable that such accuracy can be attained. By the adjustment these small outstanding discrepancies are reconciled and all values which have a check give one and the same result however computed.

Besides this regular routine work, the geodetic division prepares special publications on the theory of cartography, on triangulation, on the theory of gravity and isostatic compensation, and on various other phases of geodesy and geophysics. In this work practical applications may be found for many branches of mathematics. The whole subject of cartography comes finally under the head of the differential geometry of curves and surfaces. The subject of conformal mapping is founded directly upon the theory of the functions of a complex

variable and forms the field for one of the most interesting practical applications of this branch of pure mathematics. The attempt to treat any of the projections in this class without consideration of its foundation upon the general theory of functions of a complex variable leads always to lack of clearness and definiteness of conception. Lagrange and Gauss have contributed much towards the generalization of the treatment of this kind of mapping. The equal-area mapping is a special case of equal-area representation of one surface upon another such as is treated in extended works on differential geometry. The perspective projections are practical applications of the principles of projective geometry. It is needless to say that trigonometry and calculus, both differential and integral, are employed in all of this cartographic work. Developments in Taylor series and especially those in Fourier series are frequently employed.

When tables are computed for any one of the projections, extensive logarithmic trigonometric computations have to be performed. In this work the judgment must be continually on the alert to determine the necessary exactness of the calculations required to attain the end desired. During the period of the war two extensive tables of this kind were prepared in the Division of Geodesy; the results are given in *Lambert Projection—Tables for the United States* (U. S. Coast and Geodetic Survey, Special Publication no. 52) and *Grid System for Progressive Maps in the United States* (U. S. Coast and Geodetic Survey, Special Publication no. 59). Both of these tables were prepared at the request of the army authorities and they were intended primarily for military purposes.

The tidal problem is still very far from a satisfactory solution despite the efforts of many of the ablest mathematicians since Newton's time. For purposes of prediction a modified form of the statical theory is used. This requires something of spherical harmonics and a great deal of development in series very similar to the developments needed in the lunar and planetary theories of dynamical astronomy, though not nearly so extensive. In questions of hydrodynamics involved in the tidal theory as so far developed, use is made (among other branches) of spherical harmonics, Fourier series, functions of a complex variable, Bessel functions, partial differential equations, and functions too recently introduced or too special to have any well-known name. Even the theory of numbers has been applied indirectly to tidal problems, for some of its theorems were found useful in deciding on the number of teeth to put on the gears of a tide-predicting machine.

The routine of computations regularly made in the Division of Tides and Currents calls for frequent use of the trigonometric functions in all four quadrants. Perhaps this is as good a place as any to call to mind what most teachers of trigonometry will sadly admit to be true, namely, that many students end their formal study of that subject with the idea that the signs of the trigonometric functions in the various quadrants and the process of looking up functions of an angle greater than 90° are great mysteries, so mysterious that even the rules for these processes are difficult to memorize.

In the Division of Terrestrial Magnetism the mathematician must be thor-

oughly familiar with his trigonometry and use of logarithms and must be prepared to use the principles of calculus and Fourier analysis in the reduction and discussion of the results. As the quantities measured are subject to continuous fluctuations, the results cannot be subjected to rigid adjustment as in the case of triangulation, but the method of least squares can frequently be used to advantage for interpolation or for separating errors of observation from fluctuations of which the character is established. Some of the fluctuations of the earth's magnetism are of a periodic character and the harmonic analysis may be used to deduce from the observations the periods and amplitudes of fluctuations which may be correlated with other terrestrial or cosmical phenomena. In the graphical representation of the distribution of the earth's magnetism over a given area it is sometimes desirable to determine the regular distribution corresponding most nearly to the observed results, and this may be done by the methods of least squares, using an analytical expression involving the first and second powers of the latitude and longitude.

While much of the work of the Division of Terrestrial Magnetism is of a routine character, involving little more than the application of arithmetic and logarithms to the use of formulas, yet even for this work the computer must understand the derivation of the formulas, know how to modify them when the character of the data justifies the use of approximations, and regulate the refinement of computation by the degrees of accuracy to be expected.

In the design of instruments for the Survey, various questions of mechanics and mathematical physics arise.

If then we consider not only the daily routine but also the theory underlying that routine and the necessity for improving methods and advancing knowledge, we shall not be surprised at the number and variety of the mathematical theories that are laid under contribution. But little more than an imperfect catalogue can be given here. Time and space would fail for any detailed treatment.

In Geodesy proper we have solid analytic geometry, differential geometry and elliptic functions in connection with measurement of the terrestrial spheroid. In connection with gravity problems, we need spherical harmonics, including zonal harmonics of the second kind, and in some cases curvilinear coördinates and ellipsoidal harmonics. In some problems integral equations are involved. In questions of geophysics intimately related to geodesy proper, such as the transmission of earthquake waves, integral equations are again involved. The transmission of earthquake waves requires, of course, the mathematical theory of elasticity as does also the mechanical problem of the variation of latitude. In questions of mapping, as has already been mentioned, differential geometry and the theory of functions of a complex variable and of trigonometric series are needed.

No matter how difficult or how advanced the theory on which a formula is based, the numerical application of that formula usually comes down to the use of tables and the four rules of arithmetic. In certain cases where arithmetical work on a large scale is to be done, it is possible to utilize the services of people

who apply the rules of arithmetic to the data with little understanding of the reasons for what they are doing. Those who can do only this kind of unskilled labor—for it is mental labor corresponding to pick-and-shovel work in the field of manual labor—have been found to be unprofitable in the Coast and Geodetic Survey. They require too much supervision while at work and the supply of work that they can do is not continuous enough to ensure steady employment for them. For while there is much routine numerical work in the Coast and Geodetic Survey, there is a rather surprising amount of work that cannot be done by any cut-and-dried method. Peculiar conditions frequently arise in field work that must have special treatment in the reductions, and requests are continually coming in for information that the Survey would naturally supply, but which is not ground out by any of the regular machinery. Moreover, although a certain stability in form and method is desirable where great masses of data are handled, the results being published serially as they become available, nevertheless, no organization can keep on doing the same old thing in the same old way without suffering retrogression and decay. The Survey therefore seeks to learn and assimilate improved methods developed by other organizations doing similar work, to devise new methods on its own account and to advance knowledge in its own field.

Perhaps you will wonder, after hearing the foregoing list, not what branches of mathematics the Coast and Geodetic Survey needs, but what branches it does not need. You will find a few, however, that have not been mentioned. Perhaps the reason that some have been left out is that the right person has not come along to apply them; it may appear in the future that some branches of mathematics whose applicability to problems of mathematical physics is little known or not even suspected will turn out to be just what is needed to solve some problem in the work of the Coast and Geodetic Survey.

From their experience in the bureau, it seems to the authors desirable to lay stress on the following topics in elementary mathematics. These topics are important not alone in the work of the Coast and Geodetic Survey, but in other applications of mathematics to scientific and technical problems, and are not without cultural and disciplinary value. These concluding paragraphs are not to be taken as criticisms of present-day curricula and methods of teaching. The authors have both had some experience in teaching, and realize that, especially in pedagogical matters, it is easy to give advice, but much less easy to put it in practice.

1. *Numerical* calculation, first, with reference to *accuracy,* and second, *to economy of effort.* Very few young men just out of college or technical school have much idea of proportioning the accuracy of one part of a calculation to that of the other parts. They will cheerfully add together two quantities, one known to many decimal places, and the other known to much fewer, or they will multiply together two quantities, each known say to only three significant figures, and then retain the sixth significant figure of the product as if it meant something. It has never occurred to them that when the difference of two nearly equal quantities

is taken the number of accurate significant figures left is much reduced. These things are all simple enough, and are readily seen to be true once attention is drawn to them, but to many they seem to be considerations of a wholly new sort.

2. *Analytical trigonometry.* Somehow the signs of the trigonometric functions in the various quadrants should be taught so as to be retained, also the process of looking up functions of an angle greater than 90°, and the fact that the angle corresponding to a given function is not unique. Furthermore, a working knowledge of various trigonometric transformations is useful in applied mathematics.

3. *Approximations.* It is desirable to give students who are to apply their mathematics some instruction and training in the use of approximate formulas, in order that they may not distrust an admittedly approximate formula in those cases where it legitimately applies, and in order that, on the other hand, they may realize the limitations of such a formula and not try to make it do more than it is able to do.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

12. The Marquise du Chastellet.

One of the letters in my collection is of special interest because of two lines. The letter is as follows:

> We have arrived in good health, Monsieur, but with disquietude which your own health causes us. I hoped that M. Cléraut would have had news from you. I beg of you to give me information about yourself, and to tell me if we shall see you soon. You do not realize the interest that I have. M. de Voltaire charges me to present to you his most affectionate compliments; and as for me, I am, with very profound friendship, Monsieur, your very humble and very obedient servant
>
> Breteuil du Chastellet.
>
> Paris, August 31, 1744.
>
> I hope that you will not forget our little affair.

So Voltaire sends by the Marquise du Chastellet (to adopt her own spelling of the name) his affectionate compliments to some one whom we do not know. She was entirely frank about delivering the message. Voltaire, who six years before this had essayed to make Newton intelligible to the French people, was with her. In 1735 they had begun their work together at the Chastellet country place at Cirey,—"that pitiless bore, the unfortunate Marquis," assenting, and conveniently spending some time with his regiment and in Paris, but accepting without protest the eighteenth-century customs even when at home. In July of the year 1744 they were again at Cirey, Voltaire, as usual, in bed and hard at work. He was fifty and the Marquise was thirty-eight. They went up to Paris the latter part of August, preparing for the festival of rejoicing at Louis XV's recovery from an illness and his return from the wars. There was also Voltaire's *Princess* that was soon to be produced, and this demanded his presence at the capital—a production that brought him much acclaim.

A biographer who, being a woman, may be thought to have understood the Marquise, writes that "at his side was the woman who was the aptest pupil of Maupertuis and almost the only other person in France who understood Newtonianism save Maupertuis himself, Voltaire, and one Clairaut. The rest of the world was Cartesian." This Clairaut is the "M. Cléraut" of the letter, the Alexis-Claude Clairaut, then thirty-one years old, who had just published his *Théorie de la figure de la terre* (Paris, 1743). Five years later, while she was getting help from him in her work on Newton, he excited the jealous rage of Voltaire, who broke open a door at Cirey as a result, and then rushed down the stairs followed by the most notable woman mathematician of France and by one of the best-known men in the same field of work. Voltaire soon had occasion to repent his anger in sackcloth and ashes, for the Marquise died in 1749, at the age of forty-three. One of her Paris acquaintances, a grande dame of the salons, with the heartlessness of the age, remarked that "to die in childbed at her age is to wish to make oneself peculiar: it is to pretend to do nothing like other people." Frederick the Great, who would sacrifice all delicacy of feeling, if he ever had any, for the sake of a *bon mot*, suggested as her epitaph, "Here lies, one who lost her life in giving birth to an unfortunate infant and a treatise on philosophy." Thus came into light du Chastellet's work on Newton, together with a child of uncertain parentage, and thus passed away the most brilliant woman of both a country and a century of women of exceeding brilliancy,—intellectually, socially, and, probably in more cases than society history would lead us believe, morally as well.

13. Cassini Completes the Great Survey of France.

The first noteworthy attempt at measuring the earth, made in modern times was that of Jean Fernel, about 1528. He took the arc determined by Paris and Amiens, the two stations being located approximately on the same meridian. Considering the instruments available, his results were remarkable. He found that 1° = 57,099 toises, while the mean obtained by Lacaille and Delambre in the latter part of the eighteenth century, at which time the instruments were very satisfactory, was 57,068 toises. In 1669 and 1670 Jean Picard carried on an elaborate system of triangulation and found that 1° = 57,060 toises. This meridian was extended by Jean-Dominique Cassini (Cassini I) in 1701 and his work was continued by his son, Jacques, the results being published in Paris in 1720 and serving to set on foot the elaborate surveys which finally determined the spheroidal shape of the earth. César François Cassini de Thury, the son of Jacques, carried on the work of his father and grandfather, publishing his results in 1740, and in 1744 he announced his plan for extending the work to include the making of a map of France. He was followed by a representative of the fourth generation of the remarkable Cassini family, Jacques-Dominique, who continued the surveys for the map. In 1793 the work appeared in 180 sheets, but there still remained much to be done, and it was not until June 19, 1803, that the last of the four Cassinis was able to report to the authorities that the

great work, which may be said to have begun indirectly in 1528, and directly in 1744, was at last finished. The letter announcing this fact is in my collection and reads as follows:

To General Sanson, Director of the Depôt of War:

Count Cassini, Member of the Institute, of the General Council, and of the Electoral College of the Department of the Loire:

General, I venture to flatter myself that, under the direction of a brave and loyal soldier like yourself, I can see, at the General Depôt of War, the end of the interminable affair of the Carte Générale de la France.

You have before you or your representative the latest decree of the Council of State that governs the acts of those engaged upon the map of France. I have examined carefully the report which they have made, covering the two years under the Minister of War, and I am returning it with the recommendation that a general settlement be made in accordance therewith.

This report strengthens still more the agreements between the minister and those who have acted for the Compagnie de la Carte Générale de la France, concerning the distribution of the maps to the subscribers and to the associates.

The decree places the matter hereafter in your hands. It is for the Depôt of War to cancel the various obligations, and now it becomes the real owner of the material which has been prepared under the act relating thereto. . . .

I cannot retire from the work without taking the opportunity of assuming the honor of offering to you the testimony of the distinguished consideration which you have inspired within me and which I shall retain as long as I live.

General,

Your very humble
CASSINI
Member of the Institute.

1st Messidor, an XI (June 19, 1803).

The letter bears the usual memoranda of reference for report, and of filing. The omitted portion has no special interest, referring only to details of distribution.

Thus ended a great scientific work that had extended over so many years,—the greatest work of its kind that had ever been undertaken up to that time, now more than a century ago.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

The first discussion was suggested by Problem 2863 [*1920*, 482];[1] it appears in this department because the method of attack and the incidental results give it an independent interest. Mr. T. L. Bennett discusses certain properties of roulette curves by the method of circular coördinates. He has presented the material in such form as to imply no previous acquaintance with this method; and indeed his discussion gives an adequate introduction to the study of the properties of circular coördinates.[2]

[1] See the problem department of this issue of the MONTHLY.

[2] The idea of circular coördinates seems to have originated with Plücker. Some articles dealing with the subject are: "Ueber Kreiscoordinaten" by W. Stammer, *Journal für die reine und angewandte Mathematik*, vol. 44, 1852, pp. 295–316; "On some applications of circular coördinates" by F. Franklin, *American Journal of Mathematics*, vol. 12, 1890, pp. 161–190; "Sundry metric theorems concerning n lines in a plane" by F. H. Loud, *Transactions of the American Mathematical Society*, vol. 1, 1900, pp. 323–338; and F. Morley's article, pp. 97–115, referred to below. —EDITOR.

Mr. R. S. Underwood proposes and answers, in the second discussion, an interesting question concerning trigonometric functions: when will the sine or cosine of an angle rationally expressible in degrees be rational? It may be remarked that it is well known that $\sin\theta$ and θ, when θ is expressed in radians, can never simultaneously be even algebraic, much less rational numbers, for $\theta \neq 0$, and that a similar statement holds for each of the other elementary trigonometric functions. Proofs of these theorems are somewhat intricate and are closely related to the famous proofs of the transcendentality of e and π. With the same notation, Mr. Underwood's theorem may be said to deal with the simultaneous rationality of θ and $\sin\pi\theta$. This belongs to a simpler order of ideas and may be settled by an easy and direct method of attack. It has perhaps a more obvious interest for the field of collegiate mathematics than the more difficult question mentioned above. Mr. Underwood leaves open the nature of the result for the tangent and cotangent; without doubt the facts can be ascertaine sid bymilar methods in these cases.

I. A Theorem on Hypocycloids, by the Method of Circular Coördinates.

By T. L. Bennett, University of Illinois.

Let the plane be referred to rectangular coördinates (X, Y). The circular coördinates (x, y) of the point (X, Y) are defined to be $x = X + iY$, $y = X - iY$.

A complex number $a + ib$, for which $a^2 + b^2 = 1$, is called a *turn*. A turn may be written $e^{i\theta}$, where $\theta = \cos^{-1} a = \sin^{-1} b$. It is easily shown that the conjugate of a turn is its reciprocal, and that any product of turns is a turn. All points which have turn coördinates lie on a unit circle about the origin. A turn will be denoted by some form of the letter t.

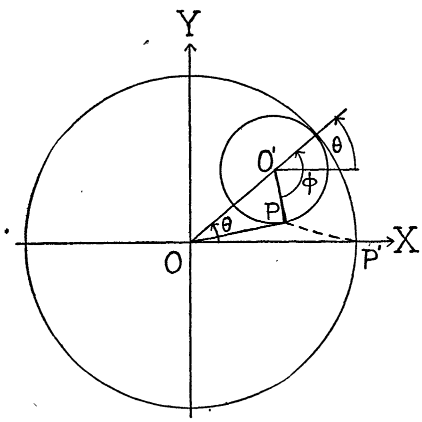
A curve may be represented by a map equation, $x = f(t)$. When t assumes turn values we get values of x, thus determining the points of the curve, because y is known as soon as x is known, being the conjugate of x. If $f(t)$ is a rational function, there is thus established a definite correspondence between the points of the unit circle and the points of the curve. If in the map equation t is replaced by any function of t which is a turn when t assumes turn values, the curve is not changed, as we have merely a different distribution of the parameter along the curve. With every map equation, $x = f(t)$, is associated another equation, namely, $\bar{x} = y = \overline{f(t)}$. If t be eliminated between these two equations, the result is the equation of the curve in circular coördinates.

The equation in circular coördinates of any straight line may be written in the form $t_1x + y = rt_1$. The coefficient t_1 is called the *clinant* of the line, and determines the direction of the line thus: the angle which the line makes with the positive real axis is found to be

$$\phi = \frac{-1}{2i}\log(-t_1).$$

If a curve is defined by the map equation $x = f(t)$, it can be shown that the clinant of the tangent at any point is $-(dy/dt)/(dx/dt)$.

Let OX and OY be the real and imaginary axes, respectively, and let the circle O', of radius r_2, roll inside the circle O, of radius r_1, with $r_1 > r_2$, and r_1/r_2 rational. Any point on the circle O', as P, will describe a hypocycloid. Let the initial position of P be P', on OX. The map equation of the locus of P is obtained as follows:



$$\text{vector } OP = \text{vector } OO' + \text{vector } O'P,$$

$$x = (r_1 - r_2)e^{i\theta} + r_2 e^{i(\theta-\phi)}.$$

Since $r_1\theta = r_2\phi$, this equation becomes

$$x = (r_1 - r_2)t + r_2 \cdot \frac{1}{t^{\frac{r_1-r_2}{r_2}}},$$

where $t = e^{i\theta}$.

As a consequence of this equation it easily follows that if $r_1/r_2 = p/q$, the cycloid so generated will be identical with the cycloid for which $r_1/r_2 = p/(p-q)$. This means that if a circle of radius $r_1 - r_2$ be rolled inside the circle O, with the initial point of contact at P', the hypocycloid so generated will be identical with that generated by P.

If $r_1/r_2 = 2/1$, we have $x = r_2[t + (1/t)]$, which is the equation of the line segment of length $4r_2$ lying along the real axis, having its center at the origin. It will be found convenient to extend this equation as follows: if a and b are any complex numbers, then from the graphical addition and multiplication of complex numbers the following facts are obvious: $x = a + b[t + (1/t)]$ is the map equation of a line segment with center at $x = a$, of length $4|b|$, inclined to the positive real axis at the angle amp b, with extremities at the points for which t equals 1 and -1.

We shall now demonstrate the following theorem: *For any odd prime p, the $\frac{1}{2}(p-1)$ distinct p-cusped hypocycloids with common vertices may be arranged in cycles, so that each is the envelope of a chord of constant length taken upon the succeeding curve of the cycle.*

Let $r_1/r_2 = p/q$. Since p is prime, and $q < p$, we may take $r_1 = p$, and $r_2 = q$. Now as p remains fixed, and q takes on all integral values from 1 to $p - 1$ inclusive, there are generated $p - 1$ cycloids. But from a preceding theorem it is seen that only half of them are distinct, and these may be generated by q taking the values from 1 to $\frac{1}{2}(p-1)$ inclusive. Hence we shall further restrict q so that $q < \frac{1}{2}p$. The cycloid for which $q = k$ will be called, for brevity, the curve $q = k$.

The general curve of the system is

$$x = (p-q)t + \frac{q}{t^{\frac{p-q}{q}}} = (p-q)T^q + \frac{q}{T^{p-q}},$$

where $t = T^q$. For any particular value of q, as k, this equation is

$$x = (p-k)T^k + \frac{k}{T^{p-k}}.$$

Consider the curve[1]

$$x = T_1^k(p-2k) + kT_1^{k-1}T + \frac{k}{T_1^{p-k-1}T}.$$

It is clear that it cuts the curve $q = k$ at the point $T = T_1$. But by computing the values of $-(dy/dt)/(dx/dt)$ for this new curve and the curve $q = k$, it is seen that the curves are tangent at the point $T = T_1$, because at this point the clinants of the two curves are the same. This new equation may be written

$$x = T_1^k(p-2k) + \frac{k}{T_1^{\frac{p}{2}-k}}\left[T_1^{\frac{p-2}{2}}T + \frac{1}{T_1^{\frac{p-2}{2}}T}\right].$$

Hence this is the equation of a line segment of length

$$4\left|\frac{k}{T_1^{\frac{p}{2}-k}}\right| = 4k,$$

with extremities given by $T_1^{(p-2)/2}T = \pm 1$, that is, at the points

$$x = T_1^k(p-2k) \pm \frac{2k}{T_1^{\frac{p}{2}-k}}.$$

These two points lie on the hypocycloid

$$x = (p-2k)T^{2k} + \frac{2k}{T^{p-2k}}, \quad [\text{for } T = \pm\sqrt{T_1}],$$

which is either the equation of the curve $q = 2k$, or the equation o: $q = p - 2k$, according as $2k$ is or is not less than $\frac{1}{2}p$.

Hence, if k be any positive integer less than $\frac{1}{2}p$, if a segment of length $4k$ be suitably taken on the tangents to the curve $q = k$, the extremities of these tangents lie either on the curve $q = 2k$ or on the curve $q = p - 2k$, as above indicated. It is then clear that the successive values of q for the sequence of curves of the theorem must be obtained by successive doubling,[2] by then reducing mod p to values between $-\frac{1}{2}p$ and $\frac{1}{2}p$, and then taking the absolute value of the result.

That the curves may be arranged in cycles follows from Fermat's Theorem: $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$; whence $k2^{(p-1)/2} \equiv \pm k \pmod{p}$. There will be one or more cycles according as $\frac{1}{2}(p-1)$ is or is not the smallest integer which will satisfy the congruence $k2^x \equiv \pm k \pmod{p}$. For example, if $p = 13$, the values of q form the single cycle 4 5 3 6 1 2, while for $p = 17$ we get the two cycles 3 6 5 7 and 8 1 2 4.

The proof for $p = 5$ is included in the foregoing as a special case.

[1] Professor Morley has called such curves "penosculants" of the original curve. See his paper "On the metric geometry of the plane n-line," *Trans. of the Amer. Math. Soc.*, vol. 1, p. 102.

[2] In the general equation of the hypocycloid it was assumed that $r_1 > r_2$. It may be readily shown that if $r_1 < r_2$, this equation represents an epicycloid, which is identical with the epicycloid generated by rolling a circle of radius $r_2 - r_1$ on the outside of the circle O, with the initial point of tangency at P'. Therefore the above analysis shows that if we start with any positive integer for q, (prime to p), then by successive doubling of q, without reducing mod p, we obtain an infinite sequence of epicycloids having the property mentioned in the theorem.

II. On the Irrationality of Certain Trigonometric Functions.

By R. S. Underwood, Purdue University.

Theorem: *If an angle, expressed in degrees, is rational and not a multiple of 30°, its sine, cosine, secant, and cosecant are irrational.*

By De Moivre's Theorem,

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n. \tag{1}$$

Equating $\cos n\theta$ to the real part of the binomial expansion, we get

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{2!} \cos^{n-2} \theta \sin^2 \theta + \frac{n(n-1)(n-2)(n-3)}{4!} \cos^{n-4} \theta \sin^4 \theta - \cdots. \tag{2}$$

(The letters m and n stand for integers throughout this discussion. For convenience we shall call an angle integral, rational, or irrational in accordance with the character of its value expressed in degrees.)

From (2) it appears that $\cos n\theta$ can always be expressed in terms of integral powers of $\cos \theta$, and that $\cos \theta$ is irrational if $\cos n\theta$ is irrational. Since $\cos (45° \pm m180°)$ is irrational, $\cos \dfrac{m180° \pm 45°}{45}$, or $\cos (4m \pm 1)°$ is irrational. Every odd number can be expressed in the form $(4m \pm 1)$, and therefore the cosine of every odd integral angle is irrational.

Testing the equation

$$\sin 30° = \sin 10°(3 - 4 \sin^2 10°) = 1/2$$

for rational roots, we find that $\sin 10°$ is irrational. Since $\cos 30°$ and hence $\cos 10°$ is irrational, it follows that the cosines of $(100° \pm m180°)$, $(80° \pm m180°)$, and $(40° \pm m180°)$, and hence of $(9m \pm 5)°$, $(9m \pm 4)°$, and $(9m \pm 2)°$, are irrational. The last angle includes the form $(9(2s) \pm 2)°$; therefore, $\cos (9m \pm 1)°$ is irrational. The only even numbers which cannot be expressed in these forms are multiples of 6. By a geometric method we can show that $\sin 18° = (\sqrt{5} - 1)/4$ and $\cos 18° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$ are irrational, and hence this is true also of the cosines of 108°, 72°, 54°, 36°, 24°, 12° and 6°. Then, since $\cos 12°$ is irrational, $\sin 6°$ (or $\sqrt{[1 - \cos 12°]/2}$) is irrational; and we can add the cosines of 96°, 84°, 48°, 42°, $66° = (180° - 48°)/2$, and $78° = (180° - 24°)/2$ to the list. Thus, in general, when n is not a multiple of 30°, $\cos n°$, and obviously $\sin n°$ as well, is irrational.

Furthermore, $\cos (m°/n)$ when m/n is a rational non-integral fraction reduced to its lowest terms and m is not a multiple of 30°, is evidently irrational since $\cos m°$ is irrational. For every such angle $m°/n$ there is a complementary angle $h°/n$ $(h \neq k30)$ whose sine is irrational, and vice versa; hence both the sine and the cosine of the general rational angle $m°/n$ $(m \neq k30)$ are irrational.

To investigate an angle of the form m/n when m is a multiple of 30 and prime

to n (which is therefore odd), we may use the following expansion of $\sin n\theta$, valid when n is odd:[1]

$$\sin n\theta = n \sin\theta - \frac{n(n^2-1^2)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta \qquad (3)$$
$$- \cdots (-1)^{(n-1)/2}2^{n-1}\sin^n\theta.$$

The coefficients in this expansion are integers, as may be seen by equating the imaginary parts in the expansion of (1).

The coefficient of the $(r+1)$th term in (3) contains the factor 2^{2r}. For, upon writing this coefficient,

$$\frac{(n-2r+1)\cdots(n-3)(n-1)n(n+1)(n+3)\cdots(n+2r-1)}{(2r+1)!},$$

we see that in the $2r$ successive even numbers contained in the numerator, there will be

$2r$ numbers containing the factor 2,
r numbers containing the factor 2^2,
at least $[r/2]$ numbers containing the factor 2^3,
at least $[r/2^2]$ numbers containing the factor 2^4,
. . . .
at least $[r/2^\mu]$ numbers containing the factor $2^{\mu+2}$.

By $[r/2]$ is meant the largest integer in $r/2$. Choose μ so that $[r/2^\mu] = 1$. The sum of the exponents of 2 in the numerator will be *at least*

$$2r + r + \left[\frac{r}{2}\right] + \left[\frac{r}{2^2}\right] + \cdots + \left[\frac{r}{2^\mu}\right].$$

In $(2r+1)!$ the sum of the exponents of 2 is *exactly*

$$r + \left[\frac{r}{2}\right] + \left[\frac{r}{2^2}\right] + \cdots + \left[\frac{r}{2^\mu}\right].$$

Hence the excess of exponents of 2 in the numerator will be at least $2r$, and the coefficient will contain the factor 2^{2r}.

Letting $2r+1=n$, we see that the exponent of 2 in the coefficient of $\sin^n\theta$ is no greater than the required $2r$. Then we can write

$$2^{n-1}\sin^n\theta - 2^{n-3}k_1\sin^{n-2}\theta + 2^{n-5}k_2\sin^{n-4}\theta - \cdots \pm n\sin\theta = \pm\sin n\theta.$$

If we put

$$x = 2\sin\theta,$$

then

$$x^n - k_1x^{n-2} + k_2x^{n-4} - \cdots \pm nx = \pm 2\sin n\theta.$$

All the coefficients in this equation except the absolute term on the right are integers. Let $\theta = m^\circ/n$, where m is an odd multiple of 30 or a multiple of 180, and prime to n. Then $2\sin n\theta = 2\sin m^\circ = 0, \pm 1, \pm 2$, and the absolute term is also an integer. Therefore the rational roots are integers, and $\sin\theta$ is half of an integer. Then $\sin\theta = 0, \pm\frac{1}{2}$, or ± 1, which is impossible.

[1] Loney: *Analytical Trigonometry*, p. 69.

This disposes of every case except when m is a multiple of 60 but not of 180. In this case $\sin n\theta = \pm \frac{1}{2} \sqrt{3}$, and by (3), $\sin \theta$ is irrational.

It has now been shown that the sine of every rational angle not a multiple of 30° is irrational. Obviously this establishes the theorem for the cosine, secant, and cosecant functions as well.

An analogous theorem, differing only in the substitution of 45° for 30°, may probably be shown to hold true for the tangent and cotangent.

RECENT PUBLICATIONS

REVIEWS

American Men of Science.

American Men of Science. A Biographical Directory. Edited by J. McK. Cattell and D. R. Brimhall. Third edition. Garrison, N. Y., The Science Press, 1921. 4to. 8 + 808 pp. Price $10.00.

The first edition of this work, published early in 1906, contained brief sketches of about 4,000 living American[1] men and women of science, the second edition, published late in 1910, about 5,500; the present edition contains some 9,500 sketches. The increase in the number of sketches measures roughly the increase of scientific workers.

In the first edition a star was prefixed to the subject of research in the case of one thousand of the sketches of students of the natural and exact sciences in the United States. In each of the twelve principal sciences[2] the names were arranged in the order of merit by ten leading students of the science. In this way the subjects of research of 80 mathematicians and 50 astronomers were starred.

In the second edition, the thousand leading men of science were determined in the same manner as in the first edition, stars being added to the subjects of research in the case of 269 new men; of these 20 were mathematicians. Their names, the names of the original 80 mathematicians, as well as of the 29 new mathematicians whose subjects of research are starred in the present edition, are given below.

It will be observed that in all editions the total number of sketches, in connection with which mathematics is starred, is 129. Of these, 11 refer to those who have died. There remain 118 living mathematicians with a star in the present edition, since a star once given in a sketch is not removed in subsequent editions. Hence 38 of these mathematicians are now no longer regarded as among the 80

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[1] This term is interpreted as applying not only to natives of the United States and of the Dominion of Canada, but also to foreigners temporarily resident in these countries; for example: O. Bolza and P. Boutroux.

[2] Mathematics, physics, chemistry, astronomy, geology, botany, zoölogy, physiology, anatomy, pathology, anthropology, and psychology.

(1, 2, 3), F. S. Woods (1, 2, 3), J. W. A. Young (1, 2, 3) [Math., Pedagogy of math.], J. W. Young (2, 3), A. Ziwet (1, 2, 3) [Math., Mech.].

Fourteen names in this list are followed by brackets [ ] enclosing the names of different fields of work, but mathematics coming first. The following 10 names are of others in connection with whom mathematics does not come first or else the subject is mathematical astronomy or mathematical physics:

J. G. Coffin (2, 3) [Mathematical physics], C. L. Doolittle (1, 2; d. 1919) [Ast., Math.], J. R. Eastman (1, 2; d. 1913) [Ast., Math.], G. W. Hill (1, 2; d. 1914) [Mathematical astronomy], F. H. Loud (1, 2, 3) [Ast., Math.], A. Macfarlane (1, 2; d. 1913) [Mathematical physics], R. C. Maclaurin (2; d. 1920) [Mathematical physics], B. O. Peirce (1, 2; d. 1914) [Mathematical physics], F. Slate (1, 2, 3) [Mathematical physics], J. B. Webb (1, 2; d. 19 ?) [Physics, Math.].

On pages 771–780 is given a very useful list of 1059 American men of science who died between January 1, 1903, and December 31, 1920. The years of birth and death are appended in each case where it was possible to determine such dates. The list includes not only the names of those whose names appeared in earlier editions, but also of others such as of G. M. Green. Certain events of very recent occurrence are recorded in the volume; for example, the death of A. Pell, January 26, 1921, and the appointment of J. R. Angell as president of Yale. It is not clear to the reviewer why a sketch of W. G. Everett, whose subject is ethics, should appear in the third edition.

Everyone interested in American science will wish to have this new *Biographical Directory* constantly at hand.

R. C. ARCHIBALD.

NOTES.

School Arithmetics is the title of a work, for grades I–VIII, by G. WENTWORTH and D. E. SMITH. (3 books, Boston, Ginn, 1920. 12mo. 6 + 282 + 16; 6 + 298 + 19; 6 + 346 + 19 pp. Price 72 + 76 + 92 cents.)

Professor J. H. M. WEDDERBURN'S article "On equations of motion of a single particle," published separately January 24, 1921, appeared in part 1 (May), volume 41 (pp. 26–33), of *Proceedings of the Royal Society of Edinburgh.*

In *Popular Astronomy*, June–July, 1921, there is a reproduction of a photograph, taken on May 6, 1921, of a group with Professor Einstein at the Yerkes Observatory. Professor A. C. LUNN, of the University of Chicago, is a member of the group.

Comptes Rendus du Congrès International des Mathématiciens à Strasbourg, was published November 1, 1921, by Imprimerie Edouard Privat, Toulouse (paper, 100 francs; cloth, 125 francs).

In *Proceedings of the London Mathematical Society*, second series, volume 20, August 20, 1921, there is an article "Arithmetic of quaternions" by L. E. DICKSON. Compare *1921*, 289.

The last number of *Mathematische Annalen*, volume 80, has been published (see *1921*, 219). It is a General register to volumes 51–80 and has a preface by H. VERMEIL. There is a portrait frontispiece of C. Neumann.

The third edition of W. F. Osgood's *Lehrbuch der Funktionentheorie*, volume 1, published by Teubner in 1920, is an anastatic reprint, the only textual changes of the second edition being in connection with the correction of a few misprints. The paper and general appearance are inferior and the volume is only about half as thick.

In the recently published *Washington University Studies*, volume 8, no. 2, the following papers are included: "The curve which with its caustic encloses the minimum area" (cf. this MONTHLY, *1920*, 225) by OTTO DUNKEL, 183–194; and "Brilliant point phenomena" (cf. this MONTHLY, *1913*, 299; *1919*, 111) by W. H. ROEVER, 131–159 + 4 plates.

Abhandlungen aus dem mathematischen Seminar der Hamburger Universität is the title of one of the nine new mathematical periodicals founded in 1921. The editors are W. BLASCHKE, E. HECKE and J. RADON. The first number (98 pages, price 25 marks) was published in September (Hamburg, Otto Meissner). It contains six memoirs by the following authors: W. Blaschke, E. Hecke, A. Ostrowski, J. Radon, K. Reidemeister, and P. Steinhagen.

We have previously noticed the *Tables of the Digamma and Trigamma Functions* by Miss ELEANOR PAIRMAN (*1921*, 265–266). During the summer of 1921 she completed the work for a doctorate at Harvard University, her thesis being entitled: "Expansion theorems for solutions of a Fredholm linear homogeneous integral equation of the second kind with kernel of special non-symmetric type."

A volume supplementary to *Œuvres de Fermat*, tomes I–IV, is being prepared —Heath, of New York, announces as in the press *Projective Geometry* by R. M. WINGER, of the University of Washington, and *Trigonometry, Plane and Spherical* by J. A. BULLARD and ARTHUR KIERNAN, of the United States Naval Academy—Wiley, of New York, published, in December, 1921, *First Course in the Theory of Equations* by L. E. DICKSON (171 pages; $1.75)—Macmillan of New York has published *Mathematics for Students of Agriculture* by S. E. RASOR.

The number of *L'Enseignement Mathématique*, published in July, 1921, contains a complete list of the publications of the International Commission on the Teaching of Mathematics. This list gives references to 310 reports published in 187 fascicules or volumes, and totalling 13,565 pages. To this total Germany contributed 5,571 pages, the United States coming next with 1,499 pages, then Great Britain with 921, Japan with 788, Switzerland with 781, Austria with 776, and similarly for twelve other countries down to Argentina with 24 pages and Roumania with 16. Among other indexes, an alphabetical list of authors is appended to the list.

In notes on perfect numbers, we have indicated (*1921*, 140–141) that the Mersenne numbers, $p = 2^n - 1$, are prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107$, and 127; that it was not known whether p was prime or composite for $n = 137, 139, 149, 157, 167, 193, 199, 227, 229, 241$, and 257; and that for all other values of $n < 257$, p was composite. A. Gérardin has indicated a method (*Comptes rendus du Congrès des Sociétés Savantes en 1920*, Sciences, pp. 53–55) showing that p is composite for $n = 137$. Hence there are now only 10 doubtful cases, $n < 257$, in connection with the Mersenne numbers.

Among parts of *Encyklopädie der mathematischen Wissenschaften* published in 1921 are the following: (*a*) II, 3, 4—"Neuere Untersuchungen über Funktionen von komplexen Variabeln" by L. Bieberbach, 379–532 (22 marks); (*b*) III, 1, 6—"Elementargeometrie und elementare nichteuklidische Geometrie in synthetischer Behandlung" (Teil 2, Schluss) by M. Zacharias, 963–1172 (23.75 marks); (*c*) III, 1, 7—"Neuere Dreiecksgeometrie" by G. Berkhan and F. Meyer, and "Systeme geometrischer Analyse" by H. Rothe, 1173–1423 (30.80 marks); (*d*) III, 2, 7—"Mehrdimensionale Räume" by G. Segre, 769–972 (22 marks); (*e*) III, 3, 5—"Dreifach orthogonale Flächensysteme" by E. Salkowski, 541–606 (9.70 marks); (*f*) II, 2, 5—"Nichtlineare Differentialgleichungen" by E. Hilb, and "Abelsche Funktionen und allgemeine Thetafunktionen" by W. Wirtinger, and Titel, Inhaltsverzeichnis und Register zu Band II, 2, 563–897 (49.30 marks); (*g*) II, 3, 5—"Arithmetische Theorie der algebraischen Funktionen" by K. Hensel and "Arithmetische Theorie der algebraischen Funktionen zweier unabhängigen Veränderlichen" by H. W. E. Jung, 533–674; (*h*) V, 1, 6—"Physikalische- und Elektrochemie" by K. F. Herzfeld and Titel und Inhaltsverzeichnis zu Band V, 1, 947–1112 + 20 (30 marks); (*i*) V, 2, 4—"Relativitätstheorie" by W. Pauli, Jr., 539–775 (36 marks).

The first number (80 pages, sm. 4to) of a new periodical, *Zeitschrift für angewandte Mathematik und Mechanik,* was published in February, 1921, by the Verlag des Vereines Deutscher Ingenieure, Berlin (Price, in Germany, 50 marks; to members of the Deutsche Mathematiker-Vereinigung in America, 72 marks a year for the six numbers). The periodical is edited by R. von Mises with the assistance of A. Föppl, G. Hamel, R. Mollier, H. Müller, L. Prandtl, and R. Rüdenberg. The first number opens with an article by the editor: "Zur Einführung, Ueber die Aufgaben und Ziele der angewandten Mathematik" (pp. 1–15). This is followed by a variety of material under the headings: Principal papers, Comprehensive reports, Short abstracts, Book-reviews, Short notices, and News.

Journal of Mathematics and Physics of the Massachusetts Institute of Technology is the title of a new periodical, the first number of which was published in November, 1921 (The Murray Printing Co., Cambridge, Mass.). The editors are F. S. Woods (professor of mathematics), H. M. Goodwin (professor of physics and electro-chemistry), and F. G. Keyes (professor of physico-chemical research); the managing editor is C. L. E. Moore (professor of mathematics). A volume

of 200–250 pages (price $3.00) consisting of three or four numbers will probably be published each year. The contents of the current number are: "A new vector method in integral equations" by F. L. Hitchcock and Norbert Wiener, 1–20; "On the geometry of motion in curved n-space" by Joseph Lipka, 21–41; "The equation of state, with applications to viscosity" by H. B. Phillips, 42–53; "Some hydrodynamic aspects of group theory" by S. D. Zeldin, 54–62.

Among the articles in the concluding number of *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1920, are the following: "Zur Biographie Johann Bolyais" by S. Wiener, 130–135 [new material concerning one of the founders of non-euclidean geometry while he was at the academy for engineers, and in military service]; "Die relative und absolute Bewegung bei Huygens" by J. A. Schouten, 136–144 [interesting for students of the history of relativity]; "Jan Versluys" by L. Schlesinger, 236–237 [Netherland mathematician, born 1845, died January 15, 1920. Among his publications listed by Schlesinger are five prize memoirs, crowned by the Mathematical Society of Amsterdam, and numerous books, some of which were widely used in schools of the Netherlands. Among his books are: *Leerboek der Stereometrie, Handboek der Meetkunde, Methoden bij het oplossen van Meetkundige Vraagstukken, Isometrie en Axonometrie, Inleiding tot de nieuwere Meetkunde van den driehoek*, and *Zes en negentij bewijzen voor het theorema von Pythagoras*. There are many books by Versluys not listed by Schlesinger].

In the new volume of the great *Encyclopædia of Religion and Ethics*, edited by James Hastings (volume XI, Sacrifice–Sudra, Edinburgh, T. & T. Black, 1921), there are articles (1) on "Science" by J. A. Thomson, regius professor of natural history at the University of Aberdeen (pages 252–261); and (2) on "Space" by G. J. Stokes, professor of philosophy and jurisprudence in University College, Cork, Ireland (pages 759–764). The sub-headings for (1) are: Definition and characteristics, aim of science, the scientific mood, the methods of science, scope of science, classification of sciences, correlation of sciences, limitations of science, science and feeling, science and philosophy, and science and religion. For (2) the sub-headings are: History, (*a*) Greek philosophy, (*b*) middle ages, (*c*) Cartesian and English philosophy, (*d*) Kant; Laws of thought and the concept of space. The article concludes with an acknowledged extract of three paragraphs from Professor Stokes's article, "The theory of mathematical inference," contributed to this MONTHLY, *1900*, 1–8.

While the last part of *Acta Mathematica*, volume 42, appeared before the close of 1920, no part of volumes 38 and 39 had then been distributed. The first of these volumes, in memory of Henri Poincaré, has now been published. It is a volume of 402 pages (price 50 Swedish kronor; on extra fine, thick, strong paper, 75 kronor). The contents are as follows: "Au lecteur," pages 1–2; "Henri Poincaré, Analyse de ses travaux scientifiques," 3–135 [revised by the author];

"Rapport sur les travaux de M. Cartan" by H. Poincaré, 137–145; "Lettres à M. Mittag-Leffler" by H. Poincaré, 147–160; "Lettres à M. Mittag-Leffler concernant le mémoire couronné du prix de S. M. le roi Oscar II" by H. Poincaré, 161–173; "Lettres à L. Fuchs" by H. Poincaré, 175–184; "Briefe an H. Poincaré" by L. Fuchs, 185–187; "Henri Poincaré, en mathématiques spéciales à Nancy" by P. Appell, 189–195; "Lettres à M. Mittag-Leffler" by P. Boutroux, 197–201; "L'œuvre mathématique de Poincaré" by J. Hadamard, 203–287; "Die Bedeutung Henri Poincaré's für die Physik" by W. Wien, 289–291; "Deux mémoires de Henri Poincaré sur la physique mathématique" by H. A. Lorenz, 293–308; "L'œuvre astronomique d'Henri Poincaré" by H. v. Zeipel, 309–385; "Henri Poincaré und die Quanten-theorie," 387–397; "Henri Poincaré" by P. Painlevé, 399–402.

"Anaximander's Book, the Earliest Known Geographical Treatise" is the title of an interesting fifty-page monograph, by W. A. HEIDEL, professor of Greek in Wesleyan University, published in *Proceedings of the American Academy of Arts and Sciences,* April, 1921. The mathematician will naturally wish to read collaterally what Sir Thomas Heath records concerning Anaximander in numerous passages of his *Aristarchus of Samos* (Oxford, 1913). From this latter source the following paragraphs, stripped of foot-note references, are extracted (pages 38–39):

"The story that Anaximander was the first to discover the *gnomon* (or sun-dial with a vertical needle) is incorrect, for Herodotos says that the Greeks learnt the use of the *gnomon* and the *polos* from the Babylonians. Anaximander may, however, have been the first to 'introduce' or make known the *gnomon* in Greece, and to show on it 'the solstices, the times, the seasons, and the equinox.' He is said to have set it up in Sparta. He is also credited with constructing a sphere to represent the heavens, as was Thales before him.

"But Anaximander has yet another claim to undying fame. He was the first who ventured to draw a map of the inhabited earth. The Egyptians had drawn maps before, but only of particular districts; Anaximander boldly planned out the whole world with 'the circumference of the earth and of the sea.' Hecataeus, a much-travelled man, is said to have corrected Anaximander's map, so that it became the object of general admiration. According to another account, Hecataeus left a written description of the world based on the map. In the preparation of the map Anaximander would of course take account of all the information which reached his Ionian home as the result of the many journeys by land and sea undertaken from that starting-point, journeys which extended to the limits of the then-known world; the work involved of course an attempt to estimate the dimensions of the earth. We have, however, no information as to his results."

Periodical Bibliographies and Abstracts for the Scientific and Technological Journals of the World is the title of a *Bulletin* (pages 131–154) published by the National Research Council in June, 1920. If ignorance of the subject in question is as glaringly in evidence in other topics as it is in connection with mathematics (page 145), the Council had been better advised to have withheld the manuscript of its *Bulletin* until it had been properly prepared—*Funds Available in 1920 in the United States for the Encouragement of Scientific Research* is the subject of an interesting 81-page *Bulletin* of the Research Council published in March, 1921. The grant of $25,000 from the General Education Board to the Mathematical Association of America is noted. Under the heading of "Mathematics" reference

is made also to the Sylvester prize of Johns Hopkins University, to the J. S. K. mathematical fellowship of Princeton University, and to the resident research fellowship in mathematics of Bryn Mawr College. Under the heading of "Science" there is a long list of medals and prizes, grants, institutional funds, and fellowships and scholarships, which are unrestricted in award. We have already referred to the Heckscher Foundation here listed, and to awards made from it for research in mathematics (*1921*, 287–288).

Among books published in 1921 are the following: *Theoretical Mechanics: An Introductory Treatise on the Principles of Dynamics* by A. E. H. Love (Third edition, Cambridge University Press; 15 + 310 pages; price 30 shillings) —*Elements of the Mathematical Theory of Electricity and Magnetism* by J. J. Thomson (Fifth edition, Cambridge University Press; 410 pages; price 25 shillings)—*A Treatise on the Integral Calculus, with Applications, Examples, and Problems* by J. Edwards (London, Macmillan; volume 1, 907 pages; price 50 shillings)—*The Reign of Relativity* by Viscount Haldane (London, Murray; 23 + 430 pages; price 21 shillings)—*A First Course in Statistics* by D. C. Jones (London, Bell; 9 + 286 pages; price 15 shillings)—*Des fondements de la géométrie* by H. Poincaré (Bibliothèque de synthèse scientifique. Paris, E. Chiron; 65 pages; price 3 francs)—*Theorie der reellen Funktionen* by H. Hahn (Berlin, Springer; volume 1, 4 + 600 pages; price 136 marks to Germans)—*Vorlesungen über Zahlen- und Funktionenlehre* by A. Pringsheim. Band I: *Zahlenlehre* (Leipzig, Teubner, 3. [Schluss-] Abteilung[1]—*Komplexe Zahlen, Rechnen mit komplexen Gliedern;* price 175.00 marks to Germans)—*Vorlesungen über algebraische Geometrie* by F. Severi. Deutsch von E. Löffler (Leipzig, Teubner; 408 pages; price, unbound, 92.40 marks). Mathematisch-physikalische Bibliothek (Leipzig, Teubner, price of each, boards, 5.30 marks): no. 40, P. Kirschberger, *Mathematische Streifzüge durch die Geschichte der Astronomie* (54 pages); no. 42, A. Witting, *Einführung in die Infinitesimalrechnung* 2. Auflage, II: *Die Integralrechnung* (50 pages)—*Gruppentheorie* by L. Baumgartner (Sammlung Göschen, Berlin, Vereinigung wissenschaftlicher Verleger; 120 pages; price 5.00 marks to Germans). *Geschichte der Elementar-Mathematik* by J. Tropfke, 2. vermehrte Auflage, Band 2: Die allgemeine Arithmetik (Berlin, V.W.V., 1921; 4 + 221 pages; price 50 marks to Germans).

The Institute of International Education was established in February, 1919, by the Carnegie Endowment for International Peace "to develop international good will by means of educational agencies, and for its specific purpose to act as a clearing house of information and advice for Americans concerning things educational in foreign countries and for foreigners concerning things educational in the United States." The administrative board selected to determine and guide the policy of the Institute consisted of "representatives of the endowed and the state universities, of the men's and women's colleges, and of international scholarship, law, finance, commerce, medicine, and journalism."

[1] Abteilung 1—*Reelle Zahlen und Zahlenfolgen* (1916, 12 + 292 pages; price 33.50 marks), Abteilung 2—*Unendliche Reihen mit reellen Gliedern* (1916, 8 + 22 pages; price 31.00 marks).

One of the results of the Institute's organization has been the arrangement for visits to this country of many foreign scholars and professors, and for our professors to lecture in foreign universities. Since the Institute pays travelling expenses both ways the arrangement may obviously be a happy one for those on sabbatic leave. In 1920–21 grants were made to 17 American professors to lecture in 9 countries: Argentina, Bohemia, China, England, France, Greece, Italy, Spain, and Turkey. The only mathematician was Professor SOLOMON LEFSCHETZ, of the University of Kansas, who lectured at the University of Rome. The Institute's *Annual Report*, of February 15, 1921, states that Doctor E. A. HORNE, professor of mathematics at the University of Patna, Patna, India, has been invited to this country by Harvard University.

Some publications of the Institute will be found invaluable for American students in certain foreign countries. We have already noticed the notable monograph of Kenneth McKenzie on *Opportunities for Higher Education in Italy* (61 pages; see *1919*, 300–301). There are also: *Opportunities for Higher Education in France* (148 pages), and G. E. MacLean's *Opportunities for Graduate Study in the British Isles* (40 pages). The *Guide Book for Foreign Students in the United States* (published July 1, 1921; 100 pages) will also be of interest to college teachers. Another similar bulletin, just published by the Bureau of Education, Washington (Bulletin, 1921, no. 6), is entitled: *Opportunities for Study at American Graduate Schools* (59 pages).

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 43, no. 2, April, 1921: "Boundary value and expansion problems: algebraic basis of the theory" by R. D. Carmichael, 69–101; "Algebraic theory of the expressibility of cubic forms as determinants, with application to Diophantine analysis" by L. E. Dickson, 102–125; "The impossibility of Einstein fields immersed in flat space of five dimensions" by E. Kasner, 126–129; "Finite representation of the solar gravitational field in flat space of six dimensions" by E. Kasner, 130–133; "On the motion of two spheroids in an infinite liquid along their common axis of revolution" by B. Datta, 134–142.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, May, 1921: "The February meeting of the American Mathematical Society" by R. G. D. Richardson, 341–351; "Note on equal continuity" by J. F. Ritt, 351–353; "A new method in Diophantine analysis" by L. E. Dickson, 353–365; "The transformation of elliptic integrals" by J. H. McDonald, 366–373; "Bachmann on Fermat's last theorem" by H. S. Vandiver, 373–376 [Review of *Das Fermatproblem in seiner bisherigen Entwickelung* (Berlin and Leipzig, 1919)]; "Two books about airplanes" by E. W. Brown, 377–381 [Review of E. B. Wilson's *Aeronautics: A class text* (New York) and of H. G. Bader's *Grundlagen der Flugtechnik: Entwerfen und Berechnung von Flugzeugen* (Berlin)]; Review by F. M. Morgan of E. Beutel's *Die Quadratur des Kreises* (Leipzig and Berlin, 1920), 382; Notes, 383–386; New Publications, 386–388.

MATHEMATICAL GAZETTE, volume 10, March, 1921: "Address on relativity" by A. S. Eddington, 228–233; "The teaching of mathematics to boys whose chief interests are non-mathematical" by S. H. Clarke, 234–238; "The early history of the Association, or the passing of Euclid from our schools and universities and how it came about" by J. M. Wilson, 239–244; "Further reminiscences" by A. A. Bourne and F. S. Marshall, 244–247; "Mathematics in the lycées" by E. M. Read, 248–254—May, 1921: "Aeroplane mathematics" by S. Brodetsky, 257–281; "Gleanings far and near," 281; Reviews, 282–288 [review of T. Muir's *The Theory of Determinants in the Historical Order of Development*, vol. 3; etc.].

MATHEMATICS TEACHER, volume 14, no. 2, February, 1921: "Outstanding pedagogical principles now functioning in high-school mathematics" by G. W. Myers, 57–63; "The geometry of the junior high school" by J. C. Brown, 64–70; "Algebraic magic squares" by H. P. McLaughlin, 71–77; "The outlook with regard to school mathematics" by W. P. Webber, 78–84; "Mathe-

matics in Stuyvesant High School" by W. E. Breckenridge, 85–87; "Articulation of junior and senior high school mathematics" by J. K. Van Denberg, 88–94; News and Notes, 95–101; Discussion, 102–103; New Books, 104–105—No. 3, March: "Terms and symbols in elementary mathematics," 107–118 [A preliminary report by the National Committee on Mathematical Requirements]; "The recitation in mathematics" by J. H. Minnick, 119–123; "Certain mathematical ideals of the junior high schools" by D. E. Smith, 124–127; "Geometry detected by Sherlock Holmes" by B. B. Hedges, 128–136; "Remarks on the report of the National Committee on Mathematical Requirements on College Entrance Requirements" by E. R. Hedrick and H. D. Gaylord, 137–142; "The problem of home work papers" by J. R. Overman, 143–146; "Teaching incommensurables" by Vera Sanford, 147–150; Round Table Discussion, 151–155; News and Notes, 156–158; Book Reviews, 159–160.

Messenger of Mathematics, volume 50, no. 5, September, 1920: "On a Diophantine problem" (third paper, continued) by H. Holden, 65–75; "Circular parts: the general case" by W. W. Johnson, 76–80—Nos. 6–7, October–November: "Factorization of N, treated as a bi-composite, special regard being paid to the sum of its digits and to the consequent possible sums of the digits of its twin factors, after casting out the nines" by D. Biddle, 81–95; "A differential equation occurring in the theory of the propagation of waves" by H. Bateman, 95–100; "Summation of q-hypergeometric series" by F. H. Jackson, 101–112—No. 8, December: "On the generating function of the series $\Sigma F(n)q^n$, where $F(n)$ is the number of uneven classes of binary quadratics of determinant $-n$" by L. J. Mordell, 113–128.

Nature, volume 107, March 31, 1921: "Electrical theory and relativity" by A. R. [review of J. H. Jeans's *The Mathematical Theory of Electricity and Magnetism* (Cambridge, 1920)], 133–134; "Mathematical text-books" by H. B. H. [review of C. Davison's *The Elements of Plane Geometry* (Cambridge, 1920), of W. G. Dunkley's *A Primer of Trigonometry for Engineers* (London, 1920), of S. B. Gates's *Pure Mathematics for Engineers* (2 vols., London, 1920), of P. J. Haler and A. H. Stuart's *A Second Course in Mathematics for Technical Students* (London, 1920), of W. P. Webber's *Elementary Applied Mathematics* (New York and London, 1920), of S. H. Stelfox's *The Laws of Mechanics* (London, 1920) and of J. W. Landon's *Elementary Dynamics* (Cambridge, 1920)], 134–136—April 28: Review of R. D. Carmichael's *The Theory of Relativity* (second edition, New York, 1920), 264; "The concept of 'space' in physics" by H. Jeffries, 267–268—May 5: "Logs and antilogs" a letter by R. T. A. I., 300–301 ["On p. 7 of *Nature* of March 3 a recommendation is mentioned that when taking out the number corresponding to a logarithm a table of antilogs should be used. Assuming the usual seven-figure work, the opposite course should be followed, because the computer can then write down five figures at once and add the remaining two by means of the difference table; no addition or crossing out is required. Thus for the logarithm 0.1234567 the log table gives 1.3287 for 1234269, and 298 in the 327 difference table gives 91, so we write 1328791. *Vice versa*, having 1.328791, what is the logarithm? The anti-table gives 12345 at once, whilst the difference 20 gives 67, so that we write 1234567. No figure requires alteration and the work is done with a minimum of mental strain.

"As one who does a great deal of computation, let me state that my order of preference for usual work is Cotsworth's multiplication table (which is better than Crelle's), then the Triumphator or Brunsviga calculating machine, then Shortrede's table, which in one volume gives both logs and antilogs; but special tables can also be usefully employed. Thus Bottomley for all four-figure work is still the best; for multiplying two figures by four, Peters's table; and for two figures by three, Zimmermann's.

"Amongst the indispensable tables should be included Zech's addition and subtraction log table, which is easy to use and accurate. For eight-figure work the best, if not the only, tables are Bauschinger's and Peters's"].

Philosophical Magazine, sixth series, volume 41, April, 1921: "On the supposed weight and ultimate fate of radiation" by O. Lodge, 549–557; "On a method of analysis suitable for the differential equations of mathematical physics" by W. L. Cawley and H. Levy, 584–607; "Motion and hyperdimensions" by F. Tavani, 647–651—May: "The physical significance of the least common multiple" by N. Campbell and E. C. C. Baly, 707–716.

Proceedings of the American Academy of Arts and Sciences, volume 56, no. 1, February, 1921: "Acoustic impedance and its measurement" by A. E. Kennelly and K. Kurokawa, 1–42 [Bibliography, 20 titles, pp. 38–39; list of symbols employed, pp. 40–42]—No. 7, April: "Anaximander's book, the earliest known geographical treatise" by W. A. Heidel, 239–288.

Revue Scientifique, volume 59, March 12, 1921: "Les progrès de l'astronomie physique" by H. Deslandres, 97–101 [First paragraph: "L'extension considérable, prodigieuse, des recherches

scientifiques est un des caractères de notre époque. Les hommes de valeur et les établissements spéciaux qui leur sont consacrés sont en nombre toujours croissant, et cette belle progression est surtout frappante en Amérique. Nous avons vu, dans les cinquante dernières années, les découvertes succéder aux découvertes, et dans toutes les sciences. La physique a été favorisée de façon toute particulière; son domaine, déjà de belle étendue, s'est agrandi de terres nouvelles à la fois très riches et très vastes; les phénomènes de radiation et les phénomènes électriques y ont une place prépondérante"].

School Science and Mathematics, volume 21, no. 5, May, 1921: "Teaching formulæ in the junior high school" by J. A. Nyberg, 409–417; "The Mathematical Association of America" by G. A. Miller, 418–422; "Diophantine analysis applied to the constructibility of regular polygons" by M. O. Tripp, 422–424; "Some observations concerning the history of science" by E. H. Johnson, 450–453; Problems and solutions, 483–488.

Science, new series, volume 53, April 29, 1921: "Euclid of Alexandria and the bust of Euclid of Megara" by F. Cajori, 414–415 [States that the portrait bust on a certain old Greek coin which is often published as that of Euclid the mathematician, really represents Euclid of Megara, the philosopher, who was formerly often confounded with his greater namesake]—May 27: "A section of the American Association on the History of Science" by L. C. Karpinski, 500–501 [Last paragraph: "History of science, using science with the inclusive meaning as in the title A. A. A. S., is surely the proper name for the new section now under way"]—June 10: "Inaugural address" by E. F. Nichols, 523–527 [as president of Massachusetts Institute of Technology].

Science Progress, volume 15, no. 4, April, 1921: "Recent advances in pure mathematics" by Dorothy Wrinch, 517–522 [contains an elementary explanation of nomography, apropos of S. Brodetsky's book on the subject (*1921*, 131–132)]; "DeMoivre's theorem" by R. Ross, 627–628 [First paragraph: "I will be very much obliged to any of our mathematical readers who will be so kind as to inform me where I can find any record of the following proposition—which shows that DeMoivre's famous theorem connected with complex numbers is only a particular case of the iteration—that is, the *operative* involution—of a real algebraic function of which one of the parameters is reduced to zero. I have known the proposition for many years, and indeed indicated it in my paper on 'Operative Involution' in *Science Progress*, No. 50, p. 288, October, 1918, in some examples at the end, and in No. 51, p. 486, January, 1919, last example; but I have searched in vain for it through my books—even in the works of Hamilton, Tait, and Joly on quaternions, on which subject it has an important bearing"]; "Highways and byways in the theory of numbers" by L. J. Mordell, 647–652 [Review of L. E. Dickson's *The History of the Theory of Numbers*, volume 2 (Washington, 1920). First sentences: "All mathematicians interested in the theory of numbers, and this means sooner or later most pure mathematicians, will welcome volume ii of Prof. Dickson's 'Chronological History.' It notes practically everything written on the subject, sums up the results of a paper in a few lines, and might serve as a model of orderly arrangement. This history adds considerably to the increasing debt of mathematicians to America, and is a real necessity in their libraries"].

Scientific Monthly, volume 12, May, 1921: "The history of mathematics" by E. W. Brown, 385–413 [Lecture delivered at Yale University, February 26, 1920. First paragraph: "The earliest dawn of science is without doubt not different from that of intelligence. But the civilized man of to-day, far removed as he is from the lowest of existing human races, is probably as far again from the being whom one would not differentiate from the animals as far as mental powers are concerned. What this difference is, neither ethnologist nor psychologist can yet tell. Perhaps the nearest approach to a definition, at least from the point of view of this article, is contained in the distinction between unconscious and conscious observation. We are familiar with both sides even in ourselves; records can be impressed on the brain and remain there apparently dormant until some stimulus brings them to fruition, and again, record and stimulus can appear together so that a train of thought is immediately started"]; "The history of science as an error breeder" by G. A. Miller, 439–443.

Technology Review, volume 23, January, 1921: "Professor Cecil Hobart Peabody" by W. Hovgaard, 12–14 + portrait.

Transactions of the American Mathematical Society, volume 22, April, 1921: "On division algebras" by J. H. M. Wedderburn, 129–135; "Oscillation theorems for the real, self-adjoint linear system of the second order" by H. J. Ettlinger, 136–143; "New proofs of certain finiteness theorems in the theory of modular covariants" by Olive C. Hazlett, 144–157; "On the convergence of certain trigonometric and polynomial approximations" by D. Jackson, 158–166; "Determination of all general homogeneous polynomials expressible as determinants with linear

elements" by L. E. Dickson, 167–179; "Pseudocanonical forms and invariants of systems of partial differential equations" by A. L. Nelson, 180–197; "Arithmetical paraphrases (II)" by E. T. Bell, 198–219; "On the zeros of solutions of homogeneous linear differential equations" by C. N. Reynolds, Jr., 220–229; "A generalization of the Fourier cosine series" by J. L. Walsh, 230–239; "Polynomials and their residue systems" (to be continued) by A. J. Kempner, 240–266.

UNIVERSITY BULLETIN, Louisiana State University, new series, volume 13, no. 2, February, 1921: *Fundamental Aspects of Mathematical Training* by S. T. Sanders, 30 pages [First paragraph: "The tendency is outstanding in secondary education to exclude from courses of study the subject that does not have 'value in relation to other topics and to time involved.' [1] 'The ideal of practicality has now entered the schools with telling force. It has been manifested in its demand for vocational training, and it is reconstructing the older cultural training by eliminations and additions. Materials once accepted without question when schools had a margin of energy are now displaced by the pressure of new demands.' [2] 'But the meaning of the practical is not that of the eighteenth century. Arithmetic now represents tools which the child needs to control his present and potential quantity situations' [3]"].

UNTERRICHTSBLÄTTER FÜR MATHEMATIK UND NATURWISSENSCHAFTEN, volume 27, February 26, 1921: "Die Bedeutung der nichteuklidischen Geometrie für den Elementar-Unterricht" by A. Schülke, 3–6; "Rechentafel zur Auflösung zweier Gleichungen ersten Grades mit zwei Unbekannten" by P. Luckey, 8–9; "Das grösste einem gegebenen Kreisabschnitte eingeschriebene Rechteck und die grössten einem gegebenen Kugelabschnitte einbeschriebenen Kreiszylinder," by —. Friese, 9–11; "Zur Konstruktion des grössten Rechtecks in einem Kreissegment" by W. Gaedecke, 11–12; "Grenzwerte symmetrischer Verbindungen der Winkelfunktionen am Dreieck" by A. Emmerich, 12.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 52, nos. 1–2, February 17, 1921: "Ein Weg zur Relativität für die Schule" by A. Schoenflies, 1–13; "Zur Theorie der komplexen Zahlen" by W. Schwan, 13–17; "Graphische Behandlung der Zinsrechnung" by A. Schülke, 17–19; "Für und wider das abgekürzte Rechnen" by K. Becker, 19–24; "Zur Reform des mathematischen Hochschulunterrichts" by E. Kamke, 24–26; "Die Möbiussche Form des Brechungsgesetzes" by R. Böger, 27–31; "Eine Bemerkung zum d'Hondtschen Wahlverfahren" by —. Behmann, 32–34; "Der Lehrsatz des Pythagoras als Sonderfall eines Höhensatzes" by A. Maennersdoerfer, 35–36; "Die Winkelmessung des Babyloniers, des Artilleristen, und des Mathematikers" by P. Luckey, 36–37; "Aufgabenrepertorium," 38–41; "Ueber den Nichtgebrauch und den Missbrauch der Mathematik bei den Begabten-prüfungen" by M. Vaërting, 41–46; "Bücherbesprechungen," 46–55.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to E. L. Dodd, 3012 West Ave., Austin, Texas

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF HUNTER COLLEGE, New York City. [*1918*, 187.]

Prior to October, 1919, meetings were held monthly, as follows:

January, 1919: "Sun dials" by Anita Rosenthal '19.

March: Reception to freshmen.

April: "Prominent mathematicians of the present day" by Edith Gaddis '21; "Paper folding," under the direction of Anita Rosenthal '19.

May: "Perfect, amicable, square, and triangular numbers" by Laura Guggenbuhl '22.

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[1] "Bulletin, 1920, No. 1, U. S. Bureau of Education."

[2] "'High School Mathematics,' by G. W. Evans, Headmaster of Charleston High School."

[3] "Bulletin, 1917, No. 10, U. S. Bureau of Education."

In October, 1919, a reorganization of the club was effected with a view to placing all responsibility for the success of the club upon the student members. In November the following officers were elected: President, Melanie Rohrer '20; vice-president, Helen Myer '20; secretary, Helen Deaken '20; treasurer, Charlotte Warnock '21, publicity manager, Edith De Hondt '20, faculty adviser, Marcia Latham, instructor.

December: Catch problems, brought by volunteers and discussed by the club.

Members were invited to drop further solutions in the Mathematics Club mail box.

Since the reorganization, the arrangement is to have three afternoon meetings each semester, the first one being purely social. In addition, two regular meetings a month are held during the noon recess hour. Outside speakers are invited each semester. Recently a "Problem Chapter" has been formed with Laura Guggenbuhl '22 in charge for the discussion and solution of problems too technical for the regular meetings. Frequent reference is made to the problems published in the MONTHLY. The following are the topics of meetings held during 1920.

January 10, 1920: "Scales of notation" by Laura Guggenbuhl '22 and Helen Myer '20.

February 23: "Magic squares" by Eva Thornton '22.

March 8: "Card tricks" by Mary Bailey '21.

March 23: "The fourth dimension" by Anna Colligan '21, Jessie McCormick '22, and Sarah Block '22.

April 5: Book reviews by Monica Gilloran '21, Edna Dramer '23, Charlotte Warnock '21, and others.

May 17: Election of officers: President, Catherine Cronin '21; vice-president, Mary Bailey '21; secretary, Dora Aronoff '21; treasurer, May Warnock '22; publicity manager, Sarah Karnis '22; faculty adviser, Marcia Latham, instructor.

October 4: "Geometrical fallacies" by Margaret Cronin '21.

November 1: "The use of the slide rule in the business world" by Sarah Mones '21.

November 15: "Business calculations in the middle ages" by Laura Guggenbuhl '22, Jessie McCormick '22, and Sarah Block '22, with lantern slides shown by Professor Lao Simons.

December 6: A demonstration of calculating machines by Mr. O'Brien, of the Dalton Adding Machine Company.

December 24: Christmas Party.

A red letter day during 1920 was a special afternoon meeting addressed by Professor David E. Smith on the teaching of mathematics in the junior high schools.

The average attendance at the noon meetings has been forty and that at the afternoon social meetings one hundred. A special effort has been made to make the social meetings popular and at the same time keep the flavor of a mathematics club. The following are some special features of different meetings: Miss Polly

Gon, the magician, who divined ages, numbers on a blackboard visible to the audience but concealed from her, throws of dice, and so forth; at another meeting the chief entertainment was a contest based on the following puzzles devised by Beatrice Meyer '22. Each of these is a mathematical term when translated: 1. C + to possess something = ———; 2. To make an attempt + to fish = ———; 3. A parrot + part of the verb *to go* = ———; etc.; students sometimes furnish original songs, which together with more familiar ones, such as "Euclid had a little book" and "Every little symbol has a meaning all its own" are sung at the close of the meetings. The club has a special bulletin board for announcements, clippings, special problems, etc.

(Report by Miss Aronoff.)

THE WHITE MATHEMATICS CLUB AT THE UNIVERSITY OF KENTUCKY, Lexington, Ky.

[*1918*, 90, 451; *1919*, 309.]

The following are programs of recent meetings of the Club.

May 7, 1919: "Monge's solution of the non-integrable complete differential equation" by Professor J. M. Davis.

May 12: "Class and one-to-one correspondence" by Frank Tuttle '20.

May 20: "Review of nilpotent algebras generated by two units, i and j, such that i^2 is not an independent unit" by Dr. G. W. Smith, instructor.

May 26: "Some linkages considered analytically" by Harvey Pettitt Gr.

June 2: "Applications of vector analysis to the theory of helices" by William Elliott Gr.

September 30, 1919: Election of officers: President, Professor Davis; secretary, Dr. Smith.

October 7: Report of mathematics meetings held at Ann Arbor, Michigan, September 2–5, 1919, by Dr. Smith.

October 14: "Quadratic transformations" by Professor P. P. Boyd.

October 21: "Discontinuities" by William Elliott Gr.

October 28: "Tests of mathematical ability" by Professor Davis; "The Committee on Mathematical Requirements" by Professor Boyd; "The coefficient of correlation" by Dr. Smith.

November 4: "To draw a circle tangent to three given circles by means of inversion" by W. E. Armentrout Gr.

November 12: "Determination of the angular velocity from the accelerations of non-collinear points of a rigid body" by Professor E. L. Rees.

November 18: "Some angles of the right triangle" by Frances Kimbrough '20.

November 25: "Singular case of conics" by Professor H. H. Downing.

December 2: "Paper folding" by Frank Tuttle '20.

December 9: "Introduction to the theory of statistics" by Dr. Smith.

December 16: "Bolshevism in mathematics" by Professor Rees.

February 17, 1920: "Wireless telegraphy" by Ernest Baulch '21.

February 24: "Quadratic transformations" by Professor Boyd.

March 2: "Magnus's transformations" by Jesse Osborn Gr.
March 9: "Some peculiar functions" by Dr. Smith.
March 16: "Some propositions about circles proved by means of properties of the radical axes" by W. E. Armentrout Gr.
March 23: "Taylor and Taylor's series" by Professor Downing.
March 30: "Some properties of integral numbers" by Frances Kimbrough '20.
April 6: "Quadratic transformations, Steiner's method" by Jesse Osborn Gr.
April 13: "Discussion of the preliminary report of the Committee on Mathematical Requirements."
April 27: "Bohr's picture of the atom" by Professor W. B. Angel, physics department.
May 11: "Cubic and quartic curves with a line of symmetry" by Professor Rees.
May 20: "The mathematical theory of investment" by Dr. Smith.
October 20, 1920: "A geometric proof of a theorem concerning the roots of a quartic" by Professor Rees. Election of officers: President, Professor Rees; secretary, Dr. Flora E. Le Stourgeon, instructor.
October 26: "Derivatives from the expansion of the function" by Professor Downing.
November 2: "Conversion of series into continued fractions" by W. E. Armentrout Gr.
November 11: "Functions of lines" by Dr. LeStourgeon.
November 16: "Curve tracing" by L. P. Rippy Gr.
November 23: "Applications of mathematics to economics" by W. E. Payne, instructor.
November 30: "The trisection of an angle and the duplication of a cube" by D. C. Duncan '22.
December 7: "Differential equations (Review of Bateman's *Differential Equations*)" by Professor Davis.
December 14: "Theory of the slide rule" by Nelson Conkwright '22.
January 13, 1921: "The Hessian, the Steinerian, the Cayleyan" by Professor Boyd.
January 18: "Curve tracing" by Professor Rees.
February 14: "Some properties of the logarithmic functions" by Professor Downing.
February 23: "Some ancient methods of computation" by Professor Boyd; "Two mathematical paradoxes" by Professor Downing; "Mathematics and religion" by Professor Davis.
March 2: "Stieltjes's integrals and linear functionals" by Dr. LeStourgeon.

(Report by Dr. LeStourgeon.)

Pi Mu Epsilon Fraternity (Inc.), Syracuse University, Syracuse, N. Y.
[*1918*, 271.]

Officers for the year 1919–20 were as follows: Director, Mrs. Mary Harwood, instructor; vice-director, J. J. Hopfield, instructor in physics; secretary, Mary

Hutchinson '20; treasurer, Edward Houghtaling '21; librarian, Marion Jarvis '20.

During the college year six meetings were held, a Christmas party, and a picnic. One graduate student and twelve undergraduates were elected members. A chapter at Ohio State University was established. The programs were as follows:

November 10, 1919: "Travels through France" by Professor W. H. Metzler; "Graduate work at the University of Edinburgh" by Dr. J. J. Nassau.

December 1: "The region at infinity" by Dr. T. S. Yang.

February 23, 1920: "Some new equations in graduations" by Dr. Nassau.

April 19: "The theory of relativity" by Dr. Nassau.

May 10: "On some special curves" by Edna Lawrence '20.

(Report by Dr. Nassau.)

THE PASCAL CIRCLE OF TRINITY COLLEGE, Washington, D. C.
[*1920*, 425.]

The officers for the year 1920–21 are as follows: President, Margaret Walsh '21; vice-president, Elizabeth Herbert '22; secretary and treasurer, Margaret Kelly '23; faculty adviser and honorary president, Professor Marie C. Mangold.

October: The name, purpose, and requisites for membership of the society were explained by the president. Recreational problems in geometry were presented.

November: "The value of mathematics to the physiologist and physician"[1] by Elizabeth Herbert '22. At this meeting it was decided to make the discussions of the year hinge on the relation of mathematics to the various other sciences, and to have three members appointed at each meeting to bring in recreational problems for the next meeting.

December: "Why automobiles skid" by Frances McFadden '23; "The relation of Mathematics to English" by Lillian Manganero '22. This refuted the frequent charge that skill in one excludes skill in the other.

(Report by Miss Walsh.)

THE JUNIOR MATHEMATICAL CLUB, University of Wisconsin, Madison, Wis.
[*1918*, 188, 457; *1920*, 224.]

The present officers are as follows: President, Phillip Dowling '21; vice-president, Gretchen Votteler '21; secretary and treasurer, Alice N. Tucker '22.

The programs for the past year are as follows:

January 22, 1920: "Integers" by Professor E. B. Skinner.

February 26: "Graphical calculations" by Professor Arnold Dresden.

March 11: "Mathematics in physics" by Harold Laird '22 and Phillip Dowling '21.

March 25: "What is a wave?" by Professor C. S. Schlichter.

[1] A paper by Professor H. B. Williams, with a title similar to this, was delivered before the Mathematical Association of America (*1920*, 95–97) in December, 1919. The main part of the paper was published in *Mathematics Teacher*, March, 1920.—EDITOR.

April 8: "Geometric probabilities" by Dr. Florence Allen, instructor.
April 22: "Squaring the circle" by Professor W. W. Hart.
May 27: Election of officers; annual reports.
October 28: "The semi-regular solids of Archimedes" by Professor E. B. Van Vleck.
November 11: "Mathematics and logic" by Professor Dresden.
December 2: "Methods of mathematics" by Victor Von Szeliski.
December 16: "Wave motion" by Professor Schlichter.
January 13, 1921: "Expansion of fundamental laws of algebra" by Professor Skinner.
January 27: "The value of π" by Grace Desimval '21.
February 24: "Interesting applications of the theory of probability" by Professor Max Mason.

(Report by Miss Tucker.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2920. Proposed by N. P. PANDYA, Sojitra, India.

Construct a triangle, having given the base, the angle between the base and the median on it, and the difference of the remaining two sides.

2921. Proposed by J. W. CLAWSON, Ursinus College, Pa.

ABC is a triangle cut by a transversal PQR, so that A, P; B, Q; and C, R are opposite vertices of a complete quadrilateral. Draw CD, PF, QE, chords in the circles circumscribing, respectively, triangles ABC, BRP, AQR, all these chords being parallel to AB.

Prove that (i) D, E, F are collinear; (ii) the line DEF passes through the Wallace point of the quadrilateral (the point of concurrency of the circles mentioned above); (iii) the line DEF intersects AB at the point of tangency to AB of the parabola which touches the four sides of the quadrilateral.

2922. Proposed by the late A. M. KENYON.

A telephone engineer desires the general solution of the following differential equation:

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{a}{\sqrt{y}}.$$

2923. Proposed by C. N. SCHMALL, New York City.

The corner of a page of a book is turned down in such a manner that the triangle formed has a constant area. Show that the locus of the corner is an oval of the curve,

$$r^2 = a^2 \sin 2\theta.$$

2924. Proposed by FLORENCE P. LEWIS, Goucher College.

Given a triangle and a conic. Through each vertex of the triangle there pass two lines harmonic to the tangents through that point and to the sides of the triangle. Prove that the six lines so found pass by threes through four points.

2925. Proposed by F. V. MORLEY, New College, Oxford, Eng.

A regular polygon of $2n + 1$ sides will have only $n - 1$ diagonals of different lengths (*e.g.*, the regular heptagon has two distinct diagonals). Call the side of such a polygon a_1, and the $n - 1$ diagonals in order of size $a_2 \cdots a_n$. Then if the circumscribed circle has radius unity, $\sum_{i=1}^{n} a_i^2 = 2n + 1$; in words, the sum of the squares of the distinct lengths obtained by joining an odd number of regularly spaced points on a unit circle is equal to the number of such points.

2926. Proposed by T. M. SIMPSON, Randolph-Macon College, Ashland, Va.

Solve the differential equation,

$$(y + x^2)dx + (x - x^2y)dy = 0.$$

2927. Proposed by PHILIP FRANKLIN, Princeton University.

Prove that the only positive integral values greater than unity which satisfy the equation $3^x - 2^y = \pm 1$ are $x = 2$, $y = 3$. (Cf. Carmichael, *Diophantine Analysis*, 1915, p. 116, exercise 69.)

NOTES.

22. Huge Numbers.—"What is the largest number that we can express by three digits?" The answer is $N = 9^{(9^9)}$, that is $9^{387,420,489}$. C. A. Laisant drew attention to this number in his *Initiation Mathématique*, Paris, 1906 (English edition, London, 1913). He there remarks that in decimal numeration this number would have 369,693,100 figures. To write it on a single strip of paper, supposing that each figure occupied a space of one fifth of an inch, the length of the strip would need to be 1,166 miles, 1,690 yards, 1 foot, 8 inches. In this connection C. E. Guillaume remarked (*Revue Générale des Sciences*, vol. 17, 1906, p. 878) that under the same conditions to write $10^{(10^{10})}$, we would need a strip of paper long enough to encircle the earth.

Writing in May, 1913, A. C. D. Crommelin stated (*Journal of the British Astronomical Association*, vol. 23, pp. 380–381) that he had come across the problem with which this note commences "in an old logarithm book." By the aid of 61-figure logarithms of certain numbers given in Hutton's tables Dr. Crommelin found $\log N = 369,693,099.6315703587\cdots$; whence the number of figures indicated above. He found the first 28 of the figures to be 428,124,773,175,747,048,036,987,115,9 and the last three to be 289.

"A knowledge of 30 figures out of 300 million," he continued, "may seem trifling, but in reality the error involved in taking all the remaining figures as zeros is only one part in a thousand quadrillions. If the number were printed with 16 figures to an inch (about the tightest packing for decent legibility), it would extend over 364.7 miles. . . . If printed in a series of large volumes we might get 14,000 figures to a page, and with 800 pages to the volume it would fill 33 volumes. There are more than twice as many digits in the number as there are letters in the whole of the *Encyclopædia Britannica*.

"To find the largest number suggested by sideral astronomy I took the following. Both Very and See have expressed the opinion that certain visible objects may be at a distance of a million light years; I imagine a solid sphere of platinum of this radius, and find how many electrons it contains. From Duncan's *The New Knowledge*, p. 65, I find that the log of the number of electrons in a cubic centimetre of water is 16.469. Taking the density of platinum as 21.5 the log of the number of electrons in a cubic inch of it is 19.016, and the log of the volume of the

huge sphere in cubic inches is 71.3362. Whence the log of the number of electrons it contains is 90.352, and the corresponding number is 225 followed by 88 zeros. At 16 figures to the inch this would take $5\frac{3}{4}$ inches. . . .

"To find the radius of a sphere of platinum that would contain 9^{9^9} electrons, we must multiply our million-light-year radius by a number whose log is 123,231,003.093, *i.e.*, the multiplier is 1239 followed by 123,231,000 zeros. In fact, that gigantic sphere would exceed the million-light-year sphere in a far higher ratio than that exceeds the size of one electron.[1] Hence we may take it as morally certain that we can write with three digits a number vastly exceeding the number of electrons in the whole of creation, which is a somewhat startling fact. Indeed, even the number 4^{4^4} (which is 13407813 followed by 147 other figures) probably exceeds the number of electrons in creation. At least it equals the number of electrons in a solid platinum sphere that exceeds the million-light-year sphere in the same proportion that that exceeds a sphere 206 inches in radius."

In May, 1915, D. G. McIntyre considered (*Journal of the British Astronomical Association*, vol. 26, pp. 46–47) the "rather pretty" problem of determining the last figures of N, three of which, 289, were given by Dr. Crommelin. He found the last eight figures to be 17177289.

In *The Observatory* for July, 1920, H. H. Turner refers to N and to his choice, as rather simpler for expression and not very different in magnitude, of the number $10^{(10^{10})}$ which, when written out fully, is unity followed by ten thousand million zeros. "The validity of the substitution remained unchallenged until the other day, when Dr. Crommelin realised that injustice had been done. He accused me of treating as comparable, let us say, a billion times the distance of Dr. Shapley's furthest star cluster and a wave-length of light; for the ratio of these would, he said, be no greater than that of $10^{10^{10}}$ to 9^{9^9}. The accusation could not be repelled, and I tendered apologies as gracefully as the magnitude of the error would allow. But the accuser had apparently not reaped his advantage to the full. A few days later I received the following post card from him:

"I greatly understated the ratio of $10^{10^{10}}$ to 9^{9^9}, which is a number of some 900 million figures, and would be several thousands of miles long if written out at 16 figures to the inch. On the other hand, the ratio I mentioned yesterday (viz., a billion times the distance of Shapley's furthest cluster to the billionth of an inch) would have something under 50 figures in it, and could be written in the width of this postcard.

"I fear the framing of a suitable acknowledgment is beyond me."

Finally, in *Journal of the British Astronomical Association*, April, 1921, J. W. Meares comments on $9!^{(9!^{9!})}$ and finds that the value of his number is greater than 10 to the power $10^{2000000}$ but less than 10 to the power $10^{2000001}$. On this Dr. Crommelin commented: "If one allows the introduction of algebraic symbols the number ∞^{∞^∞} has some claims on our attention. Perhaps I may be allowed to quote an old college rhyme:

"'There was a professor of Trinity
Who found the square root of infinity;
But in counting the digits
He was seized with the fidgets,
Dropped Science and took to Divinity.'"

One wonders if reference is here made to George Salmon, of Trinity College, Dublin, whose classic mathematical works have throughout the world been the delight of generations, but whose works on "Divinity," written after he had "dropped" mathematics, are known to few! ARC.

[1] "Indeed we should have to carry out enlargement in the ratio of a million-light-year sphere to an electron more than a million times in succession before we get a sphere of the size enlarged."

23. The Wallace Line and Wallace Point, and Some Generalizations.—In 1799, William Wallace[1] stated the theorem: Given a circle and a triangle inscribed in it, the feet of the perpendiculars on the sides of the triangle from any point of the circle are collinear.[2] Such a line is the Wallace line of the point. The first generalization of the theorem was in 1822 by Poncelet[3] who showed that the perpendiculars on the sides of the triangle may be replaced by obliques making, in cyclic order, equal angles with the sides. Both Wallace and Poncelet seem to have arrived at their results[4] by considering properties of the parabola: (*a*) the circumscribed circle of a triangle tangent to a parabola passes through its focus;[5] (*b*) the feet of the perpendiculars from a focus of a parabola on its tangents lie on the tangent at the vertex of the parabola.

As an application of his theorem, Wallace considered the question[6] of describing a parabola tangent to four given straight lines; he remarks that the focus of the parabola is determined by finding the second point of intersection of the circles circumscribing any two of the four triangles formed by the four given lines. In other words the four circles meet in a point—the Wallace point.[7] Without reference to the parabola, Wallace formulated the theorem again in 1804, It appeared also in Bland's *Geometrical Problems*, Cambridge, 1819, p. 259, before

[1] *The Mathematical Repository*, March, 1799, p. 111.

[2] This same theorem was given by Servois in February, 1814 (*Annales de Mathématiques* (Gergonne), vol. 4, p. 251). In a footnote he states the theorem in another form: "Through a point on the circumference of a circle three chords are drawn. Circles are described on these chords as diameters. The second points of intersection of these circles, taken in pairs, are collinear."

[3] *Traité des Propriétés Projectives*, 1822, p. 270.

[4] *The Mathematical Repository*, September, 1798, p. 81; Poncelet, *l.c.*, pp. 218–219.

[5] Lambert, *Insigniores Orbitae Cometorum Proprietates*, 1761, p. 5.

[6] *The Mathematical Repository*, March, 1799, p. 81.

[7] The centers of the four circles lie on a circle through the Wallace point—a result stated without proof by Steiner in 1828, *Annales de Mathématiques* (Gergonne), vol. 18, p. 302. From the above it is clear that there is scant justification for calling the Wallace point, the Miquel point, as has been done by S. Kantor (*Comptes rendus de l'Académie des Sciences de Vienne*, 1878), since Miquel derived the result in the periodical *Le Géomètre* (founded by Guillard), 1836, and later in Liouville's *Journal*, vol. 3, 1838, p. 486. So also J. L. Coolidge refers (*A Treatise on the Circle and the Sphere*, 1916, p. 87) to "the Miquel point of the four lines."

In 1871 W. K. Clifford remarked [in his *Common Sense of the Exact Sciences*, 1885, pp. 80–81] that Wallace's theorem concerning concurrent circles is the third of a series:

"If we take any two straight lines they determine a point, viz., their point of intersection.

"If we take three straight lines we get three such points of intersection; and these three determine a circle, viz., the circle circumscribing the triangle formed by the three lines.

"Four straight lines determine four sets of three lines by leaving out each in turn; and the four circles belonging to these sets of three meet in a point.

"In the same way five lines determine five sets of four, and each of these sets of four gives rise, by the proposition just proved, to a point. It has been shown by Miquel [*l.c.*], that these five points lie on the same circle.

"And this series of theorems has been shown [W. K. Clifford, 'Synthetic proof of Miquel's theorem,' *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. 5, 1871, p. 124] to be endless. Six straight lines determine six sets of five by leaving them out one by one. Each set of five has, by Miquel's theorem, a circle belonging to it. These six circles meet in the same point, and so on forever. Any even number ($2n$) of straight lines determines a point as the intersection of the same number of circles. If we take one line more, this odd number ($2n + 1$) determines as many sets of $2n$ lines, and to each of these sets belongs a point; these $2n + 1$ points lie on a circle."

it was given by Steiner in 1828 (compare problem 2898 of this MONTHLY, *1921*, 228).

Professor C. N. Mills of Tiffin, Ohio, suggested the following problem: "Given any triangle cut by a transversal through a fixed point on the base produced; through the fixed point and the intersections of the transversal with each side and the adjacent vertex at the base circles are drawn. Show that the locus of the intersection of these two circles is the circumcircle of the given triangle." From what has been given above it is clear that the point in which the circles meet is the Wallace point for the particular position of the transversal, which is the Poncelet line for that Wallace point.

Another generalization of Wallace's theorem stated without proof [1] (anonymously) in July, 1823, was as follows: (*a*) If from any point of a circle concentric with the circumscribed circle of a triangle, perpendiculars are dropped on the three sides, the area of the triangle, whose vertices are the feet of the perpendiculars, is constant. When this circle becomes the circumscribed circle the area vanishes. (*b*) If two circles concentric with the circumscribed circle are such that the sum of the squares of their radii is equal to twice the square of the radius of the circumscribed circle the two triangles formed as above are equivalent.

In 1870 Combette gave among other results: [2] (*a*) If P be an assumed point, and D, E, F its projections on the triangle ABC, the locus of P, when the area of the triangle DEF is constant, is the circumference of a circle concentric with the circumscribed circle of ABC; (*b*) For every value of the area of DEF included between zero and one fourth of ABC, the locus of P will consist of two circumferences concentric with the circumscribed circle, the one interior and the other exterior to it; (*c*) The sums of the squares of the radii of these two circumferences will be double the square of the circumscribed radius; (*d*) The locus of P does not change its nature or its center when the perpendiculars let fall on the sides become lines all making equal angles with the sides; (*e*) If the triangle ABC is replaced by a plane polygon, and the projections on its sides of a point P in the plane are joined in order, the locus of P when this area is constant is still a circle which has always the same center whatever be the value of the area;[3] (*f*) When the point P is taken in space and projected on the sides of a plane polygon, the locus of P, when the area obtained by joining the projections is constant, becomes a cylindrical surface of revolution whose axis, perpendicular to the plane of the polygon, is always the same for all values of the area.

The locus of the points whose projections on the planes of the faces of a

[1] *Annales de Mathématiques* (Gergonne), vol. 14, p. 28; Gergonne was probably the author. Analytic proofs are given on pages 280–293, the latter being by the great Sturm.

[2] *Revue des Sociétés Savantes*, vol. 5, pp. 203–233; compare J. S. Mackay, *Proceedings of the Edinburgh Mathematical Society*, vol. 9, pp. 86–87.

[3] This result was first enunciated for a *regular* polygon, without proof, by L'Huilier in *Bibliothèque Universelle*, March, 1824, p. 169. Proofs were given in *Annales de Mathématiques* (Gergonne), vol. 15, by "abonné," July, 1824, pp. 45–55; and by Sturm, February, 1825, pp. 250–252. This was generalized as above (and with further interesting results) by Steiner, in *Journal für die reine und angewandte Mathematik*, vol. 1, 1826, pp. 51–52; see also vol. 2, 1827, p. 265, and a further generalization (p. 263).

tetrahedron are coplanar is a cubic surface, S, through the edges of the tetrahedron and having the vertices of the tetrahedron as nodes.[1]—The surfaces S, for the 15 tetrahedra determined by any six planes, meet in a point P (corresponding to the Wallace point above), and the pedal planes of P for the 15 tetrahedra are coincident.—Steiner remarked, in 1845, that S is the locus of the centers of the hyperboloids for which, when their equations are in standard form,

$$(1/a^2) + (1/b^2) = 1/c^2,$$

and for which a given tetrahedron is self-polar.

Among many discussions of the surfaces, S, the following may be mentioned: by Geiser, in *Journal für die reine und angewandte Mathematik*, vol. 69, 1868, p. 199 f.; by F. E. Eckhardt, in *Mathematische Annalen*, vol. 5, 1872, pp. 30–49; by E. Janhke, *Archiv der Mathematik* (Grunert), third series, vol. 4, 1903, especially pp. 275–276; by J. Neuberg, *Archiv der Mathematik* (Grunert), third series, vol. 16, 1910, p. 18 f.; and by W. H. Salmon, in *Archiv der Mathematik* (Grunert), third series, vol. 18, 1911, pp. 154–164. ARC.

SOLUTIONS.

2719 [1918, 302]. Proposed by R. P. BAKER, University of Iowa.

Show that[2] $2x(\log x)^2 - x(x-1)(x+3)\log x + (x-1)^2(3x-1)$ is negative for $1 < x < \infty$.

SOLUTION BY OTTO DUNKEL, Washington University.

Denoting the given expression by $f(x)$, its first two derivatives may be written as follows:

$$f'(x) = 2(\log x)^2 - (x-1)(3x+7)\log x + 8(x-1)^2,$$

$$f''(x) = \frac{3x^2+2x-2}{x}\varphi(x), \qquad \varphi(x) = \frac{(13x-7)(x-1)}{3x^2+2x-2} - 2\log x.$$

It will be observed that $f(1) = f'(1) = f''(1) = 0$. It will be shown that $f''(x)$ is negative for $x > 1$ and it will then follow that $f'(x)$ is also negative, and hence $f(x)$ is likewise negative for $x > 1$. The derivative of $\varphi(x)$ may be written

$$\varphi'(x) = -\frac{2(x-1)^3(9x-4)}{x(3x^2+2x-2)^2}$$

and it is clearly negative for $x > 1$. Since $\varphi(1) = 0$, it follows that $\varphi(x) < 0$. The first factor of $f''(x)$, which may be denoted by $\psi(x)$, has the roots $-(\sqrt{7}+1)/3$ and $x_1 = (\sqrt{7}-1)/3 = .549$, and hence $\psi(x) > 0$ for $x > 1$. Therefore, $f''(x) < 0$ for $x > 1$ and the desired result is proved.

The form of $\varphi'(x)$ shows that the first of the derivatives of $f(x)$ which do not vanish for $x = 1$ is $f^{\text{vi}}(x)$. The expressions above give at once $f^{\text{vi}}(1) = -20$ and this shows that $f(x)$ is also negative for values of $x < 1$. It will now be shown that the interval for such values extends down to $x = 0$. In the interval $0 \leqq x < 1$ the derivative $\varphi'(x)$ vanishes only for $4/9$ and it is negative before and positive after this value. Hence, $\varphi(x)$ has a minimum at this point. It will be found that $\varphi(4/9) = .312$, and so $\varphi(x)$ is positive from 0 to x_1. It is negative from x_1 to 1, since it vanishes at $x = 1$ and $\varphi'(x)$ is positive at all other points of this interval. On the other hand $\psi(x)$ is negative from 0 to x_1 and positive from x_1 to 1. At x_1 the product $\psi(x)\varphi(x)$ has a finite negative value and hence $f''(x)$ is negative from 0 to 1. From this follows that $f'(x)$ is positive and $f(x)$ is negative in the interval $0 \leqq x < 1$. Thus $f(x)$ is negative at every point of the interval $0 \leqq x < \infty$ except at the point $x = 1$ where it vanishes.

[1] In *Annales de Mathématiques* (Gergonne), April, 1814, vol. 4, p. 320, the following problem was proposed (by Gergonne?) for solution: "The feet of the perpendiculars dropped on the faces of a tetrahedron from a point on the circumscribed sphere are coplanar." The incorrectness of this statement was proved by Durrande, in the same periodical, February, 1817, vol. 7, p. 255.

[2] It should be noted that this relation holds for "Naperian" but *not* for common logarithms. —EDITORS.

2812 [1920, 81]. Proposed by C. N. SCHMALL, New York City.

If $F(x, y, z)$ be a homogeneous function of x, y, z which becomes $\phi(u, v, w)$ by elimination of x, y, z by means of the equations, $\partial F/\partial x = u$, $\partial F/\partial y = v$, $\partial F/\partial z = w$; show that

$$\frac{\partial F}{\partial u}\Big/ x = \frac{\partial F}{\partial v}\Big/ y = \frac{\partial F}{\partial w}\Big/ z.$$

Solution by the Proposer.

The result is incorrectly stated. It should read,

$$\frac{\partial \phi}{\partial u}\Big/ x = \frac{\partial \phi}{\partial v}\Big/ y = \frac{\partial \phi}{\partial w}\Big/ z.$$

Let k be the degree of the homogeneous function F. Then by Euler's theorem on homogeneous functions,

$$xu + yv + zw = kF. \tag{1}$$

By partial differentiation of (1) with respect to x, we obtain

$$u + x\frac{\partial u}{\partial x} + y\frac{\partial v}{dx} + z\frac{\partial w}{\partial x} = ku,$$

or

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} + z\frac{\partial w}{\partial x} = (k-1)u. \tag{2}$$

Similarly, we get

$$x\frac{\partial u}{\partial y} + y\frac{\partial v}{\partial y} + z\frac{\partial w}{\partial y} = (k-1)v \tag{3}$$

and

$$x\frac{\partial u}{\partial z} + y\frac{\partial v}{\partial z} + z\frac{\partial w}{\partial z} = (k-1)w. \tag{4}$$

Again,

$$u = \frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial x}, \tag{5}$$

$$v = \frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial y}, \tag{6}$$

$$w = \frac{\partial F}{\partial z} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial z}. \tag{7}$$

The solution of the system (2), (3), (4) for x, y, z, and of the system (5), (6), (7) for $\partial\phi/\partial u$, $\partial\phi/\partial v$, $\partial\phi/\partial w$ gives

$$\frac{\partial \phi}{\partial u}\Big/ x = 1/(k-1); \qquad \frac{\partial \phi}{\partial v}\Big/ y = 1/(k-1), \qquad \frac{\partial \phi}{\partial w}\Big/ z = 1/(k-1).$$

Whence,

$$\frac{\partial \phi}{\partial u}\Big/ x = \frac{\partial \phi}{\partial v}\Big/ y = \frac{\partial \phi}{\partial w}\Big/ z.$$

2823 [1920, 185]. Proposed by S. A. COREY, Des Moines, Iowa.

Let TQ and PR be diameters of a circle with center O. Bisect TO at X and draw PQ. On PQ erect the perpendicular XW and on PR, the perpendicular QV. Prove that $OX\cdot PV = PW\cdot PQ$

Solution by Emma M. Gibson, Springfield, Mo.

If S is the intersection of XW and PO, the triangle XOS is isosceles (the reader is requested to draw the figure); for,

$$\angle SXO = 90° - \angle XQW = 90° - \angle SPW = \angle PSW = \angle XSO.$$

Hence $OX = OS = SP$. Since the triangles PWS and PVQ are similar,

$$PW/PV = PS/PQ = OX/PQ,$$

and hence

$$OX\cdot PV = PW\cdot PQ.$$

Also solved by H. L. Agard, T. M. Blakslee, H. N. Carleton, P. J. da Cunha, H. H. Downing, R. M. Ginnings, C. E. Mange, H. L. Olson, Louis O'Shaughnessy, Arthur Pelletier, J. B. Reynolds, A. V. Richardson, and J. H. Weaver.

2838 [1920, 273–274].

"A rope is supposed to be hung over a wheel fixed to the roof of a building; at one end of the rope a weight is fixed, which exactly counterbalances a monkey which is hanging on to the other end. Suppose that the monkey begins to climb the rope, what will be the result?"

This problem was invented by Lewis Carroll in December, 1893 (S. D. Collingwood, *The Life and Letters of Lewis Carroll* (*Rev. C. L. Dodgson*), New York, 1899, pp. 317–318), and in his diary he remarked: "Got Professor Clifton's answer [R. B. Clifton, professor of physics at Oxford] to the 'Monkey and Weight Problem.' It is very curious, the different views taken by good mathematicians. Price [Bartholomew Price, professor of physics at Oxford] says that the weight goes *up* with increasing velocity; Clifton (and Harcourt [A. G. Vernon-Harcourt, professor of chemistry at Oxford]) that it goes *up*, at the same rate as the monkey; while Sampson [probably E. F. Sampson, lecturer, tutor and censor of Christ Church, Oxford] says that it goes *down*." Yet another solution by Rev. A. Brook is given on page 268 of *The Lewis Carroll Picture Book* . . . edited by S. D. Collingwood (London, 1899), namely, that "the weight remains stationary."

The problem has been recently discussed in *School Science and Mathematics*, volume 17, December, 1917, p. 821; volume 19, December, 1919, p. 815; and volume 20, February, 1920, pp. 172–173. The editors of the Monthly invite mathematical solutions of the problem.

The following solutions were contributed by request:

I. Solution by E. V. Huntington, Harvard University.

Case 1. If we neglect the weight of the pulley and rope, the solution follows immediately from the fundamental principle of mechanics, namely: the acceleration of a particle in any direction is proportional to the net force acting on the particle in that direction.

Here there are two particles to consider: (1) the monkey, and (2) the counterpoise.

The net upward force acting on the monkey is $T - W$, where W is the weight of the monkey, and T the tension in his part of the rope. The net upward force acting on the counterpoise is $T' - W'$, where W' is the weight of the counterpoise, and T' the tension in that part of the rope. But on the hypothesis of Case 1, the tension in the rope is the same at all points, so that $T' = T$; also, W' is known to be equal to W. Hence the net upward force acting on the monkey is the same as the net upward force acting on the counterweight, so that *the accelerations of the two bodies must be equal at every instant.*

Therefore, since the two bodies may be supposed to start from rest at the same level, their motions will be precisely parallel, no matter how fast or slow the monkey may climb, up or down, or how much he may allow the rope to slip through his hands.

Case 2. If we take into account the weight of the wheel (still neglecting the weight of the rope), we shall need to use also the equation of rotation.

Let w_0 be the weight of the wheel, r its radius, and k its radius of gyration. Then the equations of motion for the three bodies, W, W', and w_0, will be:

$$T - W = (W/g)dv/dt, \qquad T' - W' = (W'/g)dv'/dt,$$

and

$$Tr - T'r = (w_0/g)k^2 d\omega/dt,$$

where v and v' are the velocities of the monkey and the counterpoise, respectively, in the upward direction in space, and ω is the angular velocity of the wheel.

The geometric conditions of the problem tell us that as long as the rope does not become slack

$$v' = r\omega, \qquad \text{and} \qquad v' + v - u = 0,$$

where u is the relative velocity of the monkey up the rope.

From these five equations, we readily find:

$$\left(W + W' + \frac{w_0 k^2}{r^2}\right)\frac{dv'}{dt} = W\frac{du}{dt} + (W - W')g,$$

which gives the required acceleration, dv'/dt, of the counterpoise, when the relative acceleration, du/dt, of the monkey with respect to the rope is known.

Integrating twice, and putting $W = W'$, we find that the distance x risen by the counterpoise, when the monkey has climbed up a length s on the rope, is

$$x = s/\left(2 + \frac{w_0k^2}{Wr^2}\right),$$

which reduces, as it should, to $x = s/2$ when $w_0 = 0$.

II. Solution by L. M. Hoskins, Stanford University.

If the problem is idealized by neglecting all friction and assuming the cord to be perfectly flexible and without mass, the solution is simple; the further assumption that the pulley is without mass simplifies it still further. The solution of this ideal problem does not furnish a satisfactory answer to the question, what will actually be the result if the monkey begins to climb the rope, for the result may depend in an important way upon the neglected factors. It is, however, useful to consider the simple idealized problem as a preliminary to a more general discussion taking account of friction and the mass of the rope.

(*a*) *Solution Neglecting Friction and Assuming the Cord and Pulley to be Without Mass.*—The initial condition is one in which both the monkey and the counterweight are at rest, each being in equilibrium under the action of two equal and opposite forces,—its own weight and the supporting pull exerted by the cord. In this initial condition the monkey is exerting a downward pull upon the cord equal to his weight; in beginning to climb he increases his pull in order to make the equal and opposite reacting pull exerted upon him greater than his weight, thus giving his center of mass an upward acceleration. Since the cord exerts upon the counterweight a pull equal to that exerted upon the monkey, the two bodies will have equal upward accelerations at every instant; and since both are initially at rest, they will always have equal velocities, and will move equal distances in any time. This agrees with the answer attributed to Clifton and Harcourt.

(*b*) *Effect of the Inertia of the Pulley.*—Let the same assumptions be made as in (*a*) except that the inertia of the pulley is not neglected. Let m denote the mass of the pulley, r its radius, mk^2 its moment of inertia about its axis of rotation, M the mass of the monkey and that of the counterweight, T and T_1 the upward pulls exerted by the cord on the counterweight and monkey respectively, a and a_1 their upward accelerations; the angular acceleration of the pulley will be a/r (assuming that the cord does not slip on the pulley). The following dynamical equations may be written:

For the monkey,

$$Ma_1 = T_1 - Mg;$$

for the counterweight,

$$Ma = T - Mg;$$

for the pulley,

$$mk^2a/r = (T_1 - T)r.$$

From these equations, $T_1 - T = M(a_1 - a) = mk^2a/r^2$; $a_1 = a(1 + mk^2/Mr^2)$. The accelerations of the two bodies are thus in a constant ratio, and the velocities acquired and distances described in any time, starting from rest, will be in the same constant ratio as the accelerations.

(*c*) *Effects of Friction and Weight of Cord.*—The weight of the cord causes the tension to vary between the pulley and each suspended body; let the values T, T_1 now refer to the points of tangency of the cord and pulley. Let F denote the difference between T and T_1 which will just maintain uniform rotation of the pulley; then F may be taken as a measure of the friction (including rigidity of the cord). Since F may vary with the velocity, let F_0 be its value for zero velocity (incipient motion). Initially the system is assumed to be at rest with $T = T_1 = Mg + W$, the last term representing the weight of cord below the pulley on each side; at the point of attachment of each of the suspended bodies the tension has the value Mg. In order to begin to climb (*i.e.*, to give to his center of mass an upward acceleration) the monkey has only to exert a pull greater than Mg; unless the pull exceeds Mg by more than F_0, $T_1 - T$ will not become greater than F_0, so that the pulley, cord and counterweight will remain at rest. The monkey will, however, move upward with increasing velocity so long as he maintains a pull which is greater than Mg by any amount whatever, and his velocity will continue undiminished even if the pull becomes equal to

Mg. If, therefore, the monkey wishes to climb the rope without disturbing the counterweight, he can do so if he is skillful enough to exert a pull greater than Mg but less than $Mg + F_0$. This conclusion is not affected by the inertia of the cord and pulley, nor by the weight of the cord.

It is conceivable, however, that the monkey might make such an exertion as to increase the tension by more than F_0; $T_1 - T$ would then become greater than F_0, so that the pulley would have an angular acceleration and the counterweight an upward acceleration. As soon as the counterweight has acquired any velocity it will continue to ascend with undiminished velocity so long as $T_1 - T$ is not less than F. The pull which the monkey would need to maintain to produce this result would be $Mg + F$ if the gravity forces remained balanced; actually the weight of the cord would immediately become unbalanced so that the requisite pull would be less than $Mg + F$ by an amount proportional to the length of cord that has passed the pulley (assuming that the free end of the rope does not reach the floor or other support). If, therefore, the object of the monkey is to raise the counterweight, he can accomplish it if he is able to exert a pull slightly greater than $Mg + F_0$ in order to start the motion. If he should relax his effort immediately, the system might be brought to rest by friction so that the effort would need to be repeated; but after the motion of the cord has proceeded far enough so that the unbalanced weight of the cord reaches the value F so as to overcome friction, the motion will continue without further effort on the part of the monkey. In this case if he wished to check the motion of the counterweight he would need to permit himself to have a downward acceleration; *i.e.*, he would have to produce the requisite decrease in the tension either by very active downward climbing or by letting go of the rope.

(*d*) *General Algebraic Solution.*—The dynamical basis of the foregoing discussion may be embodied in an equation. One method of procedure would be to write separate dynamical equations for the several bodies making up the system and then eliminate the internal forces, as was done above in (*b*). The final equation free from internal forces may, however, be written immediately by applying d'Alembert's principle to the entire system consisting of the monkey, counterweight, cord and pulley. Taking the axis of rotation of the pulley as axis of moments, let the sum of the moments of the mass-accelerations be equated to the sum of the moments of the external forces.

Let x denote the distance of the counterweight above its initial position at any instant, m' the total mass of the cord and ρ its mass per unit length; other notation being as above.

The moment of the weight of the rope is $2\rho gxr$, while the sum of the moments of all other gravity forces is zero, and the moment of the friction is $-Fr$. The only other external force is the normal axle pressure, the moment of which is zero.

The moments of the mass-accelerations are as follows: For the monkey, counterweight, and cord, $-Ma_1r$, Mar, and $m'ar$ respectively; for the pulley, mk^2a/r.

The equation is therefore

$$(M + m' + mk^2/r^2)ar - Ma_1r = 2\rho gxr - Fr,$$

or

$$a = \frac{2\rho gx + Ma_1 - F}{M + m' + mk^2/r^2}.$$

This equation holds when the counterweight is actually rising. Initially $x = 0$ and a will remain zero unless Ma_1 becomes greater than F_0; until this occurs F is to be regarded as representing the actual friction (less than the limiting value F_0) and is equal to Ma_1, so that the equation gives $a = 0$ as it should. After the counterweight has moved from its initial position so that x is no longer 0, a will have a positive value so long as $Ma_1 > F - 2\rho gx$. If $2\rho gx$ becomes $> F$, a will remain positive even if a_1 becomes 0. The equation shows, however, that a can be made negative by giving a_1 a sufficiently great negative value. The conclusions stated under (*c*) are in fact all implied by the above general equation.

The equation also covers the ideal cases (*a*) and (*b*); in the latter it is assumed that ρ, m' and F are zero, while in (*a*) m is also assumed to be zero.

Since $a = d^2x/dt^2$, the general equation is a differential equation of the second order, a_1 being a function of t which depends upon the activity of the monkey, while F is an unknown function of the velocity. If F is treated as constant the equation becomes linear, and may be solved if a_1 is constant or a known function of t.

Thus if, starting from the original balanced condition, a_1 keeps a constant value a_1' for a

certain time, the solution for this part of the motion is

$$x = \frac{Ma_1' - F}{2\rho g}\left[\cosh t\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right].$$

If, at the instant $t = t_1$, a_1 changes to the constant value a_1'', the solution for the ensuing motion is

$$x = \frac{Ma_1' - F}{2\rho g}\left[\cosh t\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right] + M\frac{(a_1'' - a_1')}{2\rho g}\left[\cosh (t - t_1)\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right].$$

Professor Huntington's case 1 was also solved by C. C. Wylie.

2863 [1920, 482]. Proposed by A. A. BENNETT, University of Texas.

From their generation as roulette curves, show that the two hypocycloids of five cusps drawn with common vertices, are such that each is the envelope of a chord of constant length suitably placed upon the other.

Show that for any odd prime p, the $(p - 1)/2$ distinct p-cusped hypocycloids with common vertices may be arranged in cycles, so that each is the envelope of a chord of constant length taken upon the succeeding curve of the cycle.

A solution of this problem appears on pages 371–373 of this issue of the Monthly.

NOTES AND NEWS.

It is to be hoped that readers of the Monthly will coöperate in contributing to the general interest of this department by sending items to H. P. Manning, Brown University, Providence, R. I.

Miss Gertrude I. McCain, of Oxford College, has been made professor of mathematics at Westminster College, New Wilmington, Pa.

Associate Professor J. V. McKelvey decided to remain at Iowa State College (compare *1921*, 285).

We are requested to state that our note regarding Associate Professor W. A. Wilson's promotion (*1921*, 332) is not in accordance with fact.

At the University of Michigan, Mr. J. P. Ballantine, of Pennsylvania State College, and Mr. W. M. Coates, of the University of Virginia, have been appointed instructors of mathematics.

Mr. J. C. Funk, of Tamalpais Polytechnic High School, Mill Valley, Cal., has been elected to the headship of the department of mathematics in the Santa Maria high school and junior college.

Associate Professor Emma L. Konantz, who has spent two years on leave of absence teaching in Peking University, has returned to her position at Ohio Wesleyan University.

Mr. H. K. Cummings, instructor of physics at the Worcester Polytechnic Institute, and experimental physicist in the research laboratories of the Acheson Graphite Company, has been appointed instructor of mathematics at Brown University.

Mr. F. W. Winters, of Mount Allison University, but recently a student instructor at Yale University, has been appointed assistant professor of mathematics at Dalhousie College, Halifax, Nova Scotia.

Assistant Professor J. D. BOND, of the Agricultural and Mechanical College, College Station, Texas, has been appointed associate professor of mathematics in the Louisiana State University.

Mrs. ETHELWYNN R. BECKWITH, formerly assistant professor of mathematics in the College for Women, Western Reserve University, has been appointed acting assistant professor of mathematics in Vassar College.

Dr. F. R. MORRIS, for three years instructor of mathematics in the University of California, has been appointed head of the department of mathematics in the State College of Fresno, which has been recently formed from the normal school and junior college, and which is working under the supervision of the university.

Publications of the Astronomical Society of the Pacific announces that Dr. H. M. JEFFERS, of the University of California, fellow at the Lick Observatory during 1920, passed examinations for the doctorate last January, and has since that time been instructor of mathematics and astronomy at the University of Iowa.

Mr. C. P. ROCKWELL, who studied pure and applied mathematics at the University of Texas, and after graduating at New York University was employed two years in the actuarial department of one of the large insurance companies, has been reappointed state actuary of the Texas department of insurance and banking.

At the University of Saskatchewan, Saskatoon, Professor G. H. LING has resumed his duties as professor of mathematics and dean of the Faculty of Arts [see *1920*, 437]. Dr. L. L. DINES, who has been a junior professor of mathematics [*1919*, 84; also *1914*, 343], was on July 1, 1921, promoted to a senior professorship. He has also been granted leave of absence for the second half of the academic year 1921–1922.

At Purdue University, Professor WILLIAM MARSHALL has been made acting head of the department of mathematics. Instructors C. S. DOAN and F. H. HODGE have been promoted to assistant professorships and Messrs. J. W. BRANSON, E. G. KELLER, J. J. KNOX, and J. H. SHOCK have been appointed to instructorships. The mathematics staff now consists of twenty-one men. Approximately eighteen hundred students are taking mathematics.

THEODORE LYMAN, who (except for service with the A. E. F.) has been teaching at Harvard University since 1902, has been appointed professor of mathematics and natural philosophy. His immediate predecessors in this chair were B. O. PEIRCE, 1888–1914, and W. C. SABINE, 1914–1919. Dr. Lyman has been director of the Jefferson Physical Laboratory since 1910, and was appointed professor of physics in 1917.

In the department of mathematics at the U. S. Naval Academy, Assistant Professor H. M. ROBERT, Jr., has been promoted to an associate professorship and Instructors G. F. ABRICH, R. P. JOHNSON, R. C. LAMB, E. S. MAYER, and J. B. SCARBOROUGH have been promoted to assistant professorships. Dr. G. H. CRESSE (cf. *1919*, 276, 420) has resigned from the staff to accept an associate professorship at the University of Arizona, and Mr. C. D. GREGORY, formerly

of the faculty at Baltimore Polytechnic Institute, and last year a graduate student at the Johns Hopkins University, has been appointed to an instructorship. The 1975 midshipmen in the three classes that receive instruction in mathematics at the Academy are handled in 140 sections.

It was announced in *Science* that R. A. MILLIKAN, professor of physics at the University of Chicago, has been appointed director of the Norman Bridge Laboratory of Physics at the California Institute of Technology, Pasadena, and chairman of the Executive Council of the Institute. He will commence his duties immediately. In order to supplement work in mathematical physics now carried on there by Professor HARRY BATEMAN, Professor H. A. LORENTZ, of the University of Leyden, will be a lecturer and research associate for two months next winter, and Dr. C. G. DARWIN, fellow and lecturer of Christ's College, Cambridge, has been appointed professor of mathematical physics at the Institute for the year 1922–1923.

Dr. HENRI LEBESGUE, recently professor at the University of Paris, has been appointed as successor to HUMBERT (*1921*, 237), professor of mathematics in the Collège de France.

HOWARD ROBERTSON PARK, who became a member of the Association in 1918 (*1918*, 381), died November 12, 1919. He was born at Mount Meigs, Ala., June 28, 1891, and graduated A.B., 1910, from Southern University, where he taught during his senior year. He attended the University of Texas, 1911–12, and summer sessions of the University of Chicago, 1916–1918. He taught in high schools of Texas, New Mexico and California before his appointment in September, 1917, as head of the department of mathematics in the Polytechnic High School, Riverside, Cal. This position he held at the time of his death.

ANNA IRWIN YOUNG, for twenty years head of the department of mathematics in Agnes Scott College, Decatur, Ga., and a charter member of our Association, died September 3, 1920. She was born in Bloom Township (now Chicago Heights), Illinois, November 25, 1873. As a student she attended Westminster College, Pa., 1892–1893; Agnes Scott Institute, Atlanta, 1893–1895; and the summer school of the University of Chicago in 1898 and 1901. In 1898 she received the degree of A.B. from Agnes Scott College, and in 1914 the degree of A.M. from Columbia University. She taught during two summers in the summer school of the University of Georgia.

GEORGE WENTWORTH, one of the authors of the Wentworth-Smith Mathematical Series, died suddenly of heart disease on August 26, 1921. He was born at Exeter, N. H., on January 8, 1868, and was the son of GEORGE A. WENTWORTH (1835–1906), author of the well-known textbooks which bore the Wentworth name, and who for a long time was at the head of the mathematics department of the Phillips Exeter Academy. George Wentworth prepared for college at Exeter and entered Harvard University. In his senior year he left college and took up commercial work in the West, but he later returned to join his father in the writing of mathematical textbooks. He developed great ability in the technique

of this work, and on his father's death he purchased the Wentworth series. In 1913 he and Professor D. E. Smith entered into partnership and founded the Wentworth-Smith Series.

Rev. JOHN BASCOMBE LOCK, fellow and bursar of Gonville and Caius College, Cambridge, England, died September 8, 1921. Born March 18, 1849, he was third wrangler in the mathematical tripos of 1872, assistant master at Eton, 1872–1884, lecturer on mathematics and tutor at Caius College, 1884–1889. He was the author of many well-known texts in arithmetic, geometry, trigonometry, and mechanics.

Colonel JOHN HERSCHEL, youngest son of Sir JOHN F. W. HERSCHEL, died on May 31, 1921, and he was buried at Upton church where lie the remains of his grandfather WILLIAM HERSCHEL; his father was buried at Westminster Abbey. He was born at Cape of Good Hope, October 29, 1837. From 1859 to 1886 he was connected with the Trigonometrical Survey of India, and the Royal Society employed him to observe spectroscopically the eclipses of the sun in 1868 and 1871. He also observed many of the southern nebulæ with the same instruments.

Dr. EMILE BOREL, professor of calculus of probabilities and mathematical physics at the University of Paris, who has received the degree "doctor honoris causâ" from the University of Dublin, is the first Frenchman to be so honored. A Canadian, two Scots and two Irishmen have received similar degrees, but in each case the award was to a man of letters.

The Smith's Prizes at Cambridge University this year have been awarded to L. A. PARS, Jesus College, for an essay on "The general theorem of relativity," and to W. M. H. GREAVES, St. John's College, for an essay on "Periodic orbits in the problem of three bodies." These two annual prizes, now amounting to about £ 23 each, were founded by Robert Smith (1689–1768), master of Trinity College, elected in 1716 to succeed Cotes as Plumian professor of astronomy at Cambridge. They have been awarded annually, since 1769, except in 1884, to "two commencing Bachelors of Arts, the best proficients in mathematics and natural philosophy," determined by essays of greatest merit. New regulations were promulgated in 1884, and again in 1909 when Raleigh prizes were awarded on a similar basis. H. BATEMAN was a Smith's prizeman for 1905, J. H. JEANS for 1901, and R. C. MACLAURIN for 1897.

On May 17, 1921, Dr. O. D. KELLOGG, associate professor of mathematics at Harvard University, lectured before the Providence Engineering Society on "Submarine listening devices."

Professors G. D. BIRKHOFF and L. E. DICKSON are to lecture in the summer school of the University of California in 1922.

We have recently made numerous references to the activities of Professor SOLOMON LEFSCHETZ, of the University of Kansas (*1920*, 339, 340, 436, 440; *1921*, 241, 288, 384). His lectures at the University of Rome, in April and early

May, 1921, were on "Analysis situs and algebraic geometry," and are to form the basis of a monograph due to appear in the Borel series.

Among the papers read at the twenty-sixth meeting of the American Astronomical Society, held at the Van Vleck Observatory, Middletown, Conn., August 30–September 1, 1921, were the following: "New measures of solar activity and the 'earth-effect'" by L. A. BAUER; "A theory for the Trojan group of asteroids" by E. W. BROWN; "Gilbert's bombardment hypothesis" by J. L. COOLIDGE; "Parallaxes of 65 stars" by J. A. MILLER.

At meetings of the Royal Society of Canada, mathematical, physical and chemical section, held at Ottawa, May 18–20, 1921, the papers presented included the following: "Division in relation to the algebraic numbers," presidential address by J. C. FIELDS; "On the reduction of the circulants to polynomial form with applications to the circulants of the 7th and 11th degrees" by J. C, GLASHAN; "The gravitation potential of an anchor ring; some tidal problems" by A. H. S. GILLSON; "The solution of plane triangles by nomographic charts" by S. D. KILLAM; "Note on the geometrical equivalent of certain invariants" by C. T. SULLIVAN.

On October 24, 1921, a bill to incorporate the American Mathematical Society in the District of Columbia was introduced into the Senate of the United States by Mr. Lodge. It was twice read and referred to the committee of the judiciary. —*Congressional Record,* October 24, 1921.

In order to provide an enduring memorial for the 127 Field Service men who lost their lives in the great war, American Field Service Fellowships for French Universities were established (compare *1921*, 44). The fellowships for 1922–23, not to exceed 25 in number, are tenable for one year by citizens of the United States, and are to be of the value of $200 plus 10,000 francs. Under certain circumstances the fellowships are renewable a second year. The 31 fields of study possible for fellows include Mathematics, Engineering, and Astronomy. Prospective applicants for these fellowships should communicate with the secretary of the Trustees, Dr. I. L. Kandel, 522 Fifth Avenue, New York City. The next awards are to be made early in 1922.

The University of Rome has become one of the great mathematical centers of the world through the following recent appointments to its chairs: FEDERIGO ENRIQUES, of the University of Bologna, FRANCESCO SEVERI, of the University of Padua, and GIUSEPPE BAGNERA, of the University of Palermo. VITO VOLTERRA, GUIDO CASTELNUOVO, and TULLIO LEVI-CIVITA, were already professors there.

Published January 26, 1922.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Sixth Summer Meeting of the Association, Wellesley College, September 6–8, 1921

Seventh Annual Meeting, Harvard University, December, 1922

The following are dates of Section meetings of the Association in 1921 (unless otherwise specified):

ILLINOIS, Univ. of Chicago, December 29, 1920

IOWA, Simpson College, Indianola, April 30, Des Moines, November 4

KANSAS, State Agricultural College, Manhattan, April 3, 1920; Topeka, January 22

KENTUCKY, Danville, April 17, 1920; University of Kentucky, May 7

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Washington, D. C., May 7; The Johns Hopkins University, December 10

MINNESOTA, St. Catherine's College, St. Paul, June 5, 1920; College of St. Thomas, St. Paul, June 4

MISSOURI, Washington University and Soldan High School, St. Louis, November 25–26; Kansas City, November, 1922

OHIO, Columbus, April 2, 1920; Columbus, March 25–26

ROCKY MOUNTAIN, Colorado College, Colorado Springs, April 2, 1920; Denver, March 25–26

TEXAS, Dallas, November 25

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By MILDRED E. CARLEN, Brown University.

Misprinted names in the text are corrected, and missing initials supplied, in this index. The authors of certain anonymous contributions are also indicated.

PAPERS, REPORTS OF MEETINGS—AUTHORS.

PAPERS, REPORTS OF MEETINGS—SUBJECTS.

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PROBLEMS—AUTHORS.

Numbers refer to pages, black-face type indicating a problem solved and solution published, italics a problem solved but solution not published, ordinary type a problem proposed.

PROBLEMS—SOLUTIONS.

Numbers in black-face type refer to problems, those in light-face to pages.

NOTES AND NEWS.

PERSONAL MENTION.

[In order to find a complete list of references to any individual, the alphabetical lists under "Papers, Reports of Meetings—Authors," "Questions and Discussions—Authors," "Recent Publications—Reviews," "Recent Publications—Notes—Books," "Recent Publications—American Doctoral Dissertations," and "Problems—Authors" should also be consulted.]

PERSONAL MENTION—NECROLOGY.

ADDENDA AND CORRIGENDA.

P. 34, l. 20, *for* "482" *read* "48_2."
P. 43, l. 11 from bottom, *for* "in August" *read* "August 3."
P. 50, l. 12, *for* "Figure 1, *Q*, shows" *read* "Figure 1 shows *Q*."
P. 55, l. 18, *for* "Fig. 4" *read* "Fig. 1."
P. 80, l. 3, *for* "133–144" *read* "133–134."
P. 80, *for* footnote 2 *read* "Such a work, *Exponentials made Easy* by M. E. J. Gheury DeBray, was published in 1921."
P. 82, l. 5, *for* "tome 2" *read* "tome 5."
P. 82, l. 6, *for* "349–271" *read* "249–271."
P. 91, l. 10 from bottom, *for* "Bourger" *read* "Bouguer."
P. 185, l. 16 from bottom, *for* "quadrilatered" *read* "quadrilateral."
P. 191, l. 7 from bottom, *for* "Englishmen" *read* "Britishers."
P. 192, l. 12 from bottom, *for* "has" *read* "have."
P. 206, l. 3 from bottom, *for* "1883" *read* "1888."
P. 219, l. 4, *for* "effected.—There" *read* "effected. There."
P. 219, l. 5, *for* "libraries. The" *read* "libraries.—The."
P. 225, l. 10, *delete* "**PURES ET APPLIQUÉS.**"
P. 267, ll. 2–3, *add:* "Legendre gave the same table (without first, second, and third differences) in his *Exercices de Calcul Intégral*, volume 1, Paris, 1811, pp. 302–306."
P. 282, l. 18 from bottom, *for* "Captain" *read* "Lieutenant Colonel."
P. 284, ll. 18–19, interchange "2^1" and "2^6" at end of lines.
P. 284, l. 3 from bottom, *for* "MATHEWS" *read* "MATHEWSON."
P. 288, l. 27, *for* "KENELLY" *read* "KENNELLY."
P. 325, l. 3 from bottom, *for* "or" *read* "on."
P. 332, l. 9 from bottom, *for* "R." *read* "F."
P. 332, *delete* ll. 24–25 regarding W. A. WILSON.
P. 356, l. 7 from bottom, *for* "this" *read* "his."
P. 375, l. 17, *for* "By $[r/2]$ is meant the largest integer in $r/2$." *read* "By $[x]$ is meant the largest integer in x."

1920.

P. 431, l. 4 from bottom should be a footnote by the Editor, referring to "solutions," l. 6 from bottom.
Agnis is spelled incorrectly as Agins and Aguis, and Dantzig as Danzig. See Index.

1919.

P. 359, l. 2 from bottom, *for* "the actual was needed" *read* "the actual tool was needed.

Published April 22, 1922.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXVIII, 1921

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THE APRIL MEETING OF THE IOWA SECTION.

The eighth regular meeting of the Iowa Section was held, in conjunction with the twenty-fifth annual meeting of the Iowa Academy of Science, at Simpson College, Indianola, on April 30, 1921.

The attendance was seventeen, including the following fifteen members of the Association:

Julia T. Colpitts, C. W. Emmons, M. E. Graber, E. C. Kiefer, G. E. King, F. M. McGaw, J. V. McKelvey, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, W. J. Rusk, B. F. Simonson, F. M. Weida, C. W. Wester.

The following officers were elected for the ensuing year: W. J. Rusk, Chairman; C. W. Emmons, Vice-chairman; and J. F. Reilly, Secretary-treasurer. It was voted to accept the proposal of the Iowa Academy of Science that the Chairman of the Iowa Section become a member of the executive committee of the Academy, with a view to a closer coöperation of the two bodies. In accepting the proposal it was understood that the Iowa Section's independence, or relation to the Association, was in no way impaired.

Professor McKelvey's committee on the reports of the National Committee on Mathematical Requirements was continued.

The following papers were presented:

(1) "Correlation between mental tests and grades in mathematics of freshman engineering students" by Professor Maria M. Roberts, Iowa State College;

(2) "Playing with the sine and projection formulas" by Professor W. J. Rusk, Grinnell College;

(3) "Certain summation formulas" by Professor J. F. Reilly, State University of Iowa;

(4) "Some properties of the function $w = \tanh z$" by Mr. F. M. Weida, State University of Iowa;

(5) "Derived solutions of differential equations" by Professor M. E. Graber, Morningside College;

(6) "The surface $z = \log_y x$" by Professor C. W. Emmons, Simpson College;

(7) "Circles mutually tangent and tangent to concentric circles at specific points" by Professor C. W. Wester, Iowa State Teachers College;

(8) "A study of certain reports of the National Committee on Mathematical Requirements." Committee report by Professor J. V. McKelvey, Chairman.

Abstracts of papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Professor Roberts reported on a study made of the college records of 337 freshman engineers who took the Thurstone engineering intelligence test. Correlations were found, first, between college mathematics and the first four tests, arithmetic, algebra, geometry and general intelligence; second, between college mathematics and the first three, arithmetic, algebra and geometry; and third,

between college and high school algebra. The correlation coefficients are found to be .40, .45 and .44, respectively, showing that the mathematics portion of the engineering test is as good a means of judging the possibilities of college freshmen in mathematics as any measure we now have. These tests are still in a formative state and will doubtless eventually be valuable to college authorities.

2. Professor Rusk gave (*a*) various proofs of the formulas for $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$, $\tan(\alpha \pm \beta)$; (*b*) proofs for the half-angle formulas; and (*c*) proofs for the Mollweide formulas, showing that they themselves are half-angle formulas.[1]

3. Professor Reilly proved the summation formula

$$\begin{aligned}\frac{n!}{m!}\sum_{m} x(x-1)\cdots(x-m+n+1) &= \frac{(n+1)!}{(m+1)!}\sum_{m+1} x(x-1)\cdots(x-m+n+1)\\ &+ \frac{(n+2)!}{2!(m+1)!}\sum_{m+1} x(x-1)\cdots(x-m+n+2)\\ &+ \frac{(n+3)!}{3!(m+1)!}\sum_{m+1} x(x-1)\cdots(x-m+n+3)\\ &+\cdots+\frac{(m+1)!}{(m-n+1)!\,(m+1)!}\end{aligned}$$

for $0 \leqq n \leqq m$, where $\sum_{m} x(x-1)(x-2)\cdots(x-n+1)$ denotes the sum of all the products containing n factors each that can be formed from the m factors $x, x-1, x-2, \cdots x-m+1$, and where $\sum_{m} x(x-1)\cdots(x+k)$ equals 1, or 0, according as $k = 1$, or $k \geqq 2$.

4. Professor Weida showed that the function $w = \tanh z$ is analytic, automorphic, single valued, continuous, and holomorphic, everywhere except at its poles. Among other things he discussed its zeros, its maximum and minimum values, and the mapping of the z-plane on the w-plane.

5. The usual course in introductory differential equations is concerned simply with the problem of formal integration and pays but little attention to the origin or development of the differential equations of physics and mechanics. In this paper Professor Graber developed the differential equations for mechanical and electrical vibrating systems, exhibiting the anatomy and structure of the equations as an aid in interpreting their solutions. A classification of linear differential equations with their derived solutions was then outlined and an abbreviated solution for the non-homogeneous linear differential equation in series form presented. The differential equations considered were all of the form $(d^ny/dx^n) + A_1(d^{n-1}y/dx^{n-1}) + \cdots A_ny = X$ where $A_1, \cdots, A_n$, and X are functions of x, and their solutions were classified by the application of the principle that if $\phi(m, x)$ is a solution and n values of m are equal to a, then the first $(n-1)$ partial derivatives of $\phi(m, x)$ with respect to m are solutions when m is replaced by a after differentiation.

[1] See p. 443.

6. A model of this surface was exhibited by Professor Emmons as a guide to the study of its properties. The ranges of x and y considered were $x \geqq 0$ and $y \geqq 0$; z may vary from negative infinity to positive infinity. Various plane sections were studied and the points of discontinuity and singularity were investigated. A system of ruled surfaces, having the given surface as an envelope, was set up.

7. Given OA_1, OA_2 fixed radii of two circles c_1, c_2, and O_1, O_2 as variable centers of two mutually tangent circles c_3, c_4 tangent to c_1, c_2 respectively at A_1, A_2; then O_1, O_2 will be the corresponding elements of a projective correspondence between the points of the lines OA_1 and OA_2. The lines O_1O_2 will be tangents to a circle c_5 tangent to OA_1 and OA_2. The locus of the point of contact of circles c_3, c_4 is a circle, c_6, concentric with c_5 and passing through A_1 and A_2. Professor Wester extended this result to non-concentric circles c_1, c_2, and special cases were discussed.

B. F. SIMONSON, *Secretary-Treasurer.*

THE MAY MEETING OF THE KENTUCKY SECTION.

The fifth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Ky., on May 7, 1921. The meeting consisted of one session with Professor H. H. DOWNING presiding.

The attendance was eighteen, including the following six members of the Association: P. P. Boyd, J. M. Davis, H. H. Downing, Elizabeth LeStourgeon, E. L. Rees, C. H. Richardson.

The members and guests were entertained at luncheon by Professor H. H. Downing, the retiring chairman of the Section. Professor C. H. RICHARDSON was elected chairman of the Section and Professor ELIZABETH LESTOURGEON, secretary. Upon the invitation of Professor Richardson, it was voted to hold the next meeting at Georgetown College.

The following papers were presented:

(1) "Objective measurements of results of teaching mathematics" by Professor J. W. BRANSON, Centre College (by invitation);

(2) "Geometric proof of a theorem concerning the roots of a quartic" by Professor E. L. REES, University of Kentucky;

(3) "The influence of modern trends in education upon mathematics as a school subject" by Mr. H. M. YARBROUGH, Western Kentucky State Normal School (by invitation);

(4) "The need and content of a course in commercial mathematics" by Professor C. H. RICHARDSON, Georgetown College;

(5) "History of arithmetic and algebra" (illustrated) by Dean P. P. BOYD, University of Kentucky.

Abstracts of the papers and discussions follow below, the numbers corresponding to the numbers in the list of titles:

1. Professor Branson's discussion was based on the assumption that practically every teacher depends upon rather vague impressions for his estimates of the progress of his students, whereas he should have some well-defined standards for measuring results objectively. A method which was recommended was the analysis of each assignment into the distinct processes involved, and a record of the success of each student in dealing with these processes. The idea was not so much to determine the grade to which the student is entitled, as to determine what processes the student has mastered as a result of previous methods of instruction with a view to improving those methods which produce poor results.

2. Professor Rees gave a geometric proof of the theorem stating the conditions which govern the nature of the roots of the quartic equation $x^4+qx^2+rx+s=0$. Having shown that the roots of the equation are the abscissas of the points of intersection of the line $y=-rx$ and the symmetric quartic $y=x^4+qx^2+s$ and that the discriminant equals a constant times the product of the vertical distances from the line $y=-rx$ to the points of tangency of the parallel tangents, the various cases of the theorem were then proved from diagrams exhibiting the different forms and positions of the quartic curve.

3. In the opinion of Mr. Yarbrough, the modern tendency to over-emphasize the so-called practical and the vocational in education leads to an indifferent attitude on the part of the student toward any mathematics that does not seem to him to be capable of functioning directly in his everyday life. The student is not encouraged to strive for intellectual training and achievement, but rather to strive for industrial efficiency alone, and this is held up to him as an ideal of education. Since only the more elementary branches of mathematics are essential to the average individual, the average student sees no value in the subject and hence his work is mediocre in character. The current system of measuring the student's progress by credit leads the student to regard "units" and "hours" rather than mental achievement as a measure of his education. As a result he elects those subjects in which he can obtain the maximum amount of credit by a minimum expenditure of effort. In order to secure interested and efficient work, teachers of mathematics must find some way to lead the student to appreciate the cultural as well as the practical value of mathematics.

4. Professor Richardson showed that there is a growing recognition on the part of leading financiers and bankers of the need for an understanding of the mathematical theory underlying their work; that in the actuarial field this is particularly true. He indicated how the mathematics departments of colleges might take advantage of this opportunity, and outlined a course in commercial mathematics which would be of advantage not only for the classes specified but to those entering upon other than commercial pursuits.

5. Dean Boyd gave an outline statement of the achievements of the Egyptians, the Babylonians, the Greeks, the Hindus, the Arabs and the Christian Europeans up to the time of Descartes. Illustrations of ancient methods and problems and some idea of the progress of mathematical thought were given. Among many slides shown were a number picturing pages from famous manuscripts.

ELIZABETH LESTOURGEON, *Secretary.*

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The ninth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held in the Drafting Hall of the U. S. Capitol, Washington, D. C., on May 7, 1921. The meeting consisted of two sessions with Professor L. S. Hulburt presiding.

The attendance was fifty-five including the following thirty-four members of the Association: O. S. Adams, J. J. Arnaud, R. N. Ashmun, H. G. Avers, Sarah Beall, A. A. Bennett, G. A. Bingley, C. C. Bramble, P. Capron, A. Cohen, G. H. Cresse, F. W. Darling, L. S. Dederick, A. Dillingham, H. English, J. B. Eppes, A. Hall, W. M. Hamilton, W. E. Heal, L. S. Hulburt, W. D. Lambert, A. E. Landry, J. J. Luck, E. S. Mayer, L. N. Morscher, C. A. Mourhess, F. D. Murnaghan, J. R. Musselman, O. J. Ramler, C. H. Rawlins, Jr., R. E. Root, C. A. Shook, T. McN. Simpson, Jr., C. E. Van Orstrand.

A luncheon was served to the members and their guests by the Coast and Geodetic Survey and the International (Canadian) Boundary Commission. Action was taken authorizing the secretary-treasurer to prepare an amendment to the constitution giving Virginia representation on the executive committee, this amendment to be submitted to the members of the Section for final action at the next meeting. The following officers were elected for 1921–1922: O. S. ADAMS, Chairman; G. R. CLEMENTS, Secretary-treasurer; F. D. MURNAGHAN, Member of the executive committee.

The next meeting will be held at Johns Hopkins University, Baltimore, Md., probably early in December.

The following papers were presented:

(1) Address of welcome by Colonel E. L. JONES, Director, United States Coast and Geodetic Survey;

(2) "Relations between the metric and projective theories of curves" by Professor T. McN. SIMPSON, Jr., Randolph-Macon College, Ashland.

(3) "Shearing stress in thick cylinders" by Professor R. E. ROOT, U. S. Naval Academy;

(4) "Mechanical prediction of tides" by Mr. W. D. LAMBERT, Coast and Geodetic Survey;

(5) "Suggestions as to improvements in text-books on mathematics" by Professor ANGELO HALL, U. S. Naval Academy;

(6) "A geometrical problem in maxima and minima" by Professor F. D. MURNAGHAN, Johns Hopkins University;

(7) "On Fermat's last theorem" by Mr. W. E. HEAL, Coast and Geodetic Survey;

(8) "An arithmetical study of regular solids" by Doctor G. H. CRESSE, U. S. Naval Academy;

(9) "An elementary method of computing logarithms" by Professor L. S. DEDERICK, U. S. Naval Academy;

(10) "A note on the history of non-euclidean geometry" by Professor ABRAHAM COHEN, Johns Hopkins University;

(11) "Imaginary points in geometry" by Professor A. A. BENNETT, Technical Staff, Army Ordnance.

Abstracts of papers numbered in accordance with the above list of titles are given below.

2. Professor Wilczynski has developed a convenient machinery for the study of projective differential properties of plane and space curves by the use of linear differential equations of the third and fourth order respectively. He has also shown that specialization of the variables employed and of the transformations permitted, makes it possible to secure metric results in a similar manner. Professor Simpson undertook briefly to indicate the method of Professor Wilczynski and to give some metric results obtained by developing it in the case of space curves defined by their intrinsic equations. He considered a number of osculants of space curves and suggested how they might be regarded as images of the radii of curvature and torsion and their derivatives. His results are published in a paper issued by the University of Chicago libraries.

3. Professor Root discussed the stresses and strains in thick cylinders, particularly under conditions present in gun tubes. The usual theory is based on a thick, hollow, right circular cylinder, with uniform external and internal pressure, and with uniform normal stress on the ends. The stress distribution at a transverse section is expressed by two fundamental equations known as Lamé's laws. Professor Root raised a question as to the application of these laws in the presence of such varying pressures as are produced in built-up guns. He showed that if internal and external pressure vary uniformly from one end of the cylinder to the other, then the stress distribution at any transverse section is given by Lamé's laws. Otherwise, these laws do not hold, the normal stress is not uniform over the section and shearing stress is present, radial on transverse planes and longitudinal on tangential planes.

4. Mr. Lambert's paper on the mechanical prediction of tides gave a brief account of the principles underlying the harmonic analysis of tidal data and of the two tide-predicting machines of the Coast and Geodetic Survey, which were exhibited later in the day to those attending the meeting. The height of the tide is expressed as the sum of a series of terms of the form $A \cos (at + \alpha)$, where t is the time from the beginning of the year for which predictions are to be made; the quantity a, called the "speed," is obtained from astronomical considerations; the quantities A and α, the former being the amplitude of the component oscillation and the latter its phase at the beginning of the year of prediction, are obtained for any given port from a discussion of the observations there by the methods of the harmonic analysis. The tide-predicting machines sum these cosine terms mechanically. The smaller or Ferrel machine, now no longer used, is said to do the work of about 40 computers. The larger machine provides for nearly twice as many cosine terms as the Ferrel machine and gives the time-derivative of the heights as well as the heights themselves, thus indicating—by

the zero values of the derivative—the exact times of high or low water. The larger machine may be said to do the work of from 80 to 100 computers.

5. The many failures to pass the examinations in arithmetic, algebra, and plane geometry, for entrance to the Naval Academy, show the low standards of mathematical study obtained in our high schools. Professor Hall suggested three remedies: (*a*) the weeding out of incompetent pupils, (*b*) the securing of good teachers, and (*c*) the improvement of text-books.

6. Dr. Murnaghan discussed the problem of the determination of the points in space the sum of whose distances, taken positively or negatively, from the four vertices of a tetrahedron is a minimum (Fermat). If tetrahedral coördinates are used, and we avail ourselves of the focal transformation $y = 1/x$, there are eight Fermat points y corresponding to the eight intersections of the three quadrics $x_1^2 + x_2^2 + 2c_{12}x_1x_2 = x_3^2 + x_4^2 + 2c_{34}x_3x_4$, etc., where c_{rs} is the cosine of the internal dihedral angle between the faces r and s of the tetrahedron. The points x are such that their pedal tetrahedra have opposite edges equal, i.e., have *congruent faces*. In the plane Fermat problem (cf. this MONTHLY, *1920*, 38–41) there are two points y and two points x; these latter are the Hessian points whose pedal triangles are *equilateral*. Reference is made to a forthcoming paper by C. M. Sparrow in the *American Journal of Mathematics*.

7. Mr. Heal's paper gave a short review of various methods that have been proposed for the demonstration of the theorem, paying particular attention to that of Kummer and expressing the belief that this method cannot lead to a general demonstration. Other methods were suggested as possible for leading to a solution of the problem.

8. Dr. Cresse obtained the number of vertices of a regular polyhedron by comparing the two following expressions for the sum of the face angles: $(V - 2)2\pi$, which holds for all polyhedrons, and $V \cdot f \cdot (s - 2)/s \cdot \pi$, which holds for a regular polyhedron, in which f faces each having s sides meet in each vertex. The multiplicity of the other elements follows at once. Similarly, in 4-space, the ratio of V to E for a regular polyhedroid is obtained by comparing two independent expressions for $\Sigma P - 2\Sigma D$, where P is the magnitude in spherical degrees of each polyhedral angle and D is the magnitude in circular degrees of each dihedral angle of the bounding polyhedron. The multiplicity of all the other elements in terms of V follows at once for the six regular polyhedroids of 4-space.

9. The purpose of Professor Dederick's paper was to present a method of computing the common logarithm of any number to the base 10, which is sufficiently elementary to be easily intelligible to a student just taking up the study of logarithms. The method is based solely on the property of a logarithm as an exponent of 10 and involves nothing more advanced than the binomial theorem for positive integral exponents.

10. Professor Cohen reviewed a recent book by Dr. G. B. Halsted (*1921*, 28–30). Easy access to this work will now enable one better to estimate the place which Saccheri deserves to occupy in the history of the development of non-euclidean geometry.

11. Professor Bennett sketched briefly some of the reasons which have led to the introduction of imaginary points into geometry. He then showed that for some purposes a "complex" point, where the coördinates are complex numbers of algebra, is not the only convenient generalization beyond the set of real points of space. Another generalization involving a special class of square matrices with quaternionic elements satisfies the usual vectorial requirements, and may be used to define a "space" in which the notion of distance continues to have its characteristic properties. The "real" points of such a space constitute an ordinary real space, so that one may interpret familiar geometric loci as having not only real points but further points in such a generalized space.

O. S. ADAMS, *Secretary-Treasurer.*

THE JUNE MEETING OF THE MINNESOTA SECTION.

The regular meeting of the Minnesota Section was held at the College of Saint Thomas, St. Paul, Minn., on Saturday, June 4. The attendance was twenty-one, including the following thirteen members of the Association:

R. M. Barton, W. O. Beal, W. H. Bussey, H. H. Dalaker, Gladys Gibbens, W. L. Hart, Dunham Jackson, R. A. Johnson, Arvid Reuterdahl, Minna Schick, F. J. Taylor, Ella Thorp, Vera L. Wright.

Dinner was served at six o'clock with the College of Saint Thomas as host. At a business meeting which followed the following officers were elected: Chairman, Professor REUTERDAHL; Secretary-treasurer, Professor BARTON; Members of the executive committee, Professor DALAKER, Miss GIBBENS, Professor C. H. GINGRICH. The report of the Secretary-treasurer was approved as presented.

The following papers were presented:

(1) "Mathematical induction" by Professor W. H. BUSSEY, University of Minnesota.

(2) "Constructions with ruler and circular disk" by Professor R. A. JOHNSON, Hamline University.

(3) "Some problems in medicine which the mathematician must solve" by Dr. R. E. SCAMMON, University of Minnesota (by invitation).

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. Ordinary mathematical induction may be described briefly as an argument from n to $n+1$. Professor Bussey's note called attention to some generalizations of ordinary mathematical induction of which the most simple may be described as an argument from n and $n+1$ together to $n+2$.

2. Professor Johnson discussed the somewhat well-known proposition that all constructions which can be effected with ruler and compasses are possible with a ruler and a circular disk whose center is not given.

3. Dr. Scammon indicated how the applications of mathematics to the problems of the medical sciences may be divided roughly in two classes. Of

these the first are the applications which come through extension of the mathematical solutions of problems in physics and chemistry to similar problems in biology. Examples of this class are the application of the mathematical laws of hydrodynamics to problems of hæmodynamics, of mathematical optics to problems of physiologic optics, and of the various mathematical expressions for stress and strain to problems of muscle and bone mechanics. This phase of the application of mathematics is of long standing in medicine, having developed in the latter part of the seventeenth century. The second class of applications arose primarily through the attempt to reduce the highly variable data of medicine to some form of graphic or numerical expression. This is a more recent development since the collection of precise medical data (other than experimental records and vital statistics) practically began with the work of Louis in the early part of the nineteenth century. The mathematical treatment of this data was first attempted by Gavarret and Quetelet a little later. The modern development of this phase has proceeded mainly along the lines laid down by Galton and Karl Pearson. Among the problems in this field at the present time are the application of the mathematical principles of variation and correlation to a large and varied mass of biologic data, and the development of suitable methods of curve fitting, coördinate analysis and establishment of empirical formulæ for this type of material.

R. M. Barton, *Secretary-Treasurer.*

GRAPHICAL SOLUTIONS OF THE QUADRATIC, CUBIC, AND BIQUADRATIC EQUATIONS.

By T. R. RUNNING, University of Michigan.

Graphical solutions of equations may in general be divided into two classes. One class consists of geometrical constructions for each particular equation. The constructions depend upon the numerical values of the coefficients. Any such construction which will give the roots of an equation with a given set of coefficients must be changed for any equation having a different set of coefficients. The other class consists of charts or diagrams from which the roots are read when the coefficients or combinations of them are used as coördinates.

In what follows the solutions are given by means of charts. Such charts are made up of straight lines and curves from which the roots (both real and imaginary) are read approximately.

There are many such graphical solutions for the real roots of the simpler algebraic equations, but, so far as the writer is aware, the imaginary roots are obtained only by geometric constructions for each particular equation.[1] Such

[1] A few references are as follows:

Klein, *Vorträge über ausgewählte Fragen der Elementargeometrie*, 1895, pp. 28–31; in Beman and Smith's translation, p. 34.

d'Ocagne, *Traité de Nomographie*, 1899.

constructions are usually tedious to carry out when a number of equations are to be solved.

A single chart will suffice for the solution of the quadratic equation, another for the solution of the cubic, and a combination of these two charts will give the roots of the biquadratic.

Solution of the Quadratic. If in the equation

$$x^2 + Ax + B = 0 \tag{1}$$

a value is assigned to x it will represent a straight line, the coördinates of whose points are A and B. In the form

$$B = -xA - x^2 \tag{2}$$

Encyklopädie der mathematischen Wissenschaften, vol. 1, 1900–1904, pp. 1020–1050.

Brand, "Méthode graphique pour déterminer les racines réelles de l'équation $x^3 + px + q = 0$," *L'Enseignement Mathématique*, vol. 8, 1906, pp. 443–448.

Noaillon, "Résolution graphique de l'équation du troisième degré," *Mathesis*, vol. 7, 1907, pp. 122–125.

Gleason, "A simple method for graphically obtaining the complex roots of a cubic equation," *Annals of Mathematics*, second series, vol. 11, 1910, pp. 95–96.

Lenhardt, "Graphische Darstellung reeler und komplexer Lösungen von Gleichungen," *Zeitschrift für mathematischen und naturwiss. Unterricht*, vol. 44, 1913, pp. 116–122.

Deming, "A new method for the graphical solution of algebraic equations," *Science*, vol. 43, 1916, pp. 576–580.

Hewes, "Nomograms of adjustment," *Annals of Mathematics*, second series, vol. 18, 1917, pp. 194–199.

[Some additional references, for the most part of an earlier date, are as follows:

Bérard, *Opuscules mathématiques et Méthodes nouvelles pour déterminer les racines des équations numériques*, 1811, p. 33.

Monge, "Solution graphique de l'équation du troisième degré," *Correspondance sur l'Ecole . . . Polytechnique*, vol. 3, pp. 201–203, May, 1815. (Bérard and Monge used a cubic parabola for their graphical solutions.)

Gergonne, "De la résolution des équations numériques du 3^me degré par la parabole ordinaire," *Annales de Mathématiques Pures et Appliquées*, vol. 9, pp. 204–210, December, 1818.

Gerono, "Note sur la construction des racines de l'équation complète du quatrième degré," *Nouvelles Annales de Mathématiques*, vol. 3, 1844, pp. 533–536.

Scheffler, "Ueber die geometrische Construction der imaginären Wurzeln einer Gleichung," *Archiv der Mathematik und Physik*, vol. 15, 1850, pp. 375–389.

Vieille, "Sur la construction des racines de l'équation du quatrième degré par l'intersection d'une parabole et d'un cercle," *Nouvelles Annales de Mathématiques*, vol. 16, 1857, pp. 453–456.

Hoppe, "Construction der reellen Wurzeln einer Gleichung vierten oder dritten Grades mittelst einer festen Parabel," *Archiv der Mathematik und Physik*, vol. 56, 1874, pp. 110–112.

Hoppe, "Construction der imaginären Wurzeln einer Gleichung vierten oder dritten Grades mittelst einer festen Parabel," *Archiv der Mathematik und Physik*, vol. 69, 1883, pp. 216–218. (In the two papers by Hoppe the solutions are by means of a fixed parabola and a variable circle.)

Hofmann, "La solution géométrique de l'équation du quatrième degré," *Nouvelles Annales de Mathématiques*, vol. 67, 1888, pp. 120–133.

Heppel, "Quartic equations interpreted by the parabola," *Proceedings of the London Mathematical Society*, vol. 22, 1891, pp. 416–423. (Extension of, and supplement to, first of Hoppe's papers.)

Phillips and Beebe, *Graphic Algebra or Geometrical Interpretation of the Theory of Equations of one Unknown Quantity*. New York, 1904.

Irwin and Wright, "Some properties of polynomial curves," *Annals of Mathematics*, second series, vol. 19, 1917, pp. 152–158.—EDITOR.]

it is seen that $-x$ is the slope and $-x^2$ the intercept. Every point represented by (A, B) for real values of x will be the intersection of two lines. Such lines are drawn in Chart I and designated by the corresponding values of x.

As an illustration take the equation

$$x^2 + 2.8x - .6 = 0.$$

In order to find the roots of this equation locate the point $(2.8, -.6)$ and read the values corresponding to the lines passing through this point. The values are -3 and $.2$. In case that no line in the chart passes through the point interpolation is necessary.

Imaginary roots are determined as follows:

$$x^2 + Ax + B = 0,$$

$$x = -\tfrac{1}{2}A \pm \sqrt{(\tfrac{1}{4}A^2 - B)}.$$

Let $\frac{1}{4}A^2 - B = -K^2$, then $B = \frac{1}{4}A^2 + K^2$ represents a system of parabolas whose vertices are $(0, K^2)$, and whose axes coincide with the B-axis. These parabolas are drawn in Chart I and designated by the corresponding values of K, the coefficient of the imaginary unit i.

To find the roots of

$$x^2 + .6x + 2.69 = 0.$$

Locate the point $(.6, 2.69)$ and the real part of the roots read from the top of Chart I is $-.3$, the imaginary part read from the parabola passing through the point is $1.61i$ approximately. The roots are

$$-.3 \pm 1.61i.$$

The process of interpolation needs no explanation. If $K = 0$, that is, if

$$B = \tfrac{1}{4}A^2$$

the parabola becomes the envelope of the straight lines in the chart and forms the boundary between the region of real roots and the region of imaginary roots. This parabola is the region of equal roots.

Solution of the Cubic. The general equation of the cubic is

$$x^3 + ux^2 + vx + w = 0. \tag{3}$$

This is reduced to a cubic lacking the term of the second degree by the substitution

$$x = z + k,$$

where $k = -u/3$. Equation (3) becomes

$$z^3 + \left(v - \frac{1}{3}u^2\right)z + w - \frac{uv}{3} + \frac{2}{27}u^3 = 0. \tag{4}$$

Let $A = v - \frac{1}{3}u^2$, and $B = w - \frac{uv}{3} + \frac{2}{27}u^3$, then (4) becomes

$$B = -zA - z^3. \tag{5}$$

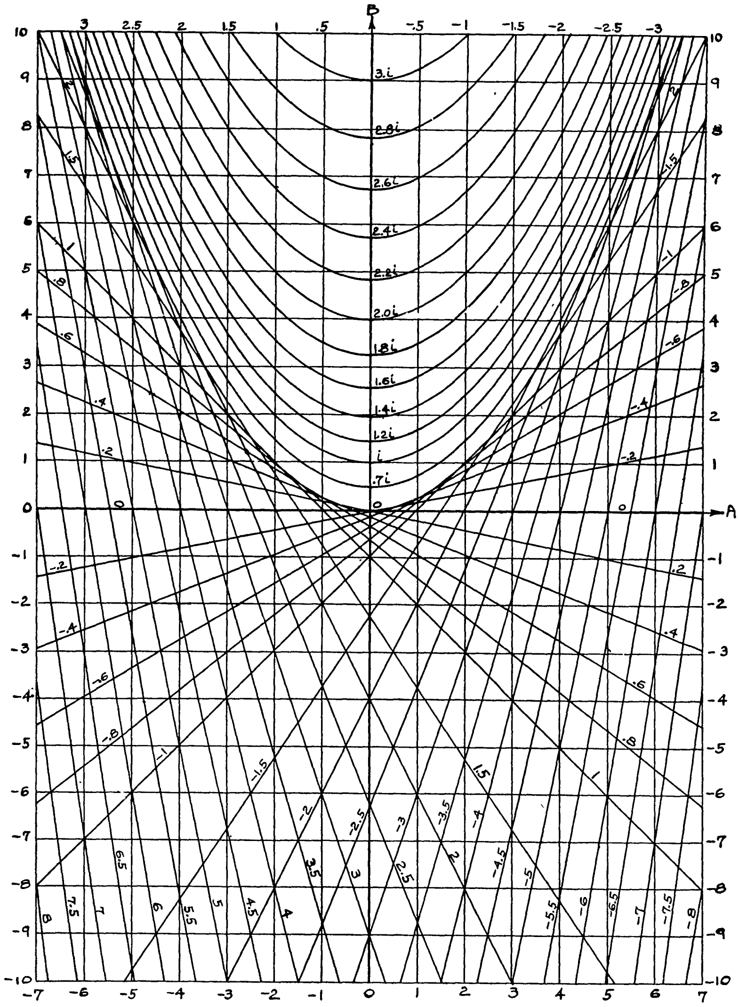

CHART I.

For given values of z equation (5) represents straight lines, the coördinates of whose points are A and B. The lines represented by this equation for different real values of z are drawn in Chart II. The real roots of a reduced cubic are the numbers designating the three lines passing through the point (A, B).[1]

[1] The method which Professor Running employs for finding the real roots of the quadratic and cubic equations was given by Lalanne in 1846. See the bibliographical footnote in connection with J. P. Ballantine's article, "A graphic solution of the cubic equation," in this MONTHLY, *1920*, 203.—EDITOR.

Take the equation

$$x^3 - 3x^2 + 1.25x + 1.5 = 0. \tag{5a}$$

By the substitution $x = z + 1$ this equation becomes

$$z^3 - 1.75z + .75 = 0. \tag{5b}$$

The roots of (5*b*) read from Chart II are -1.5, 1, and .5. The roots of (5*a*) are then $-.5$, 2, and 1.5.

In case equation (5) has imaginary roots, the three roots are of the form

$$-2a,\ a + bi, \text{ and } a - bi.$$

Also

$$A = b^2 - 3a^2, \tag{6}$$

$$B = 2a^3 + 2ab^2. \tag{7}$$

From (6)

$$a = \pm\sqrt{\frac{b^2 - A}{3}}.$$

This value of a substituted in (7) gives

$$B = \pm\frac{2}{3}\sqrt{\frac{b^2 - A}{3}}\,(4b^2 - A), \tag{8}$$

or

$$B^2 = -\frac{4}{27}(A - b^2)(A - 4b^2)^2. \tag{9}$$

For given values of b equation (9) represents cubic curves in A and B which are drawn in Chart II. For all points (A, B) along any curve the coefficient of i in the imaginary roots will be constant.

For $b = 0$ equation (9) forms the boundary between the region of imaginary roots and the region of all real roots.

As an example find the roots of

$$x^3 - 1.2x^2 + .37x - .17 = 0. \tag{10}$$

Let $x = z + .4$ and (10) becomes

$$z^3 - .11z - .15 = 0. \tag{10a}$$

The roots would be obtained by locating the point $(-.11, -.15)$ but since these coördinates are small it is better to use an equation whose roots are four times the roots of (10*a*) and divide by four. The equation is

$$z^3 - 1.76z - 9.6 = 0. \tag{10b}$$

The roots of this equation are 2.4, $-1.2 \pm 1.6i$, of (10*a*) .6, $-.3 \pm .4i$, and therefore of (10) 1, $.1 \pm .4i$. It is to be noted that the real part of the imaginary roots is one half of the negative of the real root.

An interesting property of the curves represented by (9), and which simplifies very much the process of drawing them, is the following: The segments cut off from the lines of real roots by the curves are proportional to the increments of b^2.

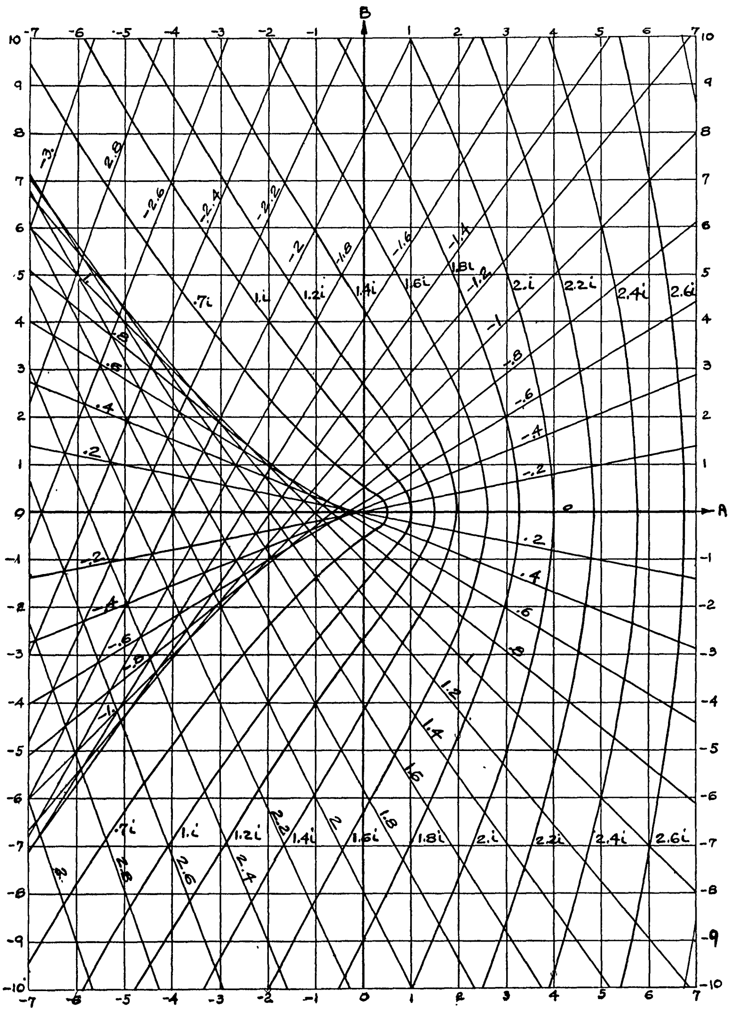

CHART II.

To show this, let A_1, B_1, and b_1 be any set of corresponding values of A, B, and b in (8). Then

$$B_1 = \pm \frac{2}{3}\sqrt{\frac{b_1^2 - A_1}{3}}\,(4b_1^2 - A_1).$$

Give to A_1 and b_1^2 the same increment K, and let B_2 be the value obtained for B.

Then

$$B_2 = \pm \frac{2}{3}\sqrt{\frac{b_1^2 - A_1}{3}}\,(4b_1^2 - A_1 + 3K),$$

and

$$B_2 - B_1 = \pm\, 2K\sqrt{\frac{b_1^2 - A_1}{3}} = 2Ka,$$

since

$$a = \pm\sqrt{\frac{b_1^2 - A_1}{3}}.$$

If now in the equation $B = - zA - z^3$ the same notation is used $B_1 = - zA_1 - z^3$, $B_2 = - zA_1 - z^3 - zK$, $B_2 - B_1 = - zK = 2Ka$, since the real root is $- 2a$. This shows that the increments of the ordinates of successive intersections of the lines representing real roots with the curves of imaginary roots are proportional to the increments of b^2 and therefore the segments are proportional to the increments. This fact enables one to locate points on the curves very rapidly. The curve $B^2 = - \frac{4}{27}A^3$ $(b = 0)$ is the envelope of the lines represented by

$$B = - zA - z^3$$

and is the discriminant of the cubic (5).

Solution of the Biquadratic. The equation of the biquadratic is.

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0. \tag{11}$$

It was said at the outset that the graphical solution of the biquadratic would be made to depend upon the solution of the quadratic and the cubic. The first step in the solution will be to factor the left hand member of (11) into two quadratic factors. This will be done graphically by means of Chart II together with Chart I. The biquadratic after factoring may be written

$$(x^2 + ax + b)(x^2 + cx + d) = 0$$

or

$$x^4 + (a + c)x^3 + (b + d + ac)x^2 + (ad + bc)x + bd = 0.$$

By equating coefficients the following equations are obtained

$$\left\{\begin{array}{rr} a + c = A, & b + d + ac = B, \\ ad + bc = C, & bd = D. \end{array}\right\} \tag{12}$$

Let $ac = K_1$, and $b + d = K_2$; then $K_1 + K_2 = B$. From the two equations $b + d = K_2$, and $bd = D$,

$$b = \tfrac{1}{2}K_2 + \tfrac{1}{2}\sqrt{K_2^2 - 4D},$$

$$d = \tfrac{1}{2}K_2 - \tfrac{1}{2}\sqrt{K_2^2 - 4D}.$$

From the two equations $a + c = A$, and $ac = K_1$,

$$a = \tfrac{1}{2}A + \tfrac{1}{2}\sqrt{A^2 - 4K_1},$$

$$c = \tfrac{1}{2}A - \tfrac{1}{2}\sqrt{A^2 - 4K_1}.$$

Substituting the values of a, b, c, and d above in the equation

$$ad + bc = C,$$

and remembering that $K_1 = B - K_2$, the cubic

$$K_2^3 - BK_2^2 + (AC - 4D)K_2 + 4BD - C^2 - A^2D = 0 \tag{13}$$

is obtained. By letting $K_2 = z + \frac{1}{3}B$, (13) becomes the auxiliary cubic

$$z^3 + (AC - 4D - \tfrac{1}{3}B^2)z + \tfrac{1}{3}(ABC + 8BD - \tfrac{2}{9}B^3 - 3C^2 - 3A^2D) = 0. \tag{14}$$

It is noticed that the coefficients of the auxiliary cubic (14) are obtained directly from the biquadratic (11) and its roots are found from Chart II. After obtaining the values of z, $b + d$ is found from the relation

$$K_2 = b + d = z + \tfrac{1}{3}B.$$

It is evident from the factored form of the biquadratic that there are always three possible values of $b + d$ real or imaginary. The numerically greatest of the real roots will always give real values of b and d.

Since $b + d = K_2$, and $bd = D$, it is seen that b and d are roots of

$$x^2 - K_2x + D = 0,$$

and are found from Chart I. The values of a and c are found in the same way. The values of a, b, c, and d must be selected so that they will satisfy the equation $ad + bc = C$.

As an example, consider the biquadratic equation

$$x^4 - 3x^3 + 5x^2 - x - 10 = 0.$$

The corresponding auxiliary cubic is

$$z^3 + 34.667z - 48.593 = 0.$$

Let $z = 3w$ and the equation becomes

$$w^3 + 3.85w - 1.8 = 0.$$

From Chart II: $w = .44$, $z = 1.32$, $K_2 = b + d = 2.99$, $bd = -10$. From Chart I: $b = -2$, $d = 5$ approximately; $a = -1$, and $c = -2$.

The biquadratic then becomes

$$(x^2 - x - 2)(x^2 - 2x + 5) = 0.$$

Again, from Chart I, the values of x are

$$-1, \quad 2, \quad 1 + 2i, \quad \text{and} \quad 1 - 2i.$$

The approximation to the roots when the coefficients are large or differ from one another by large numbers is illustrated by the solution of

$$x^4 - 9x^3 + 3{,}000x + 30{,}000 = 0.$$

The auxiliary cubic is

$$z^3 - 147{,}000z - 11{,}430{,}000 = 0.$$

Let $z = 200w$, then

$$w^3 - 3.675w - 1.429 = 0.$$

Chart II gives $w = 2.08$, $z = 416$. Chart I gives $b = 92$, $d = 325$, $a = 16.7$ $c = 25.5$, and $x = -8.3 \pm 5i$, $12.8 \pm 13i$. The error in the value of z obtained from the chart may be several units. A good approximation for the correction is easily obtained algebraically. $b + d = 417{,}579$, $bd = 30{,}000$; therefore $b = 325.378$, $d = 92.201$.

$ac = 0-(b+d) = -417.579$, $a+c = -9$; therefore $a = 16.424$, $c = -25.424$.

In the factored form the equation becomes

$$(x^2 + 16.424x + 92.201)(x^2 - 25.424x + 325.378) = 0.$$

Correct to three places of decimals

$$x = -8.212 \pm 4.976i,\ 12.712 \pm 12.798i.$$

1321, 1471, 1521, 1571, 1621, 1671, 1721, 1771, 1821.

By R. C. ARCHIBALD, Brown University.

1321—Dante Alighieri (born at Florence, 1265) died at Ravenna; his *Divine Comedy* is of mathematical importance for its statement of astronomical notions of the time.

1471—Albrecht Dürer (died, 1528), German painter and engraver, was born at Nürnberg, May 20; he was the author of various works on geometry and perspective, fortification, and human proportion ✳ In his *Zeittafeln zur Geschichte der Mathematik,* Felix Müller states[1] that Indian-Arabic numerals were in this year used for the first time in numbering leaves of a book, in Petrarch's *Liber de remediis utriusque fortunæ,* published by Arnold ther Hoernen of Cologne. This statement is erroneous, in part at least. In the library of the Annmary Brown Memorial, Providence, R. I., there is a copy of Werner Rolewinck's *Sermo in festo præsentationis* . . . ("Sermon preachable on the feast of the presentation of the blessed virgin") issued by the same publisher in 1470. It is paged in Indian-Arabic numerals half way down the right hand margins. In the printed catalogue of this library, prepared by A. W. Pollard of the British Museum, it is stated that *this* was the first printed book with pagination in Indian-Arabic numerals. The work of Petrarch mentioned above may be the *second* work of this kind. The printed catalogues of the British Museum and G. P. Winship's census of incunabula in America do not list a 1471 edition of this work of Petrarch.

[1] The same statement occurs in F. Unger, *Die Methodik der Praktischen Arithmetik in Historischer Entwickelung,* Leipzig, 1888, p. 16; and in J. Tropfke, *Geschichte der Elementar-Mathematik,* 2. Auflage, Berlin und Leipzig, 1921, p. 26.

unless it is an abridgment by Adrianus the Carthusian that is referred to. This question will elsewhere be discussed in detail by the writer.

1521—First edition of Francesco Ghaligai's *Summa de Arithmetica* published at Florence (compare Riccardi, cols. 499–502) ✳ Second edition of Anton Bartholomeo di Paxi's *Tariffa de Pexi e Mesvre,* published at Venice (first edition, 1503), is of especial interest in the history of commerce ✳ The Cambridge University Press published its first books.

1571—Adriaen Metius (Anthonisz), Dutch mathematician, was born at Metz (whence the name Metius—Cantor), December 9; author of nine minor works on mathematics and astronomy, 1591–1633 (Poggendorff), including *Opera arithmetica et geometrica,* Amsterdam, 1625 ✳ Johann Kepler, noted astronomer and mathematician, was born in Württemberg, December 27 ✳ Jacques Besson, who was professor of mathematics at Orléans about 1570, published, at Galiot du Prè, the little pamphlet *Description et Usaige du Compas Euclidien, contenant la plus-part des Observations qui se font en la Geometrie, Perspective, Astronomie, et Corographie* ✳ First edition published of Leonard Digges's excessively rare *A Geometrical Practise, named Pantometria, divided into three Bookes, Longimetra, Planimetra, and Stereometria, containing Rules manifolde for mensuration of all lines, Superficies, and Solides: with sundry straunge conclusions both by instrument and without, and also by Perspective glasses, to set forth the true description or exact plat of an whole Region: framed by Leonard Digges, lately finished by Thomas Digges his sonne. . . .* De Morgan recorded that in this book he found the earliest printed mention of the word "theodolite." The derivation of this word is unknown, and the derivation sometimes given in connection with corruptions of Arabic words is absurd (*New English Dictionary*). Tycho Brahe regarded Thomas Digges (died, 1595) as one of the greatest geniuses of that time (*Letters illustrative of the Progress of Science in England,* edited by Halliwell, pp. 6, 33.)

1621—Thomas Harriot (born at Oxford, England, 1560), died near Isleworth, July 2; his chief work *Artis analyticæ praxis,*[1] not printed till 1631, was "incomparably the best work on algebra and the theory of equations which had then been published" (Ball, *History of the Study of Mathematics at Cambridge;* but see Cantor, vol. 2, second edition, p. 790) ✳ Claude Gaspar Bachet de Méziriac, French mathematician, especially widely known for his *Problemes Plaisans et delectables* (first edition, 1612, in Harvard University Library), published at Paris his Latin translation, with commentary, of the Greek text of the arithmetic of Diophantus ✳ Johannes Bentz, of Ulm, published at Kempten what he claims to be the first treatise in German on the theory of numbers: *Manuductio ad Numerum Geometricum: Kurtze wol gegründte Anführung zu Erkañdtnuss der Natur vnd Eygenschafften allerhand Arten der figurierten oder geometrischen Zahlen;* . . . ✳ Published at Paris, Denis Henrion's *Collection ou Recueil de divers Traictez Mathematiques, à sçavoir, d'Arithmetique, d'Algebre, de la Solution de divers Problemes et Questions, tant geometriques qu'astronomiques, plusieurs Moyens pour mesurer toutes Sortes de Quantitez . . .* ✳ Published at Hanover, Bartholomæus

[1] The first printed work to use our ordinary signs for "less than" and "greater than."

Keckermann's *Systema Compendiosum totius Mathematices, hoc est, Geometriæ, Opticæ, Astronomiæ et Geographiæ . . .* ⁎ *Prælectiones tresdecim in principivm Elementorvm Evclides, Oxonii habitæ. M.DC.XX* by Sir Henry Savile, published at Oxford; in 1619 Sir Henry founded at Oxford University a chair of geometry, later filled by such men as Briggs, Wallis, Halley, and Sylvester—Hardy is the present incumbent ⁎ *Instrument pour construire sur le papier les sections coniques,* by Carlo Ladroni (compare Murhard, *Litteratur der mathematischen Wiss.*, vol. 3, p. 278) was published at Turin.

1671—John Keill, Savilian professor of astronomy at Oxford, 1708–1721, and prominent in the Newton-Leibniz controversy, was born at Edinburgh, December 1. In 1715 he published at Oxford an anonymous work: *Trigonometriæ Planæ & Sphæricæ Elementa item de Natura et Arithmetica Logarithmorum Tractatus brevis* (76 pages) ⁎ Albert Curtz, Bavarian Jesuit teacher of philosophy and mathematics, died December 19 ⁎ Jan de Witt (1625–1672) published at the Hague a Dutch tract (translated into English, 1852) of interest in the history of insurance[1] ⁎ Isaac Newton (1643–1727) wrote his notable memoir "Methodus fluxionum et serierum infinitorum," first published in English translation, by John Colson, in 1736 ⁎ Published at London the third part of "one of the classics of mechanics," John Wallis's *Mechanica: sive, de Motu, Tractatus Geometricus. Pars III. in qua, de Vecte, de Axe in Paritrochio, de Trochleâ, de Cochleâ, de Motibus Compositis, de Percussione, de Cuneo, ...*

1721—Lorenzo Lorenzini (born, 1652), a pupil of Viviani, died April 24. While in prison, where for political reasons he was confined for twenty years, he wrote the following work, published at Florence in the year of his death: *Exercitatio geometrica: in qua agitur de Dimensioni omnium Conicarum Sectionum, Curvae Parabolicae, Curvæque Superficiei Conoidis Parabolici, . . .* ⁎ John Keill (see above) died September 1 ⁎ Published at London, the "6th Edition, much inlarg'd," of Henry Philipps's *The Purchaser's Pattern, shewing the true Value of Land or Houses, by Lease or otherwise . . . also Tables of Interest and Rebate at 6 per Cent.; the Measuring of Land, Board, and Timber, and the False Methods us'd by many therein. Also the Art of Gauging much enlarg'd with many other Rules about Weights and Measurès, Tables of the Excise of Beer and Ale, Tables of Accounts, etc., etc.* ⁎ Published at London, the third edition of John Hill's *Arithmetick both in Theory and Practice, made Plain and Easie, with Interest and Annuities, Extraction of the Square and Cube Roots, the Tables and Construction of Logarithms, Arithmetical and Geometrical Progression, etc., etc.* In his *Arithmetic Books,* De Morgan refers to this as "a book of much celebrity."

1771—Jean Jacques d'Ortous de Mairan (born, 1678), mathematical physicist and secretary of the French Academy, died February 20 ⁎ Joseph Diez Gergonne (died, 1859), founder and editor of *Annales de Mathèmatiques Pures et Appliquées* (July 1810–August 1831), born at Nancy, France, June 19. He was the author

[1] An interesting sketch of "John De Wit" and his work may be found in *The Insurance Cyclopædia being a Dictionary of the definitions of terms . . . a Biographical Summary . . .*, London, volume 2, 1873.

of numerous mathematical papers ⁎ Jöns Svanberg, for thirty years professor of mathematics at the University of Upsala, was born July 6 ⁎ Jean Baptiste Cochet, rector of the University of Paris, died July 8. He was the author of *La clef des Sciences et des beaux-arts, ou la logique* (Paris, 1750), and translator of P. Varignon's *Elēments de Mathēmatiques* (Paris, 1731)* First paper by John Landen (1719–1790) on elliptic transcendents (published in the *Philosophical Transactions*): "A disquisition concerning certain fluents which are assignable by the arcs of the conic sections; where are investigated some new and useful theorems for computing the fluents" ⁎ Notable memoir "De aequationibus quadrato-cubicis disquisitio analytica" by G. Malfatti (1731–1807) published (*Atti di Siena*, vol. 4). Also the year in which Malfatti was appointed professor of mathematics at the University of Ferrara ⁎ *Dioptricae pars tertia* by Leonhard Euler (1707–1783), published at Petrograd (reprinted in *Leonhardi Euleri Opera Omnia*, 1912). Second edition of Euler's *Vollständige Anleitung zur Algebra*, Erster teil, published (first edition, 1770; last edition, *Opera Omnia*, 1911) ⁎ Published in London, Edward Noble's *The Elements of Linear Perspective, demonstrated by Geometrical Principles; with Introduction, containing so much of the Elements of Geometry as will render the whole Rationale of Perspective intelligible without any previous Mathematical Knowledge* ⁎ Published in Madrid, the first edition of Don Jorge Juan's *Examen Maritimo Theórico Práctico, ò Tratado de Mechanica aplicado á la Construccion, Conscimiento y Manejo de les Navios y demas Embarcaciones* (French edition, 1783) ⁎ Published at Leipzig, G. H. Bortz's quarto pamphlet, *De Forma Radicum ex Aequationibus Quadraticis ei simili, quam Cubicae per Methodum Cardani obtinent* ⁎ Published at Chester, England, Arthur Burns's *Geodæsia Improved; or a New and Correct Method of Surveying made exceedingly easy, with Appendix concerning the practical Methods of measuring Timber, Hay, Marl Pits, Bricklayers and Plaisterers Work* ⁎ *Transactions of the American Philosophical Society*, volume 1, published.

1821—Heinrich Edward Heine (died, 1881), for many years professor of mathematics in the University of Halle, and author of *Handbuch der Kugelfunctionen* (1861; second edition 1881), was born March 15 ⁎ Richard Townsend (died, 1884), for fourteen years professor of natural philosophy in Trinity College, Dublin, and author of *Modern Geometry of the Point, Line and Circle* (1863–65; second edition 1878–81), was born April 30 ⁎ Prince Baldassare Boncompagni (died, 1894), founder, editor, and publisher of *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche* (Rome, 1868–1887), was born May 10. He was also the editor of numerous mathematical works, and the author of many papers ⁎ Noted Russian mathematician, Pafnouty Livovitch Tchébychef (died, 1894), of the University of Petrograd, was born May 14 ⁎ August Nagel, professor of geodesy in the Dresden polytechnic school, was born May 17 ⁎ Karl Culmann (died, 1881), professor in the Zürich polytechnic school, was born July 10. He has been referred to as the "founder of graphical statics" ⁎ Jacob Heinrich Karl Durège (died, 1893), professor in the polytechnic schools of Zürich and Prague, was born July 13. G. E. Fisher and I. J. Schwatt translated into

English the fourth German edition of his *Elements of the Theory of Functions of a Complex Variable* (Philadelphia, 1896) $\ast$ Anton Winckler (died, 1892), professor of higher mathematics in the polytechnic school of Vienna, was born August 3 $\ast$ Wilhelm Johann Otto Ligowski (died, 1893), teacher of mathematics in the marine academy and marine school of Kiel, was born August 10. He is the author of *Tafeln der Hyperbelfunctionen und der Kreisfunctionen* (Berlin, 1890), and of *Sammlung fünfstelliger logarithmischer, trigonometrischer und nautischer Tafeln nebst Erklärungen und Formeln der Astronomie* (Kiel, 1873; fourth edition, Berlin, 1900) $\ast$ The celebrated English mathematician Arthur Cayley (died, 1895) was born August 16. He was a prolific author of mathematical memoirs $\ast$ The noted German physicist, Hermann Ludwig Ferdinand Helmholtz (died, 1894), was born August 31 $\ast$ First number of the *Astronomische Nachrichten* published in September $\ast$ Franz Friedrich Ernst Brünnow (died, 1891), German astronomer, was born November 18. He was director of the observatory at Ann Arbor, Mich., 1854–1864, professor of astronomy at the University of Dublin and astronomer royal for Ireland, 1865–1874 $\ast$ Astronomical observatory at Cape of Good Hope founded $\ast$ Samuel Vince, Plumian professor of astronomy and experimental philosophy at Cambridge University, 1796–1821, died. He was the author of several books including *The Elements of Conic Sections* (1781), and *A Treatise on Plane and Spherical Trigonometry* (1800) $\ast$ *Cours d'analyse de l'Ecole Polytechnique,* prèmiere partie, by A. L. Cauchy (1789–1857), was published; no other part appeared $\ast$ Published at Rome, Abate Pietro Franchini's *Saggio sulla Storia delle Matematiche, corredato di scelte Notizie Biografiche,* volume 1 $\ast$ Published at Mainz, second enlarged edition of J. J. I. Hoffmann's *Der Pythagorische Lehrsatz, mit 32 theils bekannten, theils neuen Beweisen versehen* $\ast$ Published at London, *Elementary Illustrations of the Celestial Mechanics of Laplace,* Part I, by Thomas Young (1773–1829), one of the "most eminent physicists of his time" $\ast$ Published at Edinburgh, Sir John Leslie's *Geometrical Analysis, and the Geometry of Curve Lines, being volume second of a course of Mathematics, and designed as an Introduction to the Study of Natural Philosophy.* This work was a development of part of *Elements of Geometry, Geometrical Analysis and Plane Trigonometry, with an Appendix, Notes, and Illustrations,* Edinburgh, 1809 (second edition, 1811; French translation, edited by Hachette, 1818) $\ast$ Published at Haarlem, James Lockhart's *Leerwijze om den Cubik-Wortel uit alle Getallen te trekken.*

SOME ARITHMETIC OPERATIONS WITH TRANSFINITE ORDINALS.

By ALBERT A. BENNETT, University of Texas.

(Read before the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America, December 11, 1920.)

If we write out the first few non-negative integers in order of increasing magnitude, we shall have

$$0, 1, 2, \cdots, (n-1), n.$$

This finite ordered sequence constitutes an example of an order type to which the name, $n + 1$, may be given. This is also the name of the first ordinal integer following all of those of the given set. This statement applied inductively accounts formally for all positive integers. It is also available for the introduction of transfinite ordinals. For example the sequence, $0, 1, 2, \cdots, n, \cdots$, including zero and all positive finite integers is an example of an order type to which the name, ω ("Omega"), may be given. Next after ω follows $\omega + 1$, and so forth. Two successive sets of order type ω when considered together constitute an example of the order type called $\omega 2$, but not 2ω, for reasons that we shall see. The order types $\omega 3$, $\omega 4$, etc., are obtained analogously, finally giving $\omega\omega$, called also ω^2, and so forth.

Georg Cantor, to whom is due the credit for introducing these transfinite ordinals on a rigorous basis, defined also the addition and multiplication of ordinals. The ordered sum, $a + b$ of two ordinals, a and b, is the ordinal whose order type is obtainable by placing after a sequence of the order type of a, one of the order type of b. Thus $(\omega) + (1) = \omega + 1$, $1 + \omega = \omega$, $\omega + \omega = \omega 2$, $(m) + (n) = m + n$ where m and n are finite ordinals. The ordered product ab of two ordinals, a and b, is the ordinal whose order type is obtainable by replacing each element of b by a set of the order type of a.[1] Thus 3×4 may be represented as $(\cdots)\ (\cdots)\ (\cdots)\ (\cdots)$, while 4×3 would be represented by $(\cdot\cdot\cdot\cdot)\ (\cdot\cdot\cdot\cdot)\ (\cdot\cdot\cdot\cdot)$. We see also that $(\omega)(2) = \omega 2$, and more generally $(\omega)(n) = \omega n$, while $n\omega = \omega$. The following may be verified,

$$(\omega + n)(\omega + m) = \omega^2 + \omega m + n, \qquad a(b + c) = ab + ac,$$

where m, n, $(m > 0)$ are finite ordinals, and a, b, c, any ordinals. Propositions *not* in general valid, are: $a + b = b + a$, $ab = ba$, and $(a + b)c = ac + bc$.

While addition and multiplication are always possible among ordinals, subtraction is certainly not in general admissible. Thus there is no ordinal x to satisfy the relation $x + 1 = \omega$ so that $\omega - 1$ cannot be itself an ordinal, if it is to represent an x satisfying the equation referred to. On the other hand there is an x, namely ω, for which $1 + x = \omega$. The noncommutative character of addition is what causes part of the difficulty. The following general fact may be noted. If a and b be two ordinals, and $b > a$, then there is one and only one ordinal x satisfying the equation $a + x = b$. This may be expressed in the statement that if $a < b$, $-a + b$ is a uniquely defined ordinal.

An interesting consequence of the uniqueness of left hand subtraction is the following theorem, which is believed to be new:

The highest common left hand divisor of two ordinals may be found by Euclid's algorithm.

The process may be illustrated as follows in the case of

$$\omega^3 + \omega^2 + \omega 5 + 3, \qquad \text{and} \qquad \omega^3 + \omega^2 7 + \omega 2 + 3.$$

[1] For this notation, as well as for further references, compare the bibliography in E. V. Huntington, *The Continuum and other Types of Serial Order*, second edition, 1917, p. 74.

Step	Dividend	Divisor	Remainder	Quotient
1	$\omega^3 + \omega^2 7 + \omega 2 + 3$	$\omega^3 + \omega^2 + \omega 5 + 3$	$\omega^2 6 + \omega 2 + 3$	1
2	$\omega^3 + \omega^2 + \omega 5 + 3$	$\omega^2 6 + \omega 2 + 3$	$\omega^2 + \omega 5 + 3$	ω
3	$\omega^2 6 + \omega 2 + 3$	$\omega^2 + \omega 5 + 3$	$\omega^2 + \omega 2 + 3$	5
4	$\omega^2 + \omega 5 + 3$	$\omega^2 + \omega 2 + 3$	$\omega 3 + 3$	1
5	$\omega^2 + \omega 2 + 3$	$\omega 3 + 3$	$\omega 2 + 3$	ω
6	$\omega 3 + 3$	$\omega 2 + 3$	$\omega + 3$	1
7	$\omega 2 + 3$	$\omega + 3$	0	2

Thus $\omega + 3$ is the greatest common left hand divisor. Indeed the original expressions may be written respectively as $(\omega+3)(\omega^2+\omega+5)$ and $(\omega+3)(\omega^2+\omega 7+2)$, where the greatest common left hand divisor of $\omega^2 + \omega + 5$ and $\omega^2 + \omega 7 + 2$ is 1. While ω contains each finite ordinal as a left-hand divisor, $\omega + 1$ is prime to them all.

The question remains at this stage as to whether the system of ordinals may be so extended that for each quantity of the system, the negative of the quantity also is in the system, and where addition and multiplication shall still be possible in all cases. If a somewhat generous interpretation be given to these terms, as necessitated by the fact that addition is not ordinarily commutative, the answer is, yes. This will now be demonstrated by the introduction of what appears to be a new concept in this connection.

We shall consider a system whose elements may be formally written as $a - b$, where a and b are any ordinals. The expression $a - b$ will be regarded as the formal sum of a sequence of order type a, every element of which is $+ 1$ followed by a sequence of a reverse order type, namely the reverse of b, in which, furthermore, each element is $- 1$. Thus if $a = \omega^3 n_3 + \omega^2 n_2 + \omega n_1 + n_0$ we shall write $- a$ as $- n_0 - \omega n_1 - \omega^2 n_2 - \omega^3 n_3$. The fundamental principle to be observed is that addition (among a finite number of terms of the same sign) is throughout associative. Any combination $- a + a$ may be replaced by zero, while $a - a$ will not in general be regarded as the same as zero. If $a = \omega^3 n_3 + \omega^2 n_2 + \omega n_1 + n_0$, then $- a + a$ may be written as $- n_0 - \omega n_1 - \omega^2 n_2 - \omega^3 n_3 + \omega^3 n_3 + \omega^2 n_2 + \omega n_1 + n_0$. Pairs of terms may be dropped out successively from the center until zero results.

To form the sum $(a_1 - b_1) + (a_2 - b_2)$, compare the ordinals a_2 and b_1. If a_2 be equal to or greater than b_1 it may be written as $a_2 = b_1 + c$, giving $a_1 - b_1 + b_1 + c - b_2$ which reduces to $(a_1 + c) - b_2$. If on the other hand b_1 is equal to or greater than a_2 we may write $b_1 = a_2 + d$, or $- b_1 = - d - a_2$ whence the sum becomes $a_1 - d - a_2 + a_2 - b_2$ or $a_1 - (b_2 + d)$.

Before considering multiplication of these expressions, we shall return to note a feature of ordinals. Positive ordinals are of two types, *terminating* and *non-terminating*. While $\omega + 1$, $\omega 2 + 7$, 8, each is an order type with a last element, ω, $\omega^2 + \omega 2$ and ω^ω each is an order type with no last element. The sum $a + b$ of two elements is found to be terminating if and only if the second quantity, b, is terminating, the product ab is terminating if and only if both a and b are ter-

minating. An infinite terminating ordinal, a, can be expressed uniquely as a sum $a' + n$ where n is finite, and a' is nonterminating. The finite term n is called the *termination* of a. In any product ab, where a is terminating with termination, n, the product will be terminating and have this same termination n, for every terminating b, although this termination, which is independent of the character of b so long as b is terminating, vanishes whenever b is nonterminating.

In the expression $a - b$, let l denote the lesser of the two quantities a, b, and d denote their difference. Then $a - b$ may be written $l \pm d - l$, where $\pm d$ is called the *stem*, and l, the *affix* of the expression, $a - b$. The sign of $\pm d$ depends upon the particular expression $a - b$ considered and is completely determined except for the trivial case where $d = 0$. The product $(l \pm d - l)c$ is defined as $l \pm dc - l$ when c is terminating, and as $\pm dc$ when c is nonterminating. For example

$$(\omega - 3)(\omega^2 + \omega 2 + 7) = 3 + \omega(\omega^2 + \omega 2 + 7) - 3 = \omega^3 + \omega^2 2 + \omega 7 - 3$$

$$(\omega - 3)(\omega^2 + \omega 2) = \omega(\omega^2 + \omega 2) = \omega^3 + \omega^2 2$$

$$((\omega + 1) - 4)(\omega^2 + \omega 2 + 7) = 4 + (\omega + 1)(\omega^2 + \omega 2 + 7) - 4$$
$$= \omega^3 + \omega^2 2 + \omega 7 + 1 - 4$$

$$((\omega + 1) - 4)(\omega^2 + \omega 2) = (\omega + 1)(\omega^2 + \omega 2) = \omega^3 + \omega^2 2$$

The above rule has the desirable though not essential property that if we identify $a - b$ and $(a + n) - (b + n)$, then we may also identify $(a - b)c$ with $((a + n) - (b + n))c$, where n is a finite ordinal. Thus so far as finite ordinals are concerned we may if we desire regard $n - n$ as identical with zero, although this is not desirable in some applications.

The product $(a_1 - b_1)(a_2 - b_2)$ is defined as $(a_1 - b_1)a_2 + (b_1 - a_1)b_2$. The product $(a_1 - b_1)((a_2 - b_2) + (a_3 - b_3))$ is found to be equal to $(a_1 - b_1)(a_2 - b_2) + (a_1 - b_1)(a_3 - b_3)$ if the stem of $a_1 - b_1$ is positive. In case this stem is negative, we have instead $(a_1 - b_1)(a_3 - b_3) + (a_1 - b_1)(a_2 - b_2)$, as is to be expected.

It is to be noted that the above is closely suggestive of the process by which rational fractions are introduced in the development of the number system by means of pairs of integers.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

14. MAUPERTUIS AND FREDERICK THE GREAT.

Among those whom Frederick the Great called to his court for the purpose of accomplishing what the Ptolemies had done for Alexandria, the caliphs for Bagdad, and the Medici for Florence, there were a few scientists and literati of genius, and still more of near ability. Of the latter, Maupertuis is perhaps the

best known. He was born at St. Malo in 1698, and, after falling from royal favor, died at Basel in 1759. He spent six months in England, in 1728, while Voltaire was there, and during this time was made a member of the Royal Society and became familiar with the Newtonian theory. In 1740 he was made president of the Berlin Academy, Frederick saying that it should be formed "as you alone can form it." A biographer of Voltaire thus writes of him:

"Precise, pompous, and positive; boring society with his worrying exactness upon trifles even more than society bored him; inordinately vain, and with a sensitive temper made yet more inflammable by brandy and self love; acutely conscious of his dignity, and without any sense of humor, the ex-tutor of Madame Châtelet was the sort of person with whom, sooner or later, her lover [Voltaire] was sure to disagree."

The quarrel broke soon after Voltaire reached Berlin, in 1750. It arose over a vacancy in the Academy, Maupertuis favoring Jean-Baptiste de Boyer, Marquis d'Argens, an oriental scholar, while Voltaire urged Raynal, the philosopher and historian. It was a bitter one and ended in each vanquishing the other so far as the court of Potsdam was concerned.

I have two holograph letters of Maupertuis relating to his sojourn in Potsdam. The first was written to Frederick the Great, and may some time find place in this series. The second was written when the royal favor was not so apparent. Voltaire's sarcasm had begun to have its effect. The letter is dated simply "Tuesday the 14th," which does not allow us to fix the year, and is full of apprehension as to the effect of certain remarks, doubtless Voltaire's, which had been quoted against him. The writing is indistinct in several places, and I am indebted to my colleague, Professor Méras, for assistance in arriving at the meaning, but the letter reads substantially as follows:

> You have not told me a single word about the effect produced at Potsdam by the letter of my generous and sublime defender. It has the approbation of all honest people here, and of all those who have position and taste.
>
> Someone brought me *Le Tombeau de la Sorbonne*, and I am still at it. It is very interesting, but in order to amuse [?] the malignity of the readers they have made it lose all air of truth.
>
> I do not see that Walther has answered you on the subject of the books which I requested you to ask of him. You had also promised to send me the Latin thesis on the generation of organic beings which I received from Erlangen and which I sent you at the request of Buffon. I should very much like to have it back again.
>
> Adieu, my dear friend. I think that glory is not destined for me so much as criticism, for I am less accustomed to it. . . .
>
> I am always yours,
>
> MAUPERTUIS.

Maupertuis refers to the Erlangen dissertation in his *Œuvres* (1768 edition, II, 187) and to Diderot's comment upon it. Those were days of anonymous pamphlets and of pseudonyms, and this reference illustrates the situation. The thesis was, as a matter of fact, the work of Maupertuis himself, although he speaks of it as the *Dissertatio inauguralis metaphysica de universali Naturae systemate*, and as the work of one Dr. Baumann.

The letter marks the period of the beginning of the decline of Maupertuis

at the court of Frederick. He had recently published (1752) his *Lettres,* a small volume on all sorts of schemes and things, some absurd and others trivial. Voltaire said that Maupertuis had previously been in a lunatic asylum and was still crazy. But Frederick, beginning to feel the danger of Voltaire's intimacy, took the part of Maupertuis and, on November 5, 1752, congratulated him upon his performance. It seems probable that the letter refers to this fact.

Frederick's statement was a challenge to Voltaire, and he accepted it. The result was the *Diatribe of Doctor Akakia,* one of the keenest satires in the French language. "I have no scepter," wrote Voltaire, "but I have a pen." It is said that Frederick laughed till he cried over the castigation of poor Maupertuis. He loved wit more than science, or pseudo-science, or friendship.

In the second edition of his *Lettres* Maupertuis records the official burning of the *Diatribe*; "Le Libelle fut brûlé le 24 Décembre 1752, publiquement, par le main du Bourreau, dans toutes les places de Berlin." In that delightful *Life of Voltaire* by Tallentyre, the story is told of how his secretary, Collini, seeing the crowd and watching the bonfire, mentioned the circumstance to the "old invalid of Ferney." "I'll bet it's my Doctor!" said Voltaire,—and he was right.

15. The Paradoxer and Deparcieux.

Antoine Deparcieux (to adopt his own spelling) was one of the leaders among the scholars of France who devoted their attention to applied mathematics in the middle of the eighteenth century. He was born at the little hamlet of Cessoux (Gard), near Nîmes, October 8, 1703, and died at Paris on September 2, 1768. At the age of forty-three he was admitted to the Academy of Sciences, and between 1740 and 1768 became recognized as one of the distinguished group of French savants. He wrote on trigonometry, probabilities as applied to mortality tables, and astronomy, and received numerous honors abroad as well as in his native land.

Like all mathematicians or teachers occupying positions which tend to make them known, Deparcieux was bored by that class of men about whom De Morgan wrote so delightfully in his *Budget of Paradoxes.* It is interesting to see how such a man received the suggestions of the half-demented correspondents who felt that they had solved the Fermat Theorem, squared the circle, untangled the mystery of the Hindu-Arabic numerals, or overturned the Newtonian theory.

One of the letters of Deparcieux in my collection is a reply to a communication from a certain M. Fomaigne who had discovered a "new government of the sun." The letter shows Deparcieux entering into the spirit of the argument, and indulging in some interesting bits of sarcasm. It reads in part as follows:

Compiègne, May 18, 1758.

Monsieur:

I have read the annexed memoir which informs us of a new government (nouveau gouvernement) of the sun, and of which it is difficult to understand a single thing. It seems, however, like an effort to prepare a set of tables of the rising and setting of the sun which shall be more exact than those now in use. It looks as if you wished to accomplish this by means of a certain line drawn on two sun dials, one oriental and the other occidental, or on a single one. If the Sieur Fomaigne had read anything on the subject he would surely express himself somewhat differently,

would probably have seen the matter in a different light, would know that tables of extreme accuracy already exist, and would recognize that the line which can represent different kinds of the rising of the sun (for there is one) starts from the same source. What he says about the moon has no common sense. He wishes to govern the moon, but if he is not crazy he will see that it is the moon which governs the sun.

The letter is evidently intended to follow the style and spirit of the memoir, and to run into a state of semi-aberration of mind,—a procedure which renders a translation difficult. Nevertheless it closes with the usual politeness of the Parisian:

I have the honor of being, with the most profound respect and the most sincere attachment, Monsieur, your humble and very obedient servant,

DEPARCIEUX.

One lesson we can learn from Deparcieux's letters,—namely,—that it is never necessary to punctuate. In the letter referred to there are only two marks of punctuation in three quarto pages, and in another of his letters before me there are only three,—a period and two commas.

16. A *Letter of Gassendi's written in* 1633.

Although Pierre Gassendi was best known for his work in astronomy, he was one of the brilliant circle of mathematicians which was making France the scientific center of the world in the first half of the seventeenth century. He has been called the first French disciple of Bacon, the worthy friend of Galileo and Kepler, and the precursor of Newton and Locke. He was born near Digne, in Provence, on January 22, 1592, and is one of the few infant prodigies who ever attained any eminence. He was only twenty-one when he was called to the chair of philosophy as well as to that of theology in the University of Aix. He accepted the latter, but in 1623 he resigned from his professorial duties in order to devote his time to study and travel. He visited and taught in Paris, traveled in the Low Countries, and met with many savants, such as Descartes, Mydorge, Mersenne, and Cassini I, and made friendships which lasted throughout his life. His writings cover a wide field and include works on philosophy, astronomy, theology, physics, and the calendar.

Among my autographs is a letter written from his old home at Digne, on August 2, 1633. It is addressed to "Monsieur de Peiresc, abbe & seig^{n} de Guistre, Con^{r} du Roy,"—then a king's counsellor in the local "parlement" of Aix. Nicolas-Claude Fabri de Peiresc, like Gassendi a native of Provence, was twelve years the senior of the latter. He was one of those fortunate men who establish strong bonds of friendship with scholars, and among his intimate correspondents were some of the most learned men of his time. He was a botanist, a linguist, a physiologist, a historian, and a lover of books. Louis XIII gave to him the abbey of Notre-Dame at Guistre, in the diocese of Bordeaux, and it was to him as abbot that Gassendi wrote this letter. Four years later, on June 24, 1637, he died in Gassendi's arms.

The letter is one of those which were beginning to be interchanged so freely at that time among the members of the mathematical fraternity, and offers

evidence of the nature of the work in which mathematicians and astronomers had a special interest. Such a communication brings to the present some of the atmosphere of three centuries ago, and hence is not without interest. It reads as follows:

Monsieur,

I was recently surprised at the departure of Monseigneur the Archdeacon, for he had promised me that he would go to see you. Nevertheless I did not venture to delay this departure for the purpose of writing to you. Even now I have come very near failing in my duty of sending a few words to you, along with five or six lines which I have just written to Mons. Luillier. This delay has been due to the hope of having the fortune of writing more fully by the next post. I am still waiting to observe the eclipse. *Yesterday there began some evidence of a spot on the sun, and of this I have observed the position.* I shall also take further observations on the succeeding days for the purpose of comparing it further with the moon's shadow *at the time of the eclipse,* since at that time *it will have completed only two thirds of its path over the disk of the sun.* If you wish to see it, *it is quite large* and we shall be able to speak of it sometime hereafter. If another spot appears between now and then, I shall be on the watch.

Meanwhile I kiss very humbly your hands and remain always, Monsieur, your very humble, very affectionate, and very obliged servant,

GASSENDI

Digne, August 2,
——xxxiii.

The letter gives evidence of the state of astronomy in the days when the telescope was new and imperfect and when it was only just beginning to be successfully applied to observations upon which mathematical calculations could be based for the purpose of developing the theory of celestial mechanics.

17. A Letter from Bouillaud to Hevelius, 1666.

Among my autograph letters of the seventeenth century, one of the most interesting was written by Bouillaud to Hevelius. Few more attractive mathematical portraits exist than the one of Bouillaud painted from life by Jacob Van Schappen. It is a genuine work of art, and one has but to look upon it to feel that here is the face of a scholar. Ismael Bouillaud was born at Loudun in 1605 and died at Paris in 1694. He was one of the most staunch supporters of the Copernican system and was a man of great erudition. He wrote several treatises on astronomy, translated the arithmetic (theory of numbers) of Theon of Smyrna, and published a work on spirals. He was therefore interested in pure as well as applied mathematics. This letter is written to Johannes Hevelius, the well-known astronomer of Danzig. Hevelius, or Höwelke, was six years younger than Bouillaud and died in 1687. From 1641 until his death he devoted himself to astronomy, making his own somewhat primitive telescopes and setting them up in his private observatory.

Bouillaud's letter is interesting for several reasons, but chiefly for its reference to various contemporary astronomers, to some of the work then being undertaken, and to the general question of telescopes. On the last of these subjects Hevelius held very radical opinions, believing that for accuracy in the location of heavenly bodies the old pin-hole method was better.

Bouillaud wrote in Latin, according to the custom of the time, and in a legible

hand,—which was by no means a part of this custom. In the letter he refers to the observations made by Hevelius and to a comet that had recently appeared. As to the latter he remarks that

The path of the comet can be limited and determined by the observations made by you and the illustrious Domenico Cassini.

This is Giovanni Domenico (Jean-Dominique) Cassini, then in Bologna, who was just publishing his *Opera astronomica* (Rome, 1666). In the same year the French statesman, Colbert, founded the Académie des Sciences and invited Cassini to Paris. The invitation led to a visit to France and to a brief sojourn there; but not until 1669 did Cassini take up his permanent residence in the French capital. The letter continues:

As to the matters which may be open to dispute, pro and contra, with respect to the apparent size of the head of the new star, men will maintain uncertain judgments because of the very different conditions of their observations and because of the varying degrees of accuracy and perfection of the different telescopes.

It will be recalled that it was not until 1610 that Galileo discovered the satellites of Jupiter and that only in 1663 did Gregory publish his plan for using a concave mirror in a telescope, although Descartes and Mersenne had corresponded on the subject long before, and the latter had published his ideas in 1651. Hevelius had even maintained, as already stated, that telescopes were not sufficiently perfected for accurate observations of position. Achromatic lenses were still a long way in the future, and Bouillaud's letter shows that the instrument itself was still in its infancy. He continues, referring to a letter on the subject written by Hevelius on December 5, 1665:

Since in your letter you make mention of telescopes, it will doubtless please you to know from me what has been written about the instruments made in Rome and Florence by that very skilled and famous artisan Joseph Campanus [Giuseppe Campano]. By the aid of these telescopes they have observed not only the satellites of Jupiter in the disc of the planet but also the revolution of the planet about its own axis, which is completed in about ten hours. By reason of the fitness and advantages of their location and the necessary resources for meeting the expenses of carrying out and accomplishing these things, we who are in Paris and are perfectly capable of making observations, but are without due support in the matter, are anticipated by others and the glory is taken away from us. That most illustrious Domenico Cassini has made public his observations in three letters written to a friend, and these you can obtain from our friend De Noyer and also from our friend Burattino, to whom I believe they have been sent from Italy.

The Campano mentioned was a famous maker of instruments, and the Cassini was the first of the four astronomers of this family in the Royal Observatory of Paris.

The Latin beginning and ending may be interesting to readers who have given little attention to the epistolary style of the time. The letter begins:

Amplissimo Viro Dnō Io. Heuelio Veteris Gedani Consuli Ismael Bullialdus S. P. D.

To that most learned man, Lord Johannes Hevelius, Councilor of the ancient Gedanum [Danzig], Ismael Bouillaud, Salutem Plurimam Dicit.

It closes:

Vale Vir Amplissime & me semper ama. Scribeb. Lutetiae Parisioř die 15. Ianuarii 1666.

Farewell, most learned man, and always have regard for me. Written in Paris the 15th day of January, 1666 [old style].

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

Professor Dresden discusses the minimum length from a fixed point to a fixed line, measured along a circular arc tangent to another fixed line through the given point. His result may be shown to be geometrically equivalent to the following (stated in Professor Dresden's notation): the minimum length is such that it is equal to the length of that segment of the tangent to the circle at its intersection with the line $x = b$ which falls between that line and the y-axis. Is there a way of seeing directly without the use of analysis that such an arc exists and furnishes a minimum?

Mr. Anning points out the faulty treatment of mathematical topics in general works of reference. The two works to which he refers are of course not isolated instances; in fact it is not improbable that they are among the least culpable. From three other well-known reference compendia the following excerpts from the article *Hyperbola* are chosen:

"Analytically the hyperbola is the locus of a point which moves so that. . . ."

"If two similar cones be placed apex to apex, and with the lines joining the apex and center of base in each in a straight line, then if a plane which does not pass through the apex be made to cut both cones each of the sections will be a hyperbola. . . . If the axes of coördinates be turned at right angles to their former position, two additional curves will be formed. . . . These two are called *conjugate hyperbolas*, and have the same asymptotes as the original hyperbolas."

"The straight lines . . . known as asymptotes gradually approach the curve, but actually only meet it at points infinitely distant."

Perhaps we may draw the moral that no work intended to cover the whole field of knowledge, however eminent its editorial staff, should ever be trusted for precise information on a technical subject.

The deluge of geometric proofs of the law of tangents (*1920*, 53, 465; *1921*, 71, 170) led the editor to suggest terminating discussion on this subject. Two proofs which were received before the publication of this recommendation appear below. That by Professor Bradley is in fact somewhat different in principle from the usual proofs. That by Mr. Yamanouti is closely related to the usual proofs, but cannot be read out of Professor Lovitt's consolidated figure (*1920*, 465). Finally, Professor Lovitt shows how a figure similar to that used by Mr. Yamanouti, but more general, may be utilized to prove a theorem which includes the law of tangents as a special case. Historical and bibliographical notes are added by Professor Archibald.

Similar in character is the discussion of Professor Rusk, in which a number of familiar formulæ of trigonometry are considered with regard to their deducibility from the sine, cosine, and projection formulæ.

Professor Uhler gives an account of the method he uses to make the ordinary multiplication of two large numbers convenient and fool-proof. The actual work for the multiplication of e^{π} and $e^{-\pi}$ to more than fifty places is reproduced.

Mr. Haldeman shows how a certain quartic curve may be used in order to construct a regular heptagon. Compare a similar article by the same author (*1919*, 390), in which the construction is effected by means of an equilateral hyperbola. Reference may be made also to Note 24 in the department of Problems and Solutions elsewhere in this issue.

I. The Shortest Circular Path from a Point to a Line.

By Arnold Dresden, University of Wisconsin.

Introduction. In connection with the study of a certain type of problem in the calculus of variations, I was led to consider the following question:

"Let there be given a line $x = b$ and a system of circles tangent at the origin to a line $y = mx$ $(m > 0)$ (see figure 1). We take on each of these circles the arc from the origin to the nearer of the two points of intersection with the line $x = b$ and inquire which among the circles of the system furnishes for this arc the minimum value; that is, we inquire which among the circular arcs passing through the origin with slope m will furnish the shortest path from the origin to the line $x = b$."

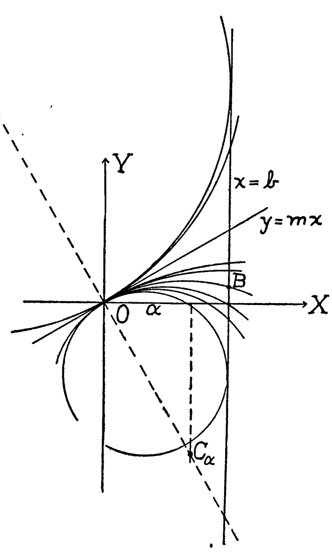

FIG. 1.

The intuitive reply to this question was that the shortest circular path is furnished by the circle which cuts the line $x = b$ at right angles. This reply proved to be incorrect. It seemed therefore worth while to publish a discussion of this question which incidentally yielded other points of interest.

1. The circles of the system are given by the equation

$$(x-\alpha)^2 + \left(y + \frac{\alpha}{m}\right)^2 = \frac{\alpha^2(m^2+1)}{m^2}, \tag{1}$$

the parameter α being the x-coördinate of the center. The points of intersection of this circle with the line $x = b$ are real if $\alpha \geqq b/(1+\mu)$ or $\alpha \leqq b/(1-\mu)$, where

$$\mu = \frac{\sqrt{1+m^2}}{m} = \operatorname{cosec} \phi,$$

$(\phi < \pi/2)$ being the inclination of the line $y = mx$. The length of the arc OB then becomes

$$L(\alpha) = \int_0^b \sqrt{1+y'^2}\,dx = \alpha\mu \int_0^b \frac{dx}{\sqrt{\alpha^2\mu^2-(x-\alpha)^2}} = \alpha\mu\left\{\arcsin\frac{b-\alpha}{\alpha\mu} + \arcsin\frac{1}{\mu}\right\},$$

this function being defined only for values of α greater than $b/(1+\mu)$ or less than $b/(1-\mu)$. The question we raised is that of determining the value of α for which $L(\alpha)$ is a minimum.

2. We introduce as a new variable the angle θ, defined by the relation

$$\theta = \text{arc sin}\,\frac{b-\alpha}{\alpha\mu} \qquad \text{or} \qquad \alpha = \frac{b}{1+\mu \sin\theta},$$

which transforms[1] the ranges $\alpha \geqq b/(1+\mu)$ and $\alpha \leqq b/(1-\mu)$ into the single continuous range $-(\pi/2) \leqq \theta \leqq (\pi/2)$. It is shown without difficulty that this angle θ represents the supplement of, or the negative, the inclination of the tangent to the circle C_α at its point of intersection with the line $x = b$.

We find then

$$L(\alpha) = L\left(\frac{b}{1+\mu\sin\theta}\right) = L_1(\theta) = \frac{b(\theta+\phi)}{\sin\theta + \sin\phi},$$

and hence

$$L'(\alpha) = L_1'(\theta)\frac{d\theta}{d\alpha} = -\frac{\sin\theta + \sin\phi - (\theta+\phi)\cos\theta}{\sin\phi\cos\theta}.$$

The maximum and minimum values of the function $L_1(\theta)$ will be found among the zeros of the function

$$F(\theta) \equiv \sin\theta + \sin\phi - (\theta+\phi)\cos\theta.$$

While for our purpose we are concerned only with the values of $F(\theta)$ on the range $(-\pi/2, \pi/2)$, we discuss the function in its entirety.

We find $F'(\theta) = (\theta+\phi)\sin\theta$, so that

$$F'(\theta) = 0, \qquad \text{for} \qquad \theta = -\phi, \qquad \text{and for} \qquad \theta = n\pi,$$

while $F'(\theta) > 0$ on the intervals

$$\cdots\ (-5\pi, -4\pi),\ (-3\pi, -2\pi),\ (-\pi, -\phi),\ (0, \pi),\ (2\pi, 3\pi),\ (4\pi, 5\pi)\ \cdots,$$

and $F'(\theta) < 0$ on the intervals

$$\cdots\ (-4\pi, -3\pi),\ (-2\pi, -\pi),\ (-\phi, 0),\ (\pi, 2\pi),\ (3\pi, 4\pi)\ \cdots.$$

Moreover since

$$F(0) = \sin\phi - \phi < 0, \qquad F(-\phi) = 0, \qquad F(\phi) = 2(\sin\phi - \phi\cos\phi) > 0,$$

we obtain for the graph of $F(\theta)$ a curve as sketched in figure 2. We conclude that there exists a value θ_1 such that $0 < \theta_1 < \phi$ and such that $F(\theta_1) = 0$.

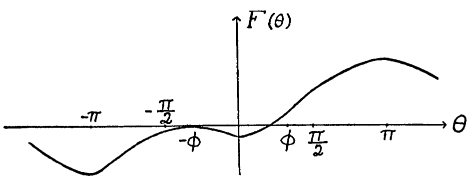

FIG. 2

Corresponding to the values $\theta = -\pi/2, -\phi, 0, \theta_1, \phi, \pi/2$, we have the values $\alpha = b/(1-\mu)$, $\pm\infty$, b, α_1, $b/2$, $b/(1+\mu)$, so that we can now obtain the graph of the function $L(\alpha)$ as in figure 3. The values $\alpha = b/(1-\mu)$ and $\alpha = b/(1+\mu)$ yield the circles of the system that are tangent to the line $x = b$ while $\alpha = \pm\infty$ yields the straight line $y = mx$.

[1] No value of α corresponds to $\theta = -\text{arc sin}\,1/\mu$; but

$$\lim_{\alpha\to}\ \theta = -\text{arc sin}\,1/\mu.$$

3. It is readily verified that

$$L\left(\frac{b}{1-\mu}\right) > L(\infty)$$

and

$$L\left(\frac{b}{1-\mu}\right) > L\left(\frac{b}{1+\mu}\right);$$

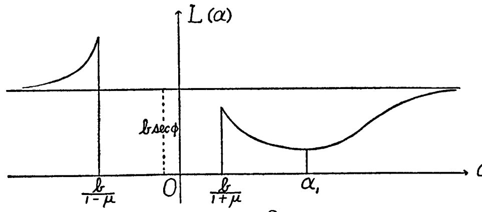

FIG. 3

but that $L[b/(1+\mu)]$ may be greater or less than $L(\infty)$. There is exactly one value of α, viz., α_1, for which $L(\alpha)$ has a minimum; this value α_1 lies between b and $b/2$, so that the shortest circular path from the origin to the line $x = b$ which departs from the origin with slope m lies between the circle which cuts this line perpendicularly and the one which passes through the point $(b, 0)$.

We have seen that $\alpha_1 = b/(1 + \mu \sin \theta_1)$, while θ_1 is the unique root of the equation $\sin \theta + \sin \phi - (\theta + \phi) \cos \theta = 0$ on the interval $(0, \pi/2)$. Hence, θ_1 is determined as a function of ϕ by the relation

$$\frac{\sin \theta_1 + \sin \phi}{\theta_1 + \phi} = \cos \theta_1.$$

From this it follows that $\lim_{\phi \to 0} \theta_1 = 0$; that is, among the circles tangent to the X-axis at the origin, the straight line furnishes the shortest path from the origin to the line $x = b$, which shows that the result obtained above is in accord with known facts.

II. Mathematical Articles in Encyclopedias.

By Norman Anning, University of Michigan.

The efforts of the Association to produce a Mathematical Dictionary that shall be a credit to American scholarship are, in the highest degree, praiseworthy. While that is in progress could not some arrangement be made with publishers who are bringing out important works of reference to have the proofs of mathematical articles read by mathematicians? The editors of the *Encyclopedia Americana* following the policy that articles on technical subjects should be written by experts have, in the main, been singularly happy in their choice of authors. With certain less technical articles they have been less fortunate. For every one who will read through Dr. Webster's excellent treatment of Hydrodynamics there are perhaps a hundred who will consult the *Encyclopedia* to find the "indispensable minimum" of information about the Hyperbola. This is, in part, what the latter will find:[1]

The length of the transverse axis is said to be a, $\cdots$. The two lines

$$\frac{x}{a} = \pm \frac{y}{b}$$

approach the hyperbola closer than any assigned limit, and are called the assymptotes (sic) of the

[1] Vol. 14, p. 601.

hyperbola. The points

$$\left(\pm a\left(\frac{\sqrt{a^2+b^2}}{a}-1\right), 0\right)$$

are called foci, $\cdots$. The ratio of the distance from a point on the curve from

$$\left(a\left(\frac{\sqrt{a^2+b^2}}{a}-1\right), 0\right)$$

to its distance from the line

$$x^a = \frac{a^2}{a^2+b^2}$$

has the constant value

$$\frac{\sqrt{a^2+b^2}}{a}\cdots.$$

The lines

$$x = \pm\frac{a^2}{a^2+b^2}$$

are known as directrices.

All in twenty-four lines.

To avoid the charge of partisanship let me add that a glance at the *eleventh* edition of the *Encyclopædia Britannica* reveals, in one short paragraph, the near facts:[1]

> Analytically the hyperbola is given by $ax^2+2hxy+by^2+2gx+2fy+c=0$ wherein $ab>h^2$. $\cdots$ In the rectangular hyperbola $a=b$; hence its equation is $x^2-y^2=0$ $\cdots$.

How can we believe and teach the doctrine that (as Bacon might have said it) "Mathematicks maketh the careful man" if such inaccurate statements are permitted in books which the public has a right to regard as sources of exact information?

III. Geometric Proofs of the Law of Tangents.

By H. C. Bradley, Massachusetts Institute of Technology.

Let ABC be the given triangle, $\angle A$ acute and greater than $\angle B$. Produce AC. Lay off $\angle ABD = \angle A$ on the same side of AB as C, forming the isosceles triangle ABD. Let E be the middle point of AB. Draw DE.

Produce BC. Lay off $CF = AC$. Draw AF, thus forming the isosceles triangle ACF. Let G be the middle point of AF. Draw GC, and produce it to meet DE at O.

Connect O with B. From O draw OH perpendicular to BC. Then

$$OH = BH \tan \angle OBH = CH \tan \angle OCH. \tag{1}$$

In the triangle BCD, CO and DO bisect the angles at C and D respectively. Hence BO bisects the angle at B. Whence $\angle OBH = \frac{1}{2}(A-B)$. Also, we have $\angle OCH = \frac{1}{2}(A+B)$.

In the triangle ABF, OE and OG are, respectively, perpendicular bisectors of the sides AB and AF. Hence OH is the perpendicular bisector of BF, and $BH = HF$. Whence $BH = \frac{1}{2}(a+b)$, and $CH = \frac{1}{2}(a-b)$.

[1] Vol. 14, p. 199.

bridge, 1707, page 122, where $\sin AEC$ is substituted for $\cos \frac{1}{2}(B - C)$ [E being the point, corresponding to D in Professor Lovitt's discussion, when AD is drawn bisecting the angle A.]

Simpson's formulæ were given by Mollweide, without reference to Simpson, in Zach's *Monatliche Correspondenz*, Gotha, volume 18, 1808, page 396.

To various geometrical proofs of the Law of Tangents already indicated in this MONTHLY (*1920*, 53–54, 465–467; *1921*, 71, 79, 170–171), might be added: one by Vignal in *Nouvelles Annales de Mathêmatiques*, volume 3, 1844, pages 456–457; and one by John Keill, the earliest I have met with, given in his anonymously published *Trigonometriæ Planæ & Sphæricæ*, Oxford, 1715, pp. 16–17.

IV. SOME FORMULAS OF ELEMENTARY TRIGONOMETRY.

BY W. J. RUSK, Grinnell College.

The formulas that are taken for granted are the sine formulas:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{1}{2R},$$

and the projection formulas:

$$a = b\cos\gamma + c\cos\beta, \qquad b = c\cos\alpha + a\cos\gamma, \qquad c = a\cos\beta + b\cos\alpha.$$

If we multiply these in order by a, b, c, and then add the first two results and subtract the third we get one of the cosine formulas; so we shall consider them as given also.

Consider the triangle ABC with $b < a$; take D on AB so that $CA = CD$; then $\angle\, DCB = \alpha - \beta$ and

$$DB = a\cos\beta - b\cos\alpha = \frac{a^2 - b^2}{c}.$$

1. Formulas for sin $(\alpha + \beta)$ and sin $(\alpha - \beta)$. We have from triangle ABC,

$$\frac{\sin\gamma}{c} = \frac{\sin(\alpha+\beta)}{a\cos\beta + b\cos\alpha} = \frac{1}{2R};$$

$$\therefore\quad \sin(\alpha+\beta) = \frac{a}{2R}\cos\beta + \frac{b}{2R}\cos\alpha$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

and from triangle CDB,

$$\frac{\sin\beta}{b} = \frac{\sin(\alpha-\beta)}{a\cos\beta - b\cos\alpha} = \frac{1}{2R},$$

or

$$\sin(\alpha-\beta) = \frac{a}{2R}\cos\beta - \frac{b}{2R}\cos\alpha,$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Again

$$\begin{aligned}\sin\gamma &= \sin(\alpha+\beta) = (c/a)\sin\alpha\\ &= [(a\cos\beta + b\cos\alpha)\sin\alpha]/a\\ &= \sin\alpha\cos\beta + (b/a)\sin\alpha\cos\alpha\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

Also

$$\begin{aligned}1 &= (a/c)\cos\beta + (b/c)\cos\alpha\\ &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)/\sin\gamma.\end{aligned}$$

$\therefore\quad \sin\gamma = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
and from triangle CDB,

$$\frac{\sin(\alpha-\beta)}{a\cos\beta - b\cos\alpha} = \frac{\sin\alpha}{a},$$

whence $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

2. **Formulas for cos** $(\alpha+\beta)$ **and cos** $(\alpha-\beta)$. We have from

$$a = b\cos\gamma + c\cos\beta,$$

$$\begin{aligned}-\cos\gamma &= \frac{c}{b}\cos\beta - \frac{a}{b},\\ &= \frac{(a\cos\beta + b\cos\alpha)\cos\beta}{b} - \frac{a}{b}.\\ &= \cos\alpha\cos\beta - \frac{a}{b}(1-\cos^2\beta)\end{aligned}$$

or $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$; and from the triangle CDB,

$$a = b\cos(\alpha-\beta) + (a\cos\beta - b\cos\alpha)\cos\beta;$$

$$\begin{aligned}\therefore\quad \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \frac{a}{b}(1-\cos^2\beta)\\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

Again from the triangle ABC, we have

$$c^2 = (a\cos\beta + b\cos\alpha)^2 = a^2 + b^2 - 2ab\cos\gamma$$

or

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta + \frac{a}{2b}(\cos^2\beta - 1) + \frac{b}{2a}(\cos^2\alpha - 1)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

And from triangle CDB, we find $\cos(\alpha-\beta)$, since

$$(a\cos\beta - b\cos\alpha)^2 = a^2 + b^2 - 2ab\cos(\alpha-\beta).$$

3. **Formulas for tan** $(\alpha+\beta)$ **and tan** $(\alpha-\beta)$. Drop the perpendicular CE upon AB in the triangle ABC. Then

$$\tan\alpha = -\tan(\beta+\gamma) = \frac{EC}{AE} = \frac{(b\cos\gamma + c\cos\beta)\sin\beta}{c - (b\cos\gamma + c\cos\beta)\cos\beta},$$

$$= \frac{\sin\beta\cos\gamma + \frac{c}{b}\cos\beta\sin\beta}{-\cos\beta\cos\gamma + \frac{c}{b}(1-\cos^2\beta)};$$

or,

$$\tan(\beta+\gamma) = \frac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma}.$$

For tan $(\alpha - \beta)$ drop DF perpendicular to BC. Then

$$\tan(\alpha-\beta) = \frac{FD}{CF} = \frac{(a\cos\beta - b\cos\alpha)\sin\beta}{a - (a\cos\beta - b\cos\alpha)\cos\beta},$$

$$= \frac{\frac{a}{b}\cos\beta\sin\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \frac{a}{b}(1-\cos^2\beta)},$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Proofs could have been obtained in the same way for the corresponding sine and cosine formulas.

4. **Half-angle formulas.** Produce the side AC to K so that $CK = a$ and apply the cosine formula to the triangle ABK. We have

$$c^2 = (a+b)^2 + 4a^2\cos^2\frac{\gamma}{2} - 2(a+b)2a\cos^2\frac{\gamma}{2}$$

$$= (a+b)^2 - 4ab\cos^2\frac{\gamma}{2}.$$

whence

$$\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}, \quad \text{and} \quad \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Now consider the triangle CDB. Produce DC to L so that $CL = a$, and apply the cosine formula to the triangle DBL. We have

$$\left(\frac{a^2-b^2}{c}\right)^2 = (a+b)^2 + 4a^2\cos^2\frac{\alpha-\beta}{2} - 2(a+b)2a\cos^2\frac{\alpha-\beta}{2}$$

$$= (a+b)^2 - 4ab\cos^2\frac{\alpha-\beta}{2}$$

or

$$4ab\cos^2\frac{\alpha-\beta}{2} = \left(\frac{a+b}{c}\right)^2[c^2 - (a-b)^2]$$

$$\therefore \quad \cos\frac{\alpha-\beta}{2} = \frac{a+b}{c}\sin\frac{\gamma}{2}; \qquad \sin\frac{\alpha-\beta}{2} = \frac{a-b}{c}\cos\frac{\gamma}{2},$$

and

$$\tan\frac{\alpha-\beta}{2}=\frac{a-b}{a+b}\cot\frac{\gamma}{2}.$$

Again if we call the perimeter of the triangle CDB, $2s_1$ and the sides a, b, c_1, it can be very easily proved that

$$s_1=\frac{a+b}{c}(s-b), \qquad s_1-a=\frac{a-b}{c}(s-c),$$

$$s_1-b=\frac{a-b}{c}s, \qquad s_1-c_1=\frac{a+b}{c}(s-a).$$

$$\therefore\quad \sin\frac{\alpha-\beta}{2}=\sqrt{\frac{(s_1-a)(s_1-b)}{ab}}=\frac{a-b}{c}\sqrt{\frac{s(s-c)}{ab}}=\frac{a-b}{c}\cos\frac{\gamma}{2},$$

and

$$\cos\frac{\alpha-\beta}{2}=\sqrt{\frac{s_1(s_1-c_1)}{ab}}=\frac{a+b}{c}\sin\frac{\gamma}{2};$$

which shows that Simpson's formulas and the tangent formula are equivalent to the half-angle formulas.

All of the above formulas have been proved on the condition that the angles are the angles of a triangle but they can be generalized in any of the usual ways.

Again from

$$a=b\cos\gamma+c\cos\beta, \qquad b=a\cos\gamma+c\cos\alpha$$

we have

$$\frac{a-b}{c}=\frac{\cos\beta-\cos\alpha}{1+\cos\gamma}, \quad\text{and}\quad \frac{a+b}{c}=\frac{\cos\beta+\cos\alpha}{1-\cos\gamma}.$$

$$\therefore\quad \frac{a-b}{c}=\frac{\sin^2\frac{\alpha}{2}-\sin^2\frac{\beta}{2}}{\cos^2\frac{\gamma}{2}}=\frac{\sin^2\frac{\alpha}{2}\left(1-\sin^2\frac{\beta}{2}\right)-\sin^2\frac{\beta}{2}\left(1-\sin^2\frac{\alpha}{2}\right)}{\cos^2\frac{\gamma}{2}},$$

$$=\frac{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{\cos^2\frac{\gamma}{2}}=\frac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{\gamma}{2}}$$

$$=\frac{2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)}{\sin\gamma}.$$

But

$$\frac{a-b}{c}=\frac{\sin\alpha-\sin\beta}{\sin\gamma}.$$

$$\therefore\quad \sin\alpha-\sin\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta).$$

It is not necessary to go through the detailed work in the case of other analogous formulas.

V. Multiplication of Large Numbers.

By H. S. Uhler,[1] Yale University.

In a recent number of the Monthly (*1921*, 114) I gave computations for $e^{\pm\pi}$, $e^{\pm\pi/2}$, $e^{\pm\pi/3}$, $e^{\pm\pi/6}$ to over fifty places of decimals. After mailing the article I set to work to form the product $e^{-\pi}e^{\pi}$, which should approximate to unity. The computation was performed in a single afternoon and evening and resulted in 1 followed by 52 zeros before the appearance of significant figures $3859\cdots$ It seems certain, therefore, that the earlier computations are correct to more than 50 decimal places,—my original tacit goal.

It may be of interest to give an account of the scheme used for multiplying two large numbers in such a manner as to avoid errors.

Buy suitable coördinate paper. Cut up long strips of cardboard about half an inch wide. Lay a strip of cardboard on the coördinate paper so as to write the digits properly spaced. Multiply the multiplicand by one of the nine digits 1, 2, $\cdots$, 9, one such result to be written on one strip of cardboard; so that nine strips will contain all the partial products to be used. No error can remain on these strips after careful inter-comparison and mutual checking, *e.g.*, $6 = 2 \times 3 = 5 + 1$, etc. Then place a properly spaced copy of the multiplier above and to the right of the large sheet of coördinate paper, and fasten it in position.

Commence with the extreme left-hand digit of the multiplier and select the corresponding strip of cardboard. Place the extreme right-hand digit of the cardboard exactly below or under the left-hand digit of the multiplier and copy on the coördinate sheet the figures on the cardboard. There is no mental work; and the eye cannot miss a single mistake, since each digit copied on the coördinate sheet is directly under the same digit on the cardboard.

Now place the next strip indicated by the multiplier one space lower and one space further to the right, and copy as before. Continue this process for successive figures of the multiplier, omitting unnecessary figures falling beyond the right side of the coördinate sheet.

No error should arise in adding the columns on the coördinate sheet because of repetition and other obvious means of checking. As a matter of fact, I have, as yet, made not a single mistake that I did not detect before proceeding further.

The illustration given shows the actual copy of the coördinate sheet as used for the multiplication of the numbers

[1] The text of this article was adapted by the editor from a private communication of Professor Uhler.—Editor.

$e^{\pi} = 23.14069\ 26327\ 79269\ 00572\ 90863\ 67948\ 54738\ 02661\ 06242\ 60021\ 16$

$e^{-\pi} = \ \ 0.04321\ 39182\ 63772\ 24977\ 44177\ 37171\ 72801\ 12757\ 28109\ 81063\ 30854$

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 8642 78365 27544 49954 88354 74343 45602 25514 56219 62126 61707 00256 2
 1296 41754 79131 67493 23253 21151 51840 33827 18432 94318 99256 05038 43
   43 21391 82637 72249 77441 77371 71728 01127 57281 09810 63308 53501 281
   17 28556 73055 08899 90976 70948 68691 20451 02912 43924 25323 41400 5124
      25928 35095 82633 49864 65064 23030 36806 76543 68658 86379 85121 00769−
       3889 25264 37395 02479 69759 63454 55521 01481 55298 82956 97768 15115+
         86 42783 65275 44499 54883 54743 43456 02255 14562 19621 26617 07003−
         25 92835 09582 63349 86465 06423 03036 80676 54368 65886 37985 12101−
          1 29641 75479 13167 49323 25321 15151 84033 82718 43294 31899 25605
             8642 78365 27544 49954 88354 74343 45602 25514 56219 62126 61707
             3024 97427 84640 57484 20924 16020 20960 78930 09676 86744 31597+
              302 49742 78464 05748 42092 41602 02096 07893 00967 68674 43160−
               38 89252 64373 95024 79697 59634 54555 21014 81552 98829 56978−
                  86427 83652 75444 99548 83547 43434 56022 55145 62196 21266+
                  25928 35095 82633 49864 65064 23030 36806 76543 68658 86380−
                   3889 25264 37395 02479 69759 63454 55521 01481 55298 82957−
                      2 16069 59131 88612 48872 08868 58586 40056 37864 05491−
                        30249 74278 46405 74842 09241 60202 09607 89300 96769−
                          864 27836 52754 44995 48835 47434 34560 22551 45622−
                          388 92526 43739 50247 96975 96345 45552 10148 15530−
                            3 45711 34611 01779 98195 34189 73738 24090 20582+
                              25928 35095 82633 49864 65064 23030 36806 76544−
                               1296 41754 79131 67493 23253 21151 51840 33827+
                                259 28350 95826 33498 64650 64230 30368 06765+
                                 30 24974 27846 40574 84209 24160 20209 60789+
                                  3 88925 26437 39502 47969 75963 45455 52101+
                                    17285 56730 55088 99909 76709 48686 91205−
                                     3457 11346 11017 79981 95341 89737 38241−
                                      216 06959 13188 61248 87208 86858 58640
                                       17 28556 73055 08899 90976 70948 68691+
                                        3 02497 42784 64057 48420 92416 02021−
                                          12964 17547 91316 74932 32532 11515+
                                           3457 11346 11017 79981 95341 89737+
                                              8 64278 36527 54449 95488 35474+
                                              2 59283 50958 26334 98646 50642+
                                                25928 35095 82633 49864 65064+
                                                  432 13918 26377 22497 74418−
                                                   25 92835 09582 63349 86465
                                                      86427 83652 75444 99549−
                                                      17285 56730 55088 99910−
                                                        864 27836 52754 44995+
                                                        259 28350 95826 33499−
                                                             8642 78365 27544+
                                                              432 13918 26377+
                                                               43 21391 82638−
                                                               25 92835 09583−
                                                                   43213 91826+
                                                                   30249 74278+
                                                                    2592 83510−
                                                                     388 92526+
                                                                      30 24974+
                                                                       2 16070−
--------------------------------------------------------------------------------
1.0000 00000 00000 00000 00000 00000 00000 00000 00000 00000 00038 59222 80390
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VI. Construction of the Regular Heptagon by a Quartic Curve.

By C. B. Haldeman, Ross, Butler County, Ohio.

Consider the quartic

$$x(a + y^2)\sqrt{-(b^2 + a^7)} = a^3y^4 - by^3 + 4a^4y^2 - 3aby + 2a^5, \tag{1}$$

which will be real when $b^2 + a^7$ is negative, and the circle

$$x^2 + y^2 = -4a, \tag{2}$$

which will be real when a is negative.

Eliminating x from (1) and (2), the result may be placed under the form

$$(y^7 + 7ay^5 + 14a^2y^3 + 7a^3y + 2b)(a^3y - 2b) = 0.$$

By the transformation $y = -2s\sqrt{-a}$ the first of these factors when placed equal to zero may be reduced to

$$7s - 56s^3 + 112s^5 - 64s^7 = \frac{b}{a^3\sqrt{-a}};$$

and because

$$7 \sin A - 56 \sin^3 A + 112 \sin^5 A - 64 \sin^7 A = \sin 7A,$$

we may take

$$s = \sin A, \qquad \frac{b}{a^3\sqrt{-a}} = \sin 7A$$

and get

$$y = -2\sqrt{-a}\,\sin\frac{1}{7}\sin^{-1}\frac{b}{a^3\sqrt{-a}},$$

$$y = -2\sqrt{-a}\,\sin\frac{1}{7}\left(2\pi + \sin^{-1}\frac{b}{a^3\sqrt{-a}}\right),$$

$$y = -2\sqrt{-a}\,\sin\frac{1}{7}\left(3\pi - \sin^{-1}\frac{b}{a^3\sqrt{-a}}\right),$$

$$y = -2\sqrt{-a}\,\sin\frac{1}{7}\left(\pi - \sin^{-1}\frac{b}{a^3\sqrt{-a}}\right),$$

$$y = 2\sqrt{-a}\,\sin\frac{1}{7}\left(\pi + \sin^{-1}\frac{b}{a^3\sqrt{-a}}\right),$$

$$y = 2\sqrt{-a}\,\sin\frac{1}{7}\left(3\pi + \sin^{-1}\frac{b}{a^3\sqrt{-a}}\right),$$

$$y = 2\sqrt{-a}\,\sin\frac{1}{7}\left(2\pi - \sin^{-1}\frac{b}{a^3\sqrt{-a}}\right).$$

The seven intersections, whose ordinates are the seven real roots of the above-mentioned factor, are the vertices of the regular heptagon required, as may be seen by reference to the above values of y.

The above factor of the seventh degree may be resolved algebraically by the transformation

$$y = x - \frac{a}{x}\cdot$$

The result is

$$y = (-b + \sqrt{b^2 + a^7})^{1/7} + (-b - \sqrt{b^2 + a^7})^{1/7}.$$

RECENT PUBLICATIONS.

REVIEWS.

Geschichte der Mathematik. II Teil, von Cartesius bis zur Wende des 18. Jahrhunderts; II. Hälfte, Geometrie und Trigonometrie. By HEINRICH WIELEITNER. Berlin, 1921, pp. vi + 220. Price, in Germany, 45 marks.

This small volume, one of the latest numbers in the well-known Sammlung Schubert, completes a work begun thirteen years ago by Professors Günther and Braunmühl,—a work much delayed by the World War. Like all volumes appearing in the Schubert series, it aims at presenting the best attainable knowledge, in a somewhat popular style, by a scholar of recognized standing, and with such a condensation of material as shall allow for its publication at a price within the reach of every teacher in Germany.

In the present work, Dr. Wieleitner has divided his material into nine chapters, as follows: I. Analytic geometry of the plane, especially as related to conic sections; II. Analytic geometry of space, including a study of surfaces; III. Higher curves in general; IV. Special curves; V. Differential geometry; VI. Perspective, and Descriptive geometry; VII. The first steps in projective geometry; VIII. Trigonometry; IX. Elementary geometry.

This is a wide range of subjects to be treated with any thoroughness in 176 pages of text, and yet it is safe to say that Dr. Wieleitner, as might have been expected from one of his scholarship and experience as a writer, has kept up the best traditions of the Sammlung Schubert. For example, he has presented in only fourteen pages the essential features of those contributions to the invention of analytic geometry made by Fermat and, to a lesser extent, by Vieta, Ghetaldi, and Cataldi, as well as those appearing in the epoch-making work of Descartes himself. In the following nineteen pages he condenses those essential topics in analytic geometry which attracted the attention of the contemporaries and the immediate successors of Descartes. Not only are the prominent features set forth, but the student is furnished with a helpful bibliography that allows him to branch out for himself,—a contribution that cannot be too highly commended.

Among the notable features of the work should be mentioned the list of curves in chapter IV. This is, of course, not so complete as the one given by Brocard in his *Notes de Bibliographie des Courbes Géométriques,* but it covers the important curves that the college student will meet in his studies, and includes a list of bibliographical references that will be of much service.

It will hardly be profitable to mention other special features of the text, or to consider the treatment of special topics. Suffice it to say that the book is worthy of the author and the series. Mention should be made, however, of the excellent bibliography at the end and of the unusually well-arranged indexes by names and by subjects.

Attention should be called frankly and distinctly to one other matter, and that is the price of the book. It may be stated at the outset that American scholars are generally desirous of seeing not only peace established among nations, but amicable relations resumed among scholars the world over. In particular, they are, in general, doing all in their power to foster good relations with their German colleagues and to arrange for an exchange of scientific literature of all kinds. But when it comes to the prices of German books they generally feel that a mistake is being made by Berlin and Leipzig publishers and by the German government. Here, for example, is a book that is published in Leipzig at 45 marks, list, subject to dealers' discount. At the current rate of exchange, this amounts to 25 cents, and yet the publisher quotes it to the American purchaser at $2.25. Americans are perfectly willing to pay the Leipzig price, and to add thereto any reasonable export duty that may be laid, together with the regular charge for postage; but it would be difficult to find any scholar, be he librarian or teacher, who feels that he should pay any such exorbitant price as this. It gives the impression that the German publishers and government expect Americans to help Germany pay for a war which America did its best to prevent and for which it has already paid out vast sums. The opinion may be wrong, but it is the opinion that is held, and it is very safe to say that the sale of German books in this country will never assume anything like its former proportions so long as this policy or any approach to this policy continues.

DAVID EUGENE SMITH.

The Sumario Compendioso of Brother Juan Diez. The Earliest Mathematical Work of the New World. By DAVID EUGENE SMITH. Boston and London, Ginn and Company, [April] 1921. sm. 4to. 7 + 65 pp. Price $4.00.

Let us first consider who the author of the *Sumario Compendioso* was.

In his *Rara Arithmetica*, Boston, 1908, page 286, D. E. Smith drew attention to this *Sumario* by "Juan Diaz Freyle" as "the first arithmetic printed in America." The author's name is later indexed under "Freyle" and not under either Diaz or Diez. In an address published in this MONTHLY, *1921*, 10–15, Professor Smith gave a general description of the *Sumario* and its setting, and quoted typical problems "listed under algebra" and "not listed under algebra." The title-page of the original work was here reproduced in facsimile, on page 11, where the author's name is "Juan Diez freyle." In the course of his sketch Professor Smith refers to the author as simply Juan Diez. In the work under review, his translation of the author's name is "Brother Juan Diez." This may be assumed as an indication of the present result of Professor Smith's investigations in this connection.

In each of the two works to which one would naturally turn for information regarding the *Sumario*, the name of the author is given as "Juan Diez Freile," and in the index of one of them the name occurs as "Diez Freile, Juan." The works in question are: (*a*) *Bibliografia Mexicana del Siglo XVI* by J. G. Icazbalceta, Primera parte, Mexico, 1886; (*b*) *La Imprenta en México* (*1539–1821*) by J. T. Medina, Tomo 1, Santiago de Chile, 1909. It were desirable, then, that Professor Smith had given the reasons for his change of opinion as to the interpretation of "Freyle." [1]

Regarding the existing copies of the *Sumario*, Professor Smith states that "there remain perhaps only four copies" and he quotes Icazbalceta (*l.c.*) who records that one copy was in the "Convento de la Merced," and that a second was sold in the Ramírez sale (1880) for £24; he adds that a third copy is in the British Museum, and that an imperfect fourth copy, from which his own photographic copy was made, is in the "Biblioteca Nacional at Madrid"; this copy lacks the last three folios.[2] Medina lists (*l.c.*) three known copies: one in the British Museum, one in "Biblioteca del Ministerio de Fomento en Madrid," and one in "Biblioteca de don Jacobo Parga en Madrid." One might infer, then, that there were six existing copies of the *Sumario*. The second copy listed by Medina seems, however, to be nothing but the copy in the "Biblioteca Nacional"; for, witness the library stamp on the title page reproduced in this MONTHLY (*l.c.*) from Professor Smith's photographic copy. Medina's third copy *may* be the one sold at the Ramírez sale. We conclude, then, that while there are at least four copies of the *Sumario*, there may be not less than five.[3]

The *Sumario* consists of 103 folios, including 2 pages for the dedication, 24 pages of mathematical text (18 pages relating directly to arithmetic, and 6 to algebra), an elaborate set of tables (about 180 pages), and the colophon. The work under review contains a facsimile of the mathematical text, and colophon, accompanied by a translation and valuable notes. Facsimiles of the title page and of one page of the tables are also given. Then there are 11 pages of introductory matter: "Mexico of the period," pages 3–4; "Printing established in Mexico," 5–6; "General description of the book," 7–8; and "Nature of the tables," 9–11. The outstanding facts in connection with this material have been already set forth in this MONTHLY (*l.c.*).

The "earliest mathematical work of the New World" must ever be one of great interest to the mathematician, who will be profoundly grateful to Professor Smith for so much of the work as he has made generally accessible and intelligible. The historian will naturally regret that such a rare book was not reproduced in its entirety, with commentary which Professor Smith is so finely equipped to provide.

The work under review was published only in an edition de luxe on Old

[1] In his review of the *Sumario*, Professor L. C. Karpinski questioned the propriety of Professor Smith's interpretation. See *School and Society*, August 13, 1921.

[2] "The University of Michigan library possesses the complete work in rotographs of the British Museum copy." L. C. Karpinski (*l.c.*).

[3] Thus Professor Karpinski made a slip in writing (*l.c.*): "Only two copies of the work appear to be known, the one incomplete in the Escurial and the other in the British Museum."

Stratford paper, and limited to 394 copies of which 210 were taken in advance of publication. It is a beautiful example of the bookmaker's art. Ginn and Company deserve heartiest congratulations.

R. C. ARCHIBALD.

Primitive Groups. By W. A. MANNING. Part I. (*Stanford University Publications, University series,* Mathematics and Astronomy, volume 1, no. 1.) Stanford University, California, 1921. Royal 8vo. 108 pages. Paper. Price $1.25.

Preface: "Some knowledge of Algebraic Numbers and of the ordinary Theory of Numbers is assumed to have been acquired by the reader by way of preparation for a serious study of the subject of which this volume treats.

"An apology may be in order for the arrangement of the subject matter. It was arranged as it is to meet the needs of actual instruction. The use of 'group characteristics,' as developed by Frobenius, should be a familiar tool in the hands of the student as early as possible. Therefore linear substitutions are taken up in the third chapter. From the point of view of strict logic this study of linear substitutions and of linear groups should be quite fully developed before those very special substitutions which we call permutations are considered. But the idea of groups of non-commutative operations can, in the author's opinion, be best gained from a few lessons on the concrete and familiar permutations of a finite number of letters. Therefore the first two chapters are intended to familiarize the learner with the simpler processes used in Group Theory, to exhibit the fundamental theorems which admit of briefly worded proof, and to prepare the way for the more difficult developments of linear groups. Moreover, since any 'abstract' group of finite order is isomorphic to some group of permutations, it would seem that sufficient generality can be attained if the phraseology of the abstract theory is ignored, as is done in this book.

"In talking of prime numbers it is admitted that it is a matter of indifference whether unity is included among the primes or not. May one be permitted the same license, if for the sake of convenience in stating certain theorems, the identical substitution alone is denied the dignity of being called a group (§ 4)? The new terms 'similar groups' (§ 16), 'open product' (§ 21) and 'uniprimitive group' (§ 37) seem useful and necessary.

"In justification of the publication of these pages in our University series, it may be stated that some of the material to be found in the volume is new. In particular, theorems II of § 37, I of § 38, and I of § 45 have not been published elsewhere.

"Among the sources from which the author has drawn inspiration and material the following treatises should be mentioned: Jordan, *Traité des Substitutions;* Weber, *Lehrbuch der Algebra;* Burnside, *Theory of Groups;* Dickson, *Linear Groups;* Miller, Blichfeldt and Dickson, *Finite Groups;* Blichfeldt, *Finite Collineation Groups;* Hilton, *Linear Substitutions.*

"But the memoirs of Jordan and of Frobenius have contributed more by way of suggestion and encouragement than any books."

Contents—Chapter I: The elementary theory of groups of permutations, 7–27; II: Transitive groups, 28–44; III: Group characteristics, 45–69; IV: Applications of group characteristics, 70–80; V: Transitive groups, 81–91; VI: Primitive groups with transitive subgroups of lower degree, 92–108.

Computing Jetons. By D. E. SMITH. (*Numismatic Notes and Monographs,* no. 9.) New York, The American Numismatic Society, 1921. 16mo. 2 + 70 pp. + 5 plates. Paper cover, price $1.50.

This monograph, embellished with 20 pages of illustrations in addition to the plates, is based upon an address delivered by the author before the American Numismatic Society, in New York City, on February 7, 1921. Introductory paragraphs: "In accepting the invitation . . . to speak upon the subject of Computing Jetons, I have naturally considered the possibility of offering something that might appeal to its members as not already familiar. Few works upon any subject relating to numismatics are so exhaustive in their special fields as the monumental and scholarly treatise of Professor Francis Pierrepont Barnard (*Casting-Counter and Counting-Board,* Oxford, 1916), and hence it may seem quite superfluous, and indeed presumptuous, to attempt to supplement such a storehouse of information.

"Professor Barnard, however, approached the subject primarily from the standpoint of a numismatist, a field in which he is an acknowledged expert, as witness the honor that has recently come to him in his appointment as curator of coins and medals in the Ashmolean Museum at Oxford, and so it has seemed to me that I might make at least a slight contribution by approaching it from the standpoint of a student of the history of mathematics. It would, in that case, be natural to consider primarily the need for, the use of, and the historical development of the jeton in performing mathematical calculations, and this is the pleasant task that I have set for myself in preparing this monograph.

"Although Professor Barnard has also considered this field, I hope to contribute something in the way of illustrative material, at least, and perhaps to make somewhat more prominent the early history of a device which, in one form or another, seems to have dominated practical calculation during a good part of the period of human industry."

Contents—Necessity for aids in computation, 3–5; The dust abacus, 6–7; Early forms of the line abacus, 7–8; The Roman counters, 8–10; The abacus in the orient, 11–14; The Gerbert abacus and jetons, 15–16; The late European line abacus, 17–29; Names for counters or jetons 30–33; The exchequer, 34–36; Method of computing with jetons, 37–63; History of minted jetons, 64–69; Summary 69–70.

Manhood of Humanity. The Science and Art of Human Engineering. By ALFRED KORZYBSKI. New York, E. P. Dutton & Co., 1921. 8vo. 17 + 264 pp. Price $3.00.

The publishers state that Professor C. J. KEYSER refers to this book[1] as follows: "a momentous contribution to the best thought of these troubled years. It is momentous in what it contains, even more so in what it suggests, and most of all, I dare say, in the excellent things it will eventually help men and women to think and say and do. Its core is a great conception, which is new; it is a conception of man in terms of Time. Like all really great ideas, it is intelligible to all and is universal in its interest and appeal. It is, I believe, destined to light the way in all the cardinal concerns of human kind."

Contents—Chapter I: Introduction (Method and processes of approach to a new concept of life), 1–26; II: Childhood of humanity, 27–45; III: Classes of life, 46–65; IV: What is man? 66–92: V: Wealth, 93–118; VI: Capitalistic era, 119–138; VII: Survival of the fittest, 139–154; VIII: Elements of power, 155–166; IX: Manhood of humanity, 167–203; X: Conclusion, 204–208. Appendices—(a) Mathematics and time-binding, 209–223; (b) Biology and time-binding, 224–254 [pages 245–250: quotations from Karpinski, Benedict and Calhoun's *Unified Mathematics* on laws of growth, the curve of healing of a wound, wave motion]; (c) Engineering and time-binding, 255–264. There are numerous references to the literature of the subject.

Latitude Developments connected with Geodesy and Cartography, with Tables including a Table for Lambert Equal-Area Meridional Projection. By O. S. ADAMS. (Department of Commerce, U. S. Coast and Geodetic Survey, special publication no. 67.) Washington, Government Printing Office, 1921. 12mo. 132 pp. Price 20 cents.

Foreword (first two paragraphs): "There are five different kinds of latitude that come under consideration in the application of mathematical analysis to questions of geodesy and cartography. It is the aim of this publication to express the difference between the geodetic or astronomic latitude and each of the various four other kinds of latitude in a series of the sines of the multiple arcs. This difference in each case is obtained in an expression in the sines of the multiple arcs of the geodetic or astronomic latitude and also in a series of the sines of the multiple arcs of the other latitude in question.

"The analysis connected with the development of both the isometric or conformal latitude and of the authalic or equal-area latitude is given in some degree of detail, since it is a good example of the application of mathematical analysis to such questions."

[1] The ideas of the book are the basis of an address, by Professor Keyser, published in *Science*, September 9, 1921, pp. 205–213.

Philosophy and the New Physics. An Essay on the Relativity Theory and the Theory of Quanta. By LOUIS ROUGIER. Philadelphia, P. Blakiston's Son & Co., 1921. 12mo. 15 + 159 pp. Price $1.75.

This work is an authorized translation by Dr. Morton Masius, professor of physics in the Worcester Polytechnic Institute, from the author's corrected text of "La matérialisation de l'énergie."

Translator's preface: "The recent remarkable developments of physical theories, especially those concerned with relativity and quanta of energy, cannot fail to have far-reaching influences on philosophical thought. Physicists, as a rule, are too much occupied with their special field to give much attention to matters of more general philosophical interest, and few philosophers possess the knowledge of science required for discussing and criticizing fruitfully the work of the physicist. Professor Rougier's very wide reading in mathematical and experimental physics has enabled him to present and interpret the new advances in Physics in a way which should prove of great interest to both philosopher and physicist. This book seems to mark a measurable advance toward a confluence of the broad streams of philosophical and scientific enquiry."

Contents—Chapter I: "The dualism of matter and energy," pages 1–21; II: "Mass and the relativity principle," 22–40; III: "Electromagnetic dynamics," 41–56; IV: "The electronic theory of matter," 57–72; V: "The inertia of energy," 73–90; VI: "The weight of energy," 91–109; VII: "The structure of energy," 110–147; VIII: "Conclusion," 148–153; Bibliography, 153–155; Index of names, 157–159.

Praktisches Zahlenrechnen. (Sammlung Göschen no. 405.) By P. WERKMEISTER. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. 16mo. 135 pp. Price 4.20 marks.

This new volume of an admirable series is bound in flexible but durable paper covers instead of stiff cloth covers used before the war. It surveys the elementary parts of a field not over familiar to American mathematicians, introducing a number of historical notes and references to the literature of the subject. The main headings of the contents are as follows—Section I: Calculation without special aids [(*a*) Exact calculation; (*b*) Approximate calculation], 9–41; II: Calculation with the aid of tables [(*a*) Exact calculation with the aid of numerical tables—multiplication or product, quarter square, etc.; (*b*) Approximate calculation with the aid of numerical and graphical tables—logarithmic, of squares and square roots, etc.], 41–60; III: Calculation with the use of mechanical aids [(*a*) Exact calculations—with calculating machines; (*b*) Approximate calculations—by means of slide rules], 60–91; IV: Graphical calculation [(*a*) Treatment of fundamental operations; (*b*) Solution of equations; (*c*) Differentiation and integration; (*d*) Calculation of errors], 91–133; Subject index, 134–135.

Three Lectures on Fermat's Last Theorem. By L. J. MORDELL. Cambridge, at the University Press, 1921. 8vo. Pamphlet, 3 + 31 pages. Price 4s.

This booklet contains lectures in practically the form in which they were delivered at Birkbeck College, London, in March, 1920. It also contains a few details omitted from the lectures. For full references on the subject the reader is referred to L. E. Dickson's "very useful paper," "Fermat's last theorem," in *Annals of Mathematics*, vol. 18, 1917, and to volume 2 of his *History of the Theory of Numbers.*

The first chapter is entitled "Statement of the theorem" and contains subheadings "Did Fermat prove his theorem?" "Analysis of another statement of Fermat," "A simplification of the problem," "The equation $x^2 + y^2 = z^2$," "The equation $x^4 + y^4 = z^4$," "The equation $x^3 + y^3 = z^3$," "The equation $x^5 + y^5 = z^5$ and $x^7 + y^7 = z^7$." The second chapter (pages 10–26) considers Kummer's work and its consequences. The brief third chapter entitled "Libri's result" has as subheadings "Sophie Germain's result" and "Wendt's form of the result."

Some Investigations in the Theory of Map Projections. By A. E. YOUNG. (R. G. S. Technical Series, no. 1.) London, Royal Geographical Society, 1920. 8 + 76 pages. 8vo. Price 6 shillings.

These exhaustive investigations are concerned mostly with the minimum error projections

invented by Airy and Clarke. Their results are extended and simplified "so that, out of the almost bewildering number of projections that have been discovered and advocated from time to time, those which are practically the best or most useful are reduced to comparatively few, the application of which has been simplified."

The first chapter (pages 1–21) deals with the minimum error zenithal projections, the second (pages 22–56) with the minimum error conical projections, the third (pages 57–65) with the conical orthomorphic projection with two standard parallels (Lambert's second) for the spheroid, the fourth (pages 66–68) with the polyconic projections, the fifth (pages 69–72) with finite errors of projections, and the sixth with the convergency of meridians. Mr. Young's survey has shown the worth of some old projections "which seem to have been hit upon by their inventors by a sort of geometrical intuition rather than by rigorous analysis." An example of this is Murdoch's remarkable third projection, dating back to 1758, and yet "the very best of the conical class." A discussion of G. W. Hill's conical projection (*Annals of Mathematics*, 1908) leads to the conclusion that it does not appear to have "any advantage over those we have investigated and it is certainly more difficult to compute."

The work is thoroughly mathematical and constitutes a most valuable contribution to the subject. It may be of interest to note in conclusion certain expansions of use in discussions of this kind: $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$, $\sin^{-1}\theta$, $\tan^{-1}\theta$, $\log_e\sin\theta$, $\log_e\sec\theta$, $\log_e\tan\theta$, $\log_e(1+\theta)$, $\log_e(1-\theta)$, $\log_e[(1+\theta)/(1-\theta)]$ and $\tan(\theta+h)$ as a power series in h with coefficients powers of $\tan\theta$. Most of the expansions are to the eighth or ninth degree in θ.

Edinburgh's Place in Scientific Progress. Prepared for the Edinburgh Meeting of the British Association [1921]. Edinburgh and London, W. & R. Chambers, 1921. 12mo. 16 + 263 pages. Price 6 shillings.

This very interesting volume, with a preface by C. G. Knott, contains brief sketches on 25 topics by 23 different authors. "Mathematics and natural philosophy" is treated by C. G. Knott, pages 1–30; "Astronomy" by R. A. Sampson, 31–32; "Actuarial Science" by A. E. Sprague, 33–35; "Meteorology" by A. Watt, 36–43; and "Engineering" by T. H. Beare. Eight portraits are inserted in the volume, and the frontispiece in colors is of John Napier of Merchiston.

Among the numerous names occurring in Doctor Knott's sketch are the following: John Napier, James Gregory, David Gregory, Colin Maclaurin, James Stirling, James Ivory, John Playfair, John Leslie, William Wallace, Philip Kelland, D. F. Gregory, George Chrystal and P. G. Tait. It is stated that "in the light of accurate history" Napier "stands preëminent as the first great scientific Scotsman."

Mathematik in der Natur. By H. Emch. Zürich, Rascher & Cie., 1921. 12mo. 86 pages. Price, paper cover, 2 francs.

This little volume was distributed free to subscribers of *Natur und Technik*, with Heft 12 of Jahrgang 1920–21. The eight chapters have the titles: Geometry in plant and animal bodies; Concerning architecture with the smallest building stones of the world; Where power to comprehend and to visualize is lacking, there mathematics always helps further; Mathematical fundamental problems of mechanics in nature; Number in plant and animal bodies; Cells, molecules, atoms, electrons. We here find brief references, in popular manner, to matters treated, for the most part, by D'Arcy W. Thompson, in his *On Growth and Form*, 1917, in more scholarly fashion. (Compare this Monthly, *1918*, 189–193, 232–238, where logarithmic spiral forms, golden section, and Fibonacci series are discussed; see also *1920*, 314.)

Grundzüge der Einsteinschen Relativitätstheorie. By August Kopff. Leipzig, S. Hirzel, 1921. 8vo. 4 + 198 pages. Price, bound, 42.50 marks.

This introduction to the Einstein theory of relativity was developed from lectures, delivered in the winter semester of 1919–20 and in the summer semester of 1920, at the University of Heidelberg, where Dr. Kopff is extraordinary professor of astronomy. "It aims," the author states in the preface, "in the simplest possible way again to set forth the fundamental investigations of this theory, in connection with which a mathematical presentation can not be avoided. Without deep penetration into the mathematical problem of the theory of relativity, one can never really

understand the underlying thought. The theory of relativity belongs to theoretical physics in the widest sense and this is mathematical description of the physical processes of nature." In the course of the book the only assumed knowledge of mathematics and physics is such as is given in the first semesters at a university.

Fundamentals of High School Mathematics. A Textbook designed to follow Arithmetic. By H. O. Rugg and J. R. Clark. Yonkers-on-Hudson, World Book Co., 1920. 15 + 368 pp. 12mo. Price $1.80. [Answer book, 16 cents.]

Attention is drawn to this work for first year high school mathematics, by teachers in the Lincoln School, New York. It is a development from their study, *Scientific Method in the Reconstruction of Ninth-Grade Mathematics* (University of Chicago Press, 1918, 8vo. 189 pages), and takes account of principles formulated by the National Committee on Mathematical Requirements. A rough draft "experimental edition" (8 + 266 pages) was published in 1918 and distributed at cost for experimental purposes only.

The Slide Rule: A Practical Manual. By C. N. Pickworth. Seventeenth edition. Manchester, Emmot & Co.; London, Emmot & Co., and Pitman & Sons, 1921. 12mo. 133 pp. Price 3 shillings and 6 pence (the New York agent, I. Pitman & Sons, charges more than double this price, namely, $1.50).

This book has been well known for twenty years, the eleventh edition appearing in 1908, the fourteenth in 1916, and the fifteenth in 1917, each of these editions containing revisions and new matter. The present edition contains eight pages more than the fourteenth, the additions including descriptions of new slide rules, and a section dealing with screw-cutting gear calculations by the slide rule.

NOTES.

In *Bulletin des Sciences Mathématiques*, July–August, 1921, there is a historical article by C. de Waard entitled "Une lettre inédite de Roberval du 6 Janvier, 1637, contenant le premier énoncé de la cycloïde."

In *Proceedings of the Benares Mathematical Society*, volume 2, part 2, 1920, is published "On mathematical research in the last 20 years," the presidential address delivered on January 31, 1921, by Dr. Ganesh Prasad, professor of mathematics in the Benares Hindu University. Compare *1921*, 31, 179, 191.

Revista Matemática Hispano-Americana, June, 1921, contains, pages 161–166, a portrait, brief biographical sketch, and bibliography of the writings (59 titles), of C. J. de la Vallée-Poussin.

The eleventh paper in *Proceedings of the Royal Society of Edinburgh*, volume 41, 1920–1921, is (pages 111–116) "Note on a continuant of Cayley's of the year 1874" by Sir Thomas Muir, the African member of our Association.

There are 81 names in the list of members of the "Gazeta Matematică" Society of Roumania, published in the first number, September, 1921, of *Gazeta Matematică*, volume 27.

The summer number of *Isis*, 1921, includes the following articles: "Two twelfth century algorisms" by L. C. Karpinski, 396–413; "History of symbols for n-factorial" by F. Cajori, 414–418.

The fourth and last volume of E. Beltrami, *Opere matematiche pubblicate per cura della Facoltà di Scienze della R. Università di Roma*, was issued from the press of Hoepli, Milan, in 1920 (554 pages; price 50 lire).

In *L'Intermédiaire des Mathématiciens,* July–August, 1920, there is an article, "Charles Ange Laisant (1841–1920)" by the editors—H. Brocard gives additional biographical and bibliographical material concerning Laisant (compare *1921,* 319) in *Sphinx-Œdipe,* August–December, 1921.

In *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Mathematisch-naturwissenschaftliche Klasse, Abteilung A. Mathematisch-physikalische Wissenschaften,* 1920, about half the numbers deal with topics of mathematical interest. Number 7, by Oskar Perron, is entitled: "Paul Stäckel†," and number 14, by Karl Bopp: "Moritz Cantor†. Gedächtnisrede, gehalten im Mathematischen Verein zu Heidelberg am 19. Juni 1920."

We are glad to note that the editors of *Journal of the Mathematical Association of Japan for Secondary Education* have found in our MONTHLY yet other articles (cf. *1921,* 79) of interest to their constituency. In the issue for April, 1921 (pages 47–50), there is a Japanese translation of Professor A. S. MERRILL's article, "The 'danger area' curve" (*1920,* 398–410). In the issue for July (pages 94–98) there is practically a complete transcript, in rearranged form, of a list, prepared by the Library Committee of our Association, of 160 mathematical books for schools and colleges (*1917,* 369–376).

The concluding number of *Bulletin of the Calcutta Mathematical Society,* volume 11 (262 pages), was published in March, 1921. This periodical, devoted to higher mathematics, pure and applied, is excellently printed and edited. At the close of 1920 the Calcutta Mathematical Society had 170 ordinary members and 25 honorary members. During 1920, 32 papers were read, and the published accounts show a surplus—a state of affairs which most mathematical societies must envy.

L'Astronomie et les Astronomes by AUGUSTE COLLARD (Bruxelles, G. Van Oest et Cie., 1921. 8 + 119 pages) is a useful bibliography with the following headings: (*a*) Dictionaries and encyclopaedias of astronomy; (*b*) Biographies of astronomers; (*c*) Treatises on astronomy, subdivided into many sections; (*d*) Histories; (*e*) Bibliographies; (*f*) Atlases; (*g*) Reviews; and (*h*) Tables. Brief notes summarizing the scope of the work are added to the titles in many cases. It is intended as a supplement to the work of Houzeau and Lancaster, 1882–1889.

The last number (published, May, 1921) of *Proceedings of the London Mathematical Society,* volume 19, contains the following obituary notices: "Hieronymus Georg Zeuthen" (1839–1920) by H. W. R[ichardson], xxxvi–xxxix; "Srinivasa Ramanujan" (1887–1920) by G. H. H[ardy], xl–lviii; "Philip Edward Bertrand Jourdain" (1879–1919) by D. M. W[rinch], lix–lx. We are especially tempted to quote from the very interesting memoir by Professor Hardy, but we must confine ourselves to references to quotations already made in this connection: *1920,* 316, 338; *1921,* 219, 224.

Proceedings of the American Academy of Arts and Sciences, volume 56, no. 10, July, 1921, is devoted to The Rumford Fund for researches in light and heat, and contains a list of the various awards and grants (to 102 individuals) from

the first in 1839 to the two hundred and thirty-first in December, 1920. This list includes: a grant in 1883 of $30 to F. N. COLE towards "experiments on Maxwell's theory of light"; a grant in 1913 of $200 to H. N. DAVIS for thermodynamical researches; 3 grants, 1877–1895, of $200–$250 each to B. O. PEIRCE; and 7 grants, 1901–1920, of $65–$500 each to A. G. WEBSTER.

The second and concluding number of the second volume (104 pages) of *Bulletin de la Société Mathématique de Grèce* (compare *1920*, 314; *1921*, 134) contains an article by E. T. BELL, of the University of Washington, entitled "Sur la forme $x^2 + 3y^2$ et l'équation modulaire pour la transformation du troisième ordre des fonctions elliptiques," 70–74. There is also (pages 100–101) a sketch of NIKOLAOS KARATSANIDES, professor of descriptive geometry and surveying at the Polytechneion (Greek Institute of Technology), Athens, who died February 18, 1920. He was born in 1852, taught in Bulgaria five years, and came in 1885 to Athens where as student, assistant, and teacher he remained for the rest of his life. He was the author of articles and of a small book in descriptive geometry for the lycea (high schools).

The seventeenth volume (1884–1900, Marc–P) of the *Royal Society Catalogue of Scientific Papers* has been recently published (Cambridge University Press, £ 9, cloth binding; £ 10 s. 10, half morocco). The sixteenth volume (I–Marbut) appeared in 1918. One more volume will probably complete the work and the listing of over 300,000 papers published during the seventeen years 1884–1900. This invaluable author-index, covering the nineteenth century in eight alphabets, is too well known to need extended comment. Four volumes of a subject-index for the same period have also appeared. These are the volumes devoted to Mathematics, Mechanics, and Physics (2). It is sometimes useful to recall that the *Royal Society Catalogue* covers more than the nineteenth century in the case of some publications, for example: *Transactions of the American Philosophical Society*, 1771–1799; *Journal des Mines*, 1794–1799; *Memorie di Matematica e Fisica della Società Italiana delle Scienze*, Modena and Verona, 1782–1799; *Bulletin des Sciences de la Société Philomathique de Paris*, 1792–1799; and *Mémoires de l'Académie Royale des Sciences de Turin*, 1784–1799.

In *Archivio di Storia della Scienza* (*1921*, 134), volume 2, no. 1, published in January, 1921, there is a supplement, page 119, to A. Mieli's bibliography of Leonardo da Vinci (*1920*, 217) and a brief notice, page 97, of F. Cajori's *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*. In nos. 2–3, June, the contents include the following: "Sur l'auteur d'un traité 'De Motu' auquel Bradwardin a fait allusion en 1328" by G. Eneström, 133–136; "Philip E. B. Jourdain, matematico e storico della scienza (1879–1919)," 167–184 [there are 107 titles in the list of his publications]; "La storia della matematica presso i Cechi" by Q. Vetter, 199–201; "Gli studi geometrici di Eudosso da Cnido" by E. Ruffini, 222–239. Eudoxus (about 408–355 B.C.) was the first to discover the 'method of exhaustions' and to give a scientific proof that the cone and the pyramid are one third of the cylinder

and prism respectively which have the same base and height. He was also the originator of the theory of proportion, covering incommensurables, as expounded in books 5 and 6 of Euclid's 'Elements'; the first five propositions of book 13, involving golden section, are probably due to him. The curve hippopede (horse-fetter), or kampyle, or spherical lemniscate, invented by him, played a fundamental rôle in the ancient system of astronomy of which he was the author. Full discussion in this connection may be found in T. L. Heath, *Aristarchus of Samos*, Oxford, 1913, pages 190–212.

Attention is directed to an important new work, *Physics, The Elements* by N. R. CAMPBELL (Cambridge University Press, 1920. Royal 8vo. 9 + 565 pages. Price 40 shillings). The first part (264 pages) deals with "The propositions of science" and there are chapters on 'Chance and probability' (pages 159–214), 'The meaning of science' (pages 215–219) and 'Science and philosophy' (pages 230–265). The second part (283 pages) discusses "Measurement," with the chapter titles: Fundamental measurement; Physical number; Fractional and negative magnitudes; Numerical laws and derived magnitudes; Units and dimensions; The uses of dimensions; Errors of measurement, methodical errors; Errors of measurement, errors of consistency and the adjustment of observations; Mathematical physics.—Another work *Physik und Hypothese. Versuch einer induktiven Wissenschaftslehre nebst einer kritischen Analyse der Fundamente der Relativitätstheorie* by HUGO DINGLER (Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1921, 8vo. 11 + 200 pages. Price, paper covers, $1.50) is a supplementary volume to the author's *Grundlagen der Physik* (same publisher, 1919. 12 + 158 pages. Price, paper covers, 16.50 marks). The treatment is philosophic, and very little mathematical symbolism occurs in the book. "But my researches have an important relation to mathematics. Only by the path here taken can we arrive at the explanation of the nature of the axioms, their proper grounding, etc." (preface). It will be recalled that Dr. Dingler is the author of *Das Prinzip der logischen Unabhängigkeit in der Mathematik* (München, Theodor Ackerman, 1915. 6 + 164 pages); in the work under review he promises a volume on the philosophy of mathematics.

Flatland. A Romance of Many Dimensions. By the Author A. Square. With illustrations by the Author. "Fie, fie, how frantically I square my talk." London, Seeley & Co., 1884. Such was the title page of a crown quarto booklet, issued anonymously in parchment wrappers (100 pages), which in one form or another has delighted and informed more than a generation of readers—mathematical and otherwise. In 1885 a much less attractive edition, of duodecimo format, was published by Roberts Brothers, Boston (155 pages); it was reprinted in 1891. In 1896 the work was taken over by Little, Brown & Company, Boston, and their first edition appeared in 1899; there were other editions in 1907, 1911, 1915 and 1919. About 2,700 copies have been printed in America and the book is still procurable. In 1908 a Dutch translation, by L. van Zanten Jzn., was published at Zalt-Bommel (138 pages); the second Dutch edition appeared at

Zalt-Bommel in 1915, and the third at Amsterdam in 1920: *Platland. Eene roman van vele afmetingen.* Met Illustraties. Door Een Vierkant. Uit het Engelsch. Vierde Druk. The format and type are similar to that of the English edition,[1] which has been long out of print.

In none of these eleven editions or reprints is there any hint as to the author of the work, namely, EDWIN ABBOTT ABBOTT,[2] English schoolmaster and theologian who graduated at St. John's College, Cambridge, where he took the highest honors in the classical, mathematical, and theological triposes. It was probably for this reason that he never heard of the Dutch translation of his book until it was brought to his attention in our correspondence last March.

Dr. Abbott is now in his eighty-third year. After holding masterships in several schools he retired in 1889 to devote himself to literary and theological pursuits. He has been a prolific writer. His *Shakespeare Grammar*, 1869, a "permanent contribution to English philology," and *How to Write Clearly*, 1872, have gone through many editions. He published a dozen other books, including two anonymous religious romances, before his single mathematical publication, where "the assumption of the author is worked with wonderful consistency, and his mathematics are thoroughly sound." His numerous other books, several of which were published anonymously, down to 1917, have dealt almost wholly with religious and theological topics. Other information regarding Dr. Abbott's work may be found in *Who's Who* and in the last edition of the *Encyclopædia Britannica*. (See *1919*, 264.)

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 43, July, 1921: "Integral products and probability" by P. J. Daniell, 143–162; "Introduction to a general theory of elementary propositions" by E. L. Post, 163–185; "Note on Schälfli's elliptic modular functions" by A. Berry, 186–188; "Associated forms in the general theory of modular covariants" by Olive C. Hazlett, 189–198; "On (2, 3) compound involutions" by T. R. Hollcroft, 199–212.

ANNALS OF MATHEMATICS, volume 22, no. 4, June (published in September), 1921: "An analytical solution of Biot's problem" by T. Hayashi, 213–216; "Minimal surfaces containing straight lines" by J. K. Whittemore, 217–225; "An extension of Green's lemma to the case of a rectifiable boundary" by E. B. Van Vleck, 226–237; "Periodic conjugate nets" by E. S. Hammond, 238–261; "On the transformation of convex point sets" by J. L. Walsh, 262–266.

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 13, February, April, 1921: "An algebra of arithmetic functions" (concluded) by F. Hallberg, 1–8; "The group theory element of the history of mathematics" by G. A. Miller, 9–12, 57–61 [Reprinted from *Scientific Monthly*, January, 1921; see *1921*, 226]; Leaves from a lecturer's diary, 13–14; Questions and Solutions, 15–40, 62–80; "Some applications of Heawood's theorem" by N. D. Rajan, 41–44; "The theory of rational transformation" by R. Vythyanathaswami, 45–56.

MATHEMATICAL GAZETTE, volume 8, July, 1921: "Greek mathematics and science" by T. L. Heath, 289–301 [Paper read March 5, 1921, at a joint meeting of the Yorkshire Natural Science Association and branches of the Classical Association and the Mathematical Association]; "Selection in arithmetical examples" by R. S. Williamson, 302–305; Note on "The sound ranging problem" by W. Hope-Jones, 306–307; Review by G. Greenhill of H. Lamb's *Higher Mechanics* (Cambridge, 1920), 309–319; Review of R. C. Archibald's *Training of Teachers of Mathematics for Secondary Schools* (Washington, 1918), 320.

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[1] There was a second English edition containing a few extra pages, but the date of this could not be determined.

[2] There is here a play on the author's name Abbott Abbott, initials A. A. = A^2, in "By the Author A Square."

Mathematics Teacher, volume 14, April, 1921: "Elective courses in mathematics for secondary schools" [A preliminary report by the National Committee on Mathematical Requirements], 161–170; "Alignment charts" by J. Lipka, 171–178; "A reorganized course in junior high-school arithmetic" by Florence M. Brooks, 179–188; "Mathematics in the Horace Mann School for Boys" by R. Beatley, 189–193; "Values in high-school mathematics" by W. J. Ryan, 194–199; "The teaching of locus problems in elementary geometry" by F. D. Aldrich, 200–205; Communications, 206–208; Discussions, 209; Notes and News, 210–215; Book Reviews, 215–216.

Messenger of Mathematics, volume 50, January, February, 1921: "The expression of Bessel functions of positive order as products, and of their inverse powers as sums of rational fractions" by A. R. Forsyth, 129–149; "Note on the transformations of the Sylow subgroups" by G. A. Miller, 149–150; "The evaluation of certain definite integrals involving trigonometrical functions by means of Fourier's integral theorem" by E. Pollard, 151–156; "The primary aberrations of a thin optical system" by T. W. Chaundy, 157–160—March: "The primary aberrations of a thin optical system" (continued) by T. W. Chaundy, 161–165; "Notes on some points in the integral calculus" by G. H. Hardy, 165–171; "On triangular-symmetric curves" by H. Hilton, 171–176—April: "The Bernoullian functions occurring in the arithmetical applications of elliptic function transformation of the seventh order" by A. Berry, 187–189; "The dihedral angles of a tetrahedron" by T. C. Lewis, 190–192.

Monist, volume 31, April, 1921: "Einsteinian space and the probable nature of being. An adventure in metaphysics" by V. A. Endersby, 271–279.

Nature, volume 107, May 12, 1921: "Prof. W. R. Brooks" (1844–1921) by W. F. Denning, 340 [compare *1921*, 334]—May 19: "Symbols in vector analysis" by R. H. Nisbet, 362; Review of H. Hilton's *Plane Algebraic Curves* (Oxford, 1920), 388–389—June 2: Review by "J. F. T." of A. A. Robb's *Time and Space* (Cambridge, 1921), 422—June 9: "An algebraical identity $4X = Y^2 - 37Z^2$" by G. B. Mathews, 456 [Quotation: "Let p be any ordinary odd prime, and let $X = (x^p - 1)/(x - 1)$; then there is an algebraical identity $4X = Y^2 \pm pZ^2$, where Y, Z are polynomials of degree $\frac{1}{2}(p - 1)$ and $\frac{1}{2}(p - 3)$ respectively; and the sign of the ambiguity is $+$ or $-$ according as p is of the form $4n + 3$ or $4n + 1$. The cases up to $p = 31$ inclusive have been published;[1] the result for $p = 37$ has just been communicated to me by Pundit Oudh Upadhyaya, research scholar of the University of Calcutta."]—June 23: "Stellar parallax" by F. Dyson, 527–528—July 7: "What relativity in science implies" by H. W. Carr, 578–580 [review of Viscount Haldane's *The Reign of Relativity* (London, 1921)]; "The displacement of spectral lines by a gravitational field" by H. J. Priestley, 585; "An algebraical identity" by H. S. Pocklington and J. Cullen, 587 [compare Mathews's note above: gives the solutions $p = 37$ to $p = 61$, remarking that the first case where Legendre's rule fails is $p = 41$.]—July 14: Review by S. Brodetsky of L. Silberstein's *Elements of Vector Algebra* (London, 1919), 617—July 21: "An algebraical identity" by W. E. H. Berwick, 652 [First sentence: "With reference to the letters in *Nature* of June 9 and July 7 . . . on the polynomials satisfying the identity

$$Y^2(x) - (-1)^{(p-1)/2} pZ^2(x) = 4(x^p - 1)/(x - 1),$$

may I point out that $Y(x)$, $Z(x)$ are tabulated as far as $p = 101$ in Dr. Hermann Teege's inaugural dissertation, *Ueber die $\frac{1}{2}(p - 1)$ gliedrigen Gaussischen Perioden* (Kiel, Peters, 1900)?"]—August 4–11: Review of W. W. Bryant's *Kepler* (Pioneers of Progress. Men of Science, London, 1920), 713 [See *1921*, 263–264. The review: "Mr. Bryant's account of Kepler's life and work, though very readable, is not altogether satisfactory. The description of how the first two laws of Kepler were found is not clearly expressed and is incorrect in many details. When alluding to Kepler's ideas on gravity it should have been pointed out that his force was tangential to the orbit and not directed to the sun. Of the work on the harmony of the world we are told that 'the fifth book contains a great deal of nonsense.' That Kepler distinctly states that the harmony is only a mathematical conception, and that there is not really any music of the spheres, is not mentioned. The portrait given as a frontispiece is not of Kepler."]; "Remarks on simple relativity and the relative velocity of light" (to be continued) by O. Lodge, 716–719, 748–751—August 25: "Remarks on gravitational relativity" by O. Lodge, 814–818—September 8: Review of *Mediaeval Contributions to Modern Civilisation* (edited by F. J. C. Hearnshaw, London, 1921), 34–35; Review by H. Hilton of W. A. Manning's *Primitive Groups* (Stanford University, 1921) 39 [Last sentence:

[1] G. B. Mathews, *Theory of Numbers*, 1892, p. 218; he gives in this connection a certain rule, which Legendre in his *Théorie des Nombres* erroneously stated as applying to all cases; Mathews remarks that the rule fails for $p = 61$.

"If the remaining Stanford University publications all come up to the standard of this first number, they will indeed fulfil a useful purpose"]—September 15: "Speech through the aether" by O. Lodge, 88–90.

La Nature, volume 49, June 11, 1921: "La specola vaticana. Observatoire du Vatican" by S. Meunier, 369–374.

Popular Astronomy, volume 29, June–July, 1921: "Sherburne Wesley Burnham" by E. E. Barnard, 309–324 [Portrait frontispiece of Burnham]—August–September, "Arthur Searle" (portrait frontispiece) by Margaret Harwood, 377–381; "Proposed periods in the history of astronomy in America" (to be continued) by W. C. Rufus, 393–404 [Introductory period, 1490–1600; colonial period, 1600–1780; apparently stationary period, 1780–1830; popular period, 1830–1860; new astronomy, 1860–1890; correlation period 1890–. Quotation: "Thomas Hariot, English mathematician and astronomer, accompanied a band of colonists sent out by Raleigh (the second expedition to Virginia) which settled upon Roanoke Island, North Carolina, 1585, and remained about one year. This accorded with the custom of the time, as a scientist frequently was selected and sent out with an exploring or colonizing party. Hariot afterward wrote, 'A Brief and True Report of the New Found Land of Virginia.' His function was that of 'discouverer' and his equipment included mathematical instruments, a sea compass, a loadstone, 'a perspective glass whereby was showed many strange things,' and spring clocks. During his stay in America he recorded the observation of a comet."]

Proceedings of the London Mathematical Society, series 2, volume 20, no. 1, May 26, 1921: "Address by the retiring president: Some problems in wireless telegraphy" by H. M. Macdonald, 51–58 [Address delivered November 14, 1918].

Revue de Mathématiques Spéciales, volume 13, June, 1921: "Sur les développements de cones et cylindres" by M. Chenevier, 489–493.

Revue Générale des Sciences, volume 32, May 15, 1921: Review by P. Boutroux of Girolamo Saccheri's *Euclides Vindicatus*, edited and translated by R. d'Adhemar, 273–275 [First three paragraphs: "M. Bouasse, professeur à la Faculté des Sciences de Toulouse, a placé en tête de son livre sur la *Résistance des Matériaux* une Préface véhémente et qui pourrait provoquer quelque ahurissement. Cette Préface a pour titre: 'De l'inutilité des Mathématiques pour la formation de l'esprit.'

"Nous voici donc en présence de 500 pages bourrées de formules algébriques, de différentielles, d'intégrales, et la Préface de cette théorie physique est de nature à discréditer la science mathématique aux yeux d'un étudiant débutant, comprenant mal la pensée, au tour paradoxal, de l'auteur.

"Je voudrais, non point faire une 'réponse' à M. Bouasse, dont j'admire le grand talent, mais débrouiller ce qu'il dit d'excellent et mettre en garde contre les conclusions fausses qu'on pourrait facilement tirer de ses écrits, si vivants et lumineux."]

Revue Scientifique, volume 59, May 14, 1921: "Projet de calendrier mensuel fixe" by René Baire, 233–240—June 25, 1921: "Théorie spéciale de la relativité" by —. Hondros, 329–337.

Science, new series, volume 53, June 17, 1921: "A decade of American mathematics" by O. D. Kellogg, 541–548 [Address delivered as retiring vice-president of Section A of the American Association for the Advancement of Science, December 29, 1920]—Volume 54, July 8, 1921: "Newcomb on extra-mundane life" by G. C. Comstock, 29–30—July 29: "Thomas Hariot," 85–86 [reprinted from *Nature*]—August 5: "A defense of Professor Newcomb's logic" by W. W. Campbell, 113—August 19: "Mathematics in Spanish-speaking countries" by G. A. Miller, 154—September 2: "On the significance of an experimental difference, with a probability table for large deviations" by A. Bull, 200–202—September 9: "The nature of man" by C. J. Keyser, 205–213 [Address at the annual meeting of the Phi Beta Kappa Society, Columbia University, May 31, 1921]—September 30: "A new definition of pure mathematics" by G. A. Miller, 300–301 [By H. Poincaré in *Acta Mathematica*, volume 39, 1921 (see *1921*, 381)]; "Bibliography of relativity" by F. E. Brasch, 303–304; "Einstein's cosmological equations" by E. Kasner, 304–305.

Science Progress, volume 16, July, 1921: "The operative roots of the circle-function" by R. Ross, 116–132.

Scientific Monthly, volume 12, June, 1921: "The debt of mathematics to the Chinese people" by G. Loria and R. B. McClenon, 517–521 [Translated and condensed by R. B. McClenon from an article by G. Loria in *Bollettino della Mathesis*, April, 1920]—Volume 13, July, 1921: "Hermann von Helmholtz" by L. C. Karpinski, 24–32 [Born August 21, 1821; centennial review of his work]—August: "Swiss geodesy and the United States coast survey" by F. Cajori, 117–129 [Sigma Xi address delivered at Northwestern University, December 13, 1920]—September: "A

few questionable points in the history of mathematics" by G. A. Miller, 232–237—October: "Evariste Galois" by G. Sarton, 363–375 [First paragraph: "No episode in the history of thought is more moving than the life of Evariste Galois—the young Frenchman who passed like a meteor about 1828, devoted a few feverish years to the most intense meditation, and died in 1832 from a wound received in a duel, at the age of twenty. He was still a mere boy, yet within these short years he had accomplished enough to prove indubitably that he was one of the greatest mathematicians of all times. When one sees how terribly fast this ardent soul, this wretched and tormented heart were consumed one can but think of the beautiful meteoric showers of a summer night. But this comparison is misleading, for the soul of Galois will burn on throughout the ages and be a perpetual flame of inspiration. His fame is incorruptible; indeed the apotheosis will become more and more splendid with the gradual increase of human knowledge."]

Scribner's Magazine, volume 69, June, 1921: "Science and style" by G. Sarton,[1] 755–759 [Last paragraph: "A scientific training would slowly inculcate a greater fear of error, a deeper respect for truth, and hence would inspire any would-be writer with a deeper sense of responsibility. Any author should be considered as guilty of indiscretion so long as he had not proved that his knowledge, conviction, and power of expression gave him, indeed, a right to speak. Besides, he should be repeatedly made to realize that the attainment of the highest style implies absolute devotion. One must be ready to spend one's whole substance; anything short of that would be mean. If one does not write with one's own blood, what is the use of writing at all?"]—Volume 70, July: "Giant stars" (with illustration) by G. E. Hale, 1–15; "Mathematics" by Florence Waterbury, 108 [This is the following poem:

"The throbbing heart in Music's breast;
Stern Architecture's soul;
The rope that whirls across dark space
And lassoes flying stars."].

Sphinx-Œdipe, volume 16, May, 1921: "Notice sur Charles Ange Laisant" (continued), 65–68 [List of honors and principal writings].

Tôhoku Mathematical Journal, volume 19, nos. 1–2, May, 1921: "Ein Mittelwertsatz für Funktionen mehrerer Veränderlichen" by G. Pólya, 1–3; "On the projective description of cyclides" by A. Emch, 4–10; "The circle and the straight line nearest to n given points, n given straight lines or a given curve" by S. Nakajima, 11–20 [First sentence: "Professor Coolidge (*Annals of Mathematics*, vol. 21, 1919, p. 94) proved a theorem concerning the circle nearest to n given points. I will try to replace n given points by a continuous curve, and to find the circle nearest to the curve."]; "On the roots of a polynomial satisfying a certain differential equation of order 2" by K. Oishi, 21–24; "Extension au tétraèdre d'une propriété du cercle podaire" by V. Thébault, 25–26; "Sur l'orthopôle d'une droit et sur les cercles podaires relatifs à un triangle" by V. Thébault, 27–41; "Illustrative examples of domains of rationality and their groups" by G. A. Miller, 42–53; "Die Vergleichung von verschiedenen Definitionen des Krümmungsradius der Kurve" by T. Kubota, 54–64; "Über die kleinsten ganzen Funktionen, deren sämtliche Derivierten im Punkte $z = 0$ ganzzahlig sind" by G. Pólya, 65–68; "Pascal-Brianchon theorems for higher curves and surfaces" by T. Ota, 69–88; "Vector algebra in general relativity" by C. E. Weatherburn, 89–104; "Class numbers and the form $xy + yz + zx$" by E. T. Bell, 105–116; "Note on the summability of the double Fourier's series of discontinuous functions" by K. Shibata, 117–125; "Some problems in a theory of interest, and some integral equations" [in Japanese] by T. Hayashi, 126–135; Shorter notices and reviews, 136–143; Miscellaneous notes, 143–147.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 52, nos. 3–4, March 22, 1921: "Anschaulich-experimentelle Herleitung der Gausschen Fehlerkurve" by G. Pólya, 57–65; "Ueber die Bestimmung der Mondentfernung durch Schwerenmessungen" by H. Teege, 66–72; "Die Konstanz des Winkelverlustes an den Ecken des Polyeders. Eine neue Fassung des Eulerschen Satzes" by K. Bochow, 74–75 [The space analogue of the theorem, "The sum of the exterior angles of any plane polygon is 2π," is "The total angle-loss of all the angles of a polyhedron is constantly equal to 4π", derived from the Descartes-Euler relation]; "Amerikanische Schulmathematik" by H. Wieleitner, 77–78 [mainly a review of *Unified Mathematics* (1918) by Karpinski, Benedict and Calhoun]; "Paul Stäckel zum Ge-

[1] There is an anonymous article, "Sur le style mathématique," in *L'Education Mathématique*, July, 1921, pp. 150, 157–158.—Editor.

dächtnis" (1862–1919) by W. Lorey, 85–88 [with portrait]—Nos. 5–6, May 20: "Die Mathematik des Lyzeums und Oberlyzeums" by E. Fettweis, 97–103; "Ueber einige Anwendungen des Satzes vom arithmetischen und geometrischen Mittel" by H. Dörrie, 103–108; "Beitrag zur Frage nach der Rationalität der Wurzeln der kubischen und der biquadratischen Gleichung" by H. Greinacher, 113–116; "Oskar Lesser" (1867–1920) by G. Wolff, 131–136 [Biographical sketch, list of publications, and portrait]—Nos. 7–8, July 20: "Beweis des Tsabît für den pythagoreischen Lehrsatz" by the late J. E. Böttcher, and edited by R. Hunger, 153–160; "Anschauliche Beweise für den erweiterten pythagoreischen Lehrsatz" by R. Hunger, 160–167.

AMERICAN DOCTORAL DISSERTATIONS.

R. A. ARMS, *The notion of number and the notion of class*, Philadelphia, 1917. 74 pp. (University of Pennsylvania, 1917.)

C. C. BRAMBLE, "A collineation group isomorphic with the group of the double tangents of the plane quartic." Pp. 351–365. [Reprinted from *American Journal of Mathematics*, 1918.] (Johns Hopkins University, 1917).

J. A. BULLARD, "On the structure of finite continuous groups." Pp. 430–450. [Reprinted from *American Journal of Mathematics*, 1917.] (Clark University, 1917.)

SARAH E. CRONIN, *Geometric properties completely characterizing all the curves in a plane along which the constrained motion is such that the pressure is proportional to the normal component of the acting force*, Lancaster, Pa., New Era, 1917. 8 + 34 pp. (Columbia University, 1917.)

W. VAN N. GARRETSON, "On the asymptotic solution of the non-homogeneous linear differential equation of the nth order. A particular solution." Pp. 341–350. [Reprinted from *American Journal of Mathematics*, 1918.] (University of Michigan, 1916).

G. H. HALLETT, JR., "Linear order in three dimensional Euclidean and double elliptic spaces," *Annals of Mathematics*, second series, vol. 21, pp. 185–202, 1920. (University of Pennsylvania, 1919.)

E. S. HAMMOND, "Periodic conjugate nets," *Annals of Mathematics*, second series, vol. 22, pp. 238–261, 1921. (Princeton University, 1920.)

C. M. HEBBERT, "Some circular curves generated by pencils of stelloids and their polars." 14 pp. [Reprinted from the *Tôhoku Mathematical Journal*, 1918.] (University of Illinois, 1917.)

H. G. HOTZ, *First year algebra scales*, New York City, Teachers College, Columbia University, 1918. 87 pp. (Columbia University, 1918.)

E. P. LANE, "Conjugate systems with indeterminate axis curves," *American Journal of Mathematics*, vol. 43, 1921, pp. 52–68. (University of Chicago, 1918.)

ANNE and ELIZABETH LINTON, *Pascal's Mystic Hexagram, Its History and Graphical Representation*, Philadelphia, Pa., 1921. 4to (12 x 11 in.) 44 pp. + 18 plates—half of which are folding. (University of Pennsylvania, 1921.)

B. E. MITCHELL, *Complex conics and their real representation*, Lancaster, Pa., New Era, 1917. 3 + 44 pp. (Columbia University, 1917.)

H. C. M. MORSE, Certain types of geodesic motion on a surface of negative curvature. Published as: (*a*) "A one-to-one representation of geodesics on a surface of negative curvature," *American Journal of Mathematics*, vol. 43, 1921, pp. 33–41; (*b*) "Recurrent geodesics on a surface of negative curvature," *Transactions of the American Mathematical Society*, vol. 22, 1921, pp. 84–100. (Harvard University, 1917.)

G. W. MULLINS, *Differential invariants under the inversion group*, Lancaster, Pa., New Era, 1917. 22 pp. (Columbia University, 1917.)

F. D. MURNAGHAN, "The lines of electric force due to a moving electron." Pp. 147–162. [Reprinted from *American Journal of Mathematics*, 1917.] (Johns Hopkins University, 1916.)

J. R. MUSSELMAN, "The set of eight self-associated points in space." Pp. 69–86. [Reprinted from *American Journal of Mathematics*, 1918.] (Johns Hopkins University, 1916.)

H. F. PRICE, "Fundamental regions for certain finite groups in S_4." Pp. 108–112. [Reprinted from *American Journal of Mathematics*, 1918.] (Princeton University, 1917.)

O. J. RAMLER, *On the three-cusped hypocycloids fulfilling certain assigned conditions*, Washington, D. C., 1918. 4 + 22 pp. (Catholic University of America, 1918.)

C. H. RAWLINS, "Complete systems of concomitants of the three-point and the four-point in elementary geometry." Pp. 155–173. [Reprinted from *American Journal of Mathematics*, 1918.] (Johns Hopkins University, 1916.)

C. N. REYNOLDS, "On the zeros of solutions of linear differential equations," *Transactions of the American Mathematical Society*, vol. 22, 1921, pp. 220–229. (Harvard, 1919.)

J. N. Rice, *On the in-and-circumscribed triangles of the plane rational quartic curve*, Washington, D. C., 1917. 32 pp. (Catholic University of America, 1917.)

P. R. Rider, "The space problem of the calculus of variations in terms of angle." Pp. 241–256. [Reprinted from *American Journal of Mathematics*, 1917.] (Yale University, 1915.)

J. F. Ritt, "On a general class of linear homogeneous differential equations of infinite order with constant coefficients, preceded by a note on the resolution into partial fractions of the reciprocal of an entire function of genus zero." Pp. 21–49. [Reprinted from *Transactions of the American Mathematical Society*, 1917.] (Columbia University, 1917.)

A. J. Scarlett, *The surface tension of mixed liquids*, Easton, Pa., Eschenbach Printing Company, 1917. 34 pp. (Columbia University, 1917.)

C. P. Sousley, "Invariants and covariants of the Cremona cubic surface." Pp. 135–146. [Reprinted from *American Journal of Mathematics*, 1917.] (Johns Hopkins University, 1915).

J. M. Stetson, "Conjugate systems of curves on a surface both of whose Laplace transforms are lines of curvature." Pp. 106–126. [Reprinted from *Annals of Mathematics*, 1917.] (Princeton University, 1914).

Dorothy Stimson, *The gradual acceptance of the Copernican theory of the universe*, Hanover, N. H., 1917. 147 pp. (Columbia University, 1917).

H. G. Tyler, *Equations for imperfect gases*, Boston, F. H. Gilson Company, 1917. 5 + 30 pp. (New York University, 1916.)

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W. H. Wilson, "On a certain general class of functional equations." 20 pp. [Reprinted from *American Journal of Mathematics*, 1918.] (University of Illinois, 1917.)

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. P. Manning.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

2928.

Show that if through the end P (opposite from the origin) of the loop of (a) the folium of Descartes, $x^3 + y^3 = 3axy$, (b) the strophoid or logocyclic curve, $x(x^2 + y^2) + a(x^2 - y^2) = 0$, a straight line be drawn meeting the curve again in Q and R, then QR always subtends a right angle at the origin (compare *1916*, 90–92; also Basset, *Treatise on Cubic and Quartic Curves*, Cambridge, 1901, p. 82). Are these results particular cases of a general result for a certain class of cubic curves?

2929. Proposed by R. E. GAINES, University of Richmond.

Denote by A, O and B, respectively, the points $(-1, 0)$, $(0, 0)$ and $(1, 0)$ on the curve $y^2 = x^3 - x$, and let P be a variable point on the curve. Let PA and PO meet the curve again in Q and R, respectively, and let BQ and BP meet AR in M and N, respectively. Prove that QN is perpendicular to PM.

2930. Proposed by R. E. GAINES, University of Richmond.

If in reducing p/q (p and q integers, $q > p$) to a decimal the remainder $q - p$ ever appears, then the fraction will give a repeating decimal the number of digits in whose repetend will be exactly twice the number of digits in the quotient already obtained, and the remaining digits may, without further division, be obtained by subtracting the quotient already found from a succession of 9's.

2931. Proposed by R. C. ARCHIBALD, Brown University.

From the equality $\sec(\pi/14) + \sec(3\pi/14) - \sec(5\pi/14) = 0$ find a relation between the lengths of the side and diagonals of a regular inscribed heptagon.

2932. Proposed by R. C. ARCHIBALD, Brown University.

De Ville gave in 1629 the following construction for an approximation to the side of a regular polygon of n sides inscribed in a given circle: On a diameter AB construct an equilateral triangle ABC; divide AB into n equal parts, join C to D, the first point of division of AB, and let CD produced intersect the circumference in E; then the chord of the arc AF, the double of AE, will be approximately equal to the required side of the polygon. Construct a table of values for $n = 5$ to $n = 20$ exhibiting the extent, in minutes and seconds, to which the above construction is in error in connection with the central angle subtended by the constructed side of the polygon. [Somewhat similar tables have been made for constructions by Bosse and Bernard.]

2933.

Dudeney's Problem, 1902; With ruler and compasses only, divide an equilateral triangle into four rectilinear pieces which may be put together so as to form a square.

2934. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

The base of a variable triangle is fixed, the opposite vertex describing a given straight line. Find the locus of the center of the nine-point circle and the envelope of the Euler line. [*Editorial Note*—In his inaugural dissertation at the University of Bern, Otto Jugi discussed (*Ueber den Feuerbach'schen Kreis in variablen Dreiecken*, Langenthal, 1903) the locus of the center of this nine-point circle when the vertex of the triangle moved (a) along any straight line, and in particular when this line is parallel to the base; (b) along a circle, an ellipse, a hyperbola, a parabola; (c) along a cubic parabola, Neil's parabola, a cissoid, a strophoid; and (d) along a lemniscate. In a number of cases the envelope of the nine-point circle was also found.]

2935.

Find the integral solutions of the equation $x! + 1 = y^2$.

[Proposed in *Gazeta Matematică*, April, 1921.]

2936. Proposed by J. P. BALLANTINE, University of Michigan.

A person in drinking from a conical drinking glass, tips it at a constant angular rate. At what angle will the delivery be the maximum and at what angle will the surface of the water be a maximum.

2937. Proposed by C. F. GUMMER, Queen's University.

A straight uniform stone wall AB is to be rebuilt in another position CD, the intersection of AB and CD being interior to CD but not to AB. How should the material be moved for the average horizontal distance through which it is carried to be as short as possible? If CD passes through B, show that the lower limit to the average distance may be expressed by means of logarithms and algebraic functions.

2938. Proposed by C. F. GUMMER, Queen's University.

If $a, b, \cdots, i$ are real numbers $\geqq 0$, and if $\begin{vmatrix} a^r & b^r & c^r \\ d^r & e^r & f^r \\ g^r & h^r & i^r \end{vmatrix}$ is equal to zero for five real values of r other than zero, prove that the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ has either two rows or two columns proportional, or a single row or column of zeros.

2939. Proposed by C. F. GUMMER, Queen's University.

Show that the determinant

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n & -b_1 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & a_{n-1} & -b_2 & -b_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_1 & -b_n & -b_{n-1} & \cdots & -b_1 \\ b_1 & b_2 & \cdots & b_n & a_1 & 0 & \cdots & 0 \\ 0 & b_1 & \cdots & b_{n-1} & a_2 & a_1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & b_1 & a_n & a_{n-1} & \cdots & a_1 \end{vmatrix}$$

has the property that every n-rowed determinant of the n left-hand columns is equal to its cofactor.

2940. Proposed by R. E. GILMAN, Brown University.

Find the average number of operations required to obtain m white balls from an urn containing p white and q black balls ($m < p$) as follows: For the first operation m balls are drawn simultaneously and examined, and such as are found to be black, are returned to the urn. Each subsequent operation is like the first save that the number of balls drawn is equal to the number replaced in the urn in the preceding operation.

NOTES

24. Problems discussed by Huygens.—In the last published volume (14, 1920) of *Oeuvres Complètes de Christian Huygens* (cf. *1921*, 79) many problems are discussed. Among them are the following four: (*a*) [pages 208–209] dated 1655, "Given the sum of the sides of a right angled triangle and the difference of the segments of the hypotenuse made by a perpendicular dropped from the right angle, to construct the triangle"; (*b*) [pages 271–272] dated 1657, "Given two circles and a straight line, to describe a circle tangent to the given circles and with an arc cut off by the given line so as to contain an angle equal to a given angle"; (*c*) [pages 498–500] dated 1662, "To inscribe a regular heptagon in a circle"; (*d*) [pages 521–523] dated 1666, "To find the integer which, when divided by three given integers, the remainders are three given integers."

Problem (*a*)—After algebraic analysis Huygens gives a resulting geometrical construction and proof. An earlier and different discussion, by synthetic geometry, was published in 1607 by M. Ghetaldi in his *Variorum Problematum Collectio*, Venice, 1607, pp. 25–27; see also Oughtred, *Clavis Mathematicae*, third edition, Oxford, 1652, pp. 86–87. The more general problem, "Given the vertical angle, the sum of the two including sides, and the difference of the segments of the base made by a perpendicular dropped from the vertex, to construct the triangle," was solved by Renaldini in his *De Resolutione et Compositione Mathematica*, Padua, 1668, pp. 319, 529, and by T. Simpson, in his *A Treatise of Algebra*, London, 1745, pp. 292–293.

*

Problem (*b*)—This problem was proposed to Huygens by R. F. de Sluse, in a letter dated [1] October 23, 1657, and Huygens here shows merely how the solution of the problem may be reduced to the solution of a quadratic equation.

In a letter to Fermat, dated July 29, 1654, Pascal refers[2] to two problems

[1] *Oeuvres Complètes de Christian Huygens*, vol. 2, 1889, pp. 72 and 80.

[2] *Oeuvres de Fermat*, vol. 2, Paris, 1894, p. 298.

which he had solved by drawing only circles and straight lines; the second of these problems was as follows: "De trois cercles, trois lignes, [trois] quelconques étant donnés, trouver un cercle qui, touchant les cercles et les points, laisse sur les lignes un arc capable d'angle donné." But the particular one of these problems which we are considering was proposed by Pascal to Sluse, who then passed it on to Huygens. Some indication of Sluse's discussion is given in *Oeuvres de Blaise Pascal* publiées par L. Brunschvieg, P. Boutroux, et F. Gazier (vol. 7, Paris, 1914, pages 246–247; see however pages[1] 233–252; also tome 3, 1908, p. 302).

This problem is, however, only a particular case of the following which can be solved with ruler and compasses only: Describe a circle which shall cut three given coplanar circles under given angles. The first formulation and solution of this problem seems to have been by F. Neumann[2] in 1825. In 1826 Steiner referred to the problem as one for which he had a solution[3] but, so far as known, this was never published. Plücker's discussion and construction appeared[4] in 1827. The problem has also been dealt with in Adolf Anderssen's program[5] of 1864; and by Griffiths,[6] 1871–1874; by Fiedler,[7] 1882; by Laquière,[8] 1883; by Tarry,[9] 1889; by M. Fouché,[10] 1892; by Casey,[11] 1893; by Lachlan,[12] 1893; and by many others.

[1] Pages 248–255 contain a letter "de Sluse a Brunetti." This same letter is given in *Oeuvres de Fermat*, vol. 2, 1894, pp. 315–320 as of "Fermat a Carcavi."

[2] F. Neumann, *Isis*, Berlin, 1826; see E. Kötter, "Die Entwickelung der synthetischen Geometrie," *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 5, 1901, p. 115.

[3] J. Steiner "Einige geometrische Betrachtungen," *Journal für die reine und angewandte Mathematik*, vol. 1, 1826, p. 163; *Jacob Steiner's Gesammelte Werke*, vol. 1, 1881, p. 21. Similarly for the following problem on these same pages: To describe a sphere which shall cut each of four given spheres S_1, S_2, S_3, S_4, respectively, under the angles α_1, α_2, α_3, α_4.

[4] *Annales de Mathématiques Pures et Appliquées*, vol. 18, pp. 43–45, August, 1827; also Plücker, *Analytisch-geometrische Entwickelungen*, Essen, vol. 1, 1828, pp. 120–122. See also *Nouvelles Annales de Mathématiques*, 1870, pp. 371–375.

[5] *Ueber die Aufgabe einen Kreis zu konstruiren, der drei gegebene Kreise unter den Winkeln α, β, γ schneidet*, Osterprogramm des königlichen Friedrich Gymnasiums, Berlin (?), 1864.

[6] J. Griffiths: (*a*) "On the problem of finding the circle which cuts three given circles at given angles," *Proc. London Math. Soc.*, vol. 3, pp. 269–278, 1871; he found also equations for the groups of circles cutting three given small circles on a sphere at given angles. (*b*) "On the cartesian equation of the circle which cuts three given circles at given angles," *Proc. London Math. Soc.*, vol. 5, pp. 33–35, 1874.

[7] W. Fiedler, *Cyklographie oder Construction der Aufgaben über Kreise und Kugeln*, Leipsic, 1882, pp. 169–172.

[8] E. M. Laquière, "Détermination et construction nouvelle du cercle qui coupe trois cercles sous trois angles donnés et de la sphère qui coupe quatre sphères sous des angles donnés," *Nouvelles Annales de Mathématiques*, 1883, vol. 42, pp. 348–352.

[9] G. Tarry, "Sur un problème classique," *Journal de Mathématiques Elémentaires* (Bourget), 1889, pp. 217–222; he considers also Steiner's problem of a sphere cutting four given spheres under given angles. The argument of that part of the article relating to circles is given in E. Rouché et C. de Comberousse, *Traité de Géométrie*, 6e édition, part 1, Paris, 1891, pp. 288–290; 7e éd., pp. 304–306.

[10] M. Fouché, "Sur les cercles qui touchent trois cercles donnés ou qui les coupent sous un angle donné," *Nouvelles Annales de Mathématiques*, 1892, pp. 346–349.

[11] J. Casey, *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections*, second edition, Dublin, 1893, pp. 108–109.

[12] R. Lachlan, *An Elementary Treatise on Modern Pure Geometry*, London, 1893, pp. 239–241.

Problem (*c*)—In seeking to construct a regular inscribed heptagon Huygens was led to a consideration of the cubic equation

$$z^3 - 2z^2 - z + 1 = 0, \tag{1}$$

or, as reduced by the removal of its second term,

$$y^3 - \frac{7}{3}y - \frac{7}{27} = 0, \tag{2}$$

and the solution of this equation[1] he tried, apparently, to effect by the intersection of curves.

The editors of Huygens's *Oeuvres* note that when Schooten discussed this problem several years earlier,[2] he was led to the equation

$$x^6 - 7x^4 + 14x^2 - 7 = 0, \tag{3}$$

for the heptagon, and to

$$x^3 - x^2 - 2x + 1 = 0, \tag{4}$$

for the tetradecagon, where x represents in each case, for one root, the ratio of a side of the polygon to the radius of the circumscribed circle. Equation (4) appears also in connection with the heptagon problem itself, as we should study it, since the roots of this equation, $2 \cos (\pi/7)$, $-2 \cos (2\pi/7)$, $2 \cos (3\pi/7)$, are the abscissas of the vertices of a heptagon inscribed in a circle of radius 2.

These same equations were found earlier still, in an endeavor to solve our problem of the heptagon. Equation (4) occurs in work of an unknown Arab, who flourished[3] about 980 A.D., and was found also by Vieta (1540–1603), whose works, edited by Schooten,[4] were published in 1646. Equation (3) was given by Kepler in his *Harmonice Mundi Liber I*, Linz, 1619, where, in proposition[5] 45, considerable space is also devoted to a discussion of approximate constructions for the heptagon. Kepler was convinced that its construction with ruler and compasses alone was impossible.

In his *Treatise of Angular Sections*, Oxford, 1684, John Wallis found (p. 48) the equation $RccH \doteq 7RccA - 14RqqAc + 7RqAqc - Aqqc$, that is, $R^6H = 7R^6A$

[1] z is the length of the longer diagonal of the heptagon whose side is of unit length; $y = z - 2/3$.

[2] F. van Schooten: (*a*) *Exercitationum Mathematicarum*, Liber V, Leyden, 1657, pp. 467–473; (*b*) *Vyfde Bouck der Mathematische Oeffeningen*, Amsterdam, 1660, pp. 436–442.

[3] F. Woepcke, *L'Algèbre d'Omar Alkhayâmi*, Paris, 1851, pp. 126–127.

[4] *Francisci Vietæ Opera Mathematica*, Leyden, 1646, "protasis IV," pp. 359–364. The discussion here may be considered as indicating the following construction: "Given a circle with center A and diameter CAB; on CB produced take a point D such that $DB \cdot CD^2 = AD \cdot AB^2$; with D as center and AB as radius describe a circle to cut the given circle in E; then the arc BE is the seventh part of the circumference." Or, "to construct an isosceles triangle such that the length of the bisector of an exterior base angle (between the base and a side produced) is equal to the length of the side." (Compare *Nouvelles Annales de Mathématiques*, vol. 9, 1850, pp. 51–53, 151, 233–234.)

[5] Also *Joannis Kepleri Astronomi Opera Omnia*, edidit Ch. Frisch, vol. 5, 1864, pp. 101–107, 471–473. In this MONTHLY (*1914*, 14, 148) J. Q. McNatt and S. A. Joffe were led, in discussion of the regular inscribed heptagon, to equation (3), and they found, by Horner's method, $x^2 = .7530203962821\cdots$; so that x is approximately equal to $\sqrt{3}/2$, half the side of an inscribed equilateral triangle—a result to which we shall return later.

$- 14R^4A^3 + 7R^2A^5 - A^7$. Wallis then makes the following notable statement (p. 49): "The Seven Roots of this Equation; are, so many straight Lines from some one Point of the Circumference, to the Seven Angles of an inscribed Regular Heptagon." "Of these Roots (putting H. Affirmative,) the two least are Affirmative; the two next, are Negative; the two next to these, are again Affirmative; and, the greatest Negative." [In the equation above, R is the radius, and H is "the Subtense of the Septuple Arch," *i.e.*, $H = 2R \sin (\pi/7)$; one value of A is $2R \sin (\pi/7^2)$. In other words, Wallis's equation is really the equation which we obtain when we put $\pi/7^2$ for θ in the trigonometric identity, $- \sin 7\theta = 2^6 \sin^7 \theta - 7 \cdot 2^4 \sin^5 \theta + 14 \cdot 2^2 \sin^3 \theta - 7 \sin \theta$. Comparison may be made with Marqfoy's discussion referred to in the first footnote at the bottom of this page.]

Maria G. Agnesi found in her *Instituzioni Analitiche* (Milan, 1748, vol. 1, pp. 279–284; English edition by J. Colson, London, 1801, pp. 168–171) various equations to which she was led in the discussion of the problem of the regular inscribed heptagon. Two of these are: (*a*) $x^6 - 7r^2x^4 + 14r^4x^2 - 7r^6 = 0$ (where r is the radius of the circle and one value of x is the side of the heptagon) which is practically identical with equation (3) of Kepler and Schooten; (*b*) $8x^3 - 4rx^2 - 4r^2x + r^3 = 0$ (where one value of x is the distance of the side of the heptagon from the center of the circle, that is, $x = \cos (\pi/7)$); to be more accurate, the latter equation was given by Agnesi with incorrect signs before the second and fourth terms—which seems to have been first noticed by P. Frisi (*Opera*, Milan, vol. 1, 1782, p. 177). Grunert showed (*Archiv der Mathematik und Physik*, vol. 17, 1851, p. 360) that this equation reduced, by a transformation, to

$$u^3 - \frac{7}{12} r^2 u + \frac{7}{216} r^3 = 0,$$

an equation whose roots are minus one half of those of equation (2) multiplied by the radius r.

In this survey the next in chronological order would be Sir William Rowan Hamilton, who was led, 1862–1864, to equations we shall presently number (5) and (6); consideration of his work will be reserved for the concluding part of this topic.

In 1873 Affolter gave[1] a geometrical construction for the inscribed heptagon, assuming that a cardioid (or the general pedal curve of a circle) could be used, in one step, for the trisection of an angle.[2] He found the equation, identical

[1] F. G. Affolter, "Construction des regulären Sieben- und Dreizehn-Ecks," *Mathematische Annalen*, vol. 6, pp. 593–595. Since $\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta$ the equation $64x^7 - 112x^5 + 56x^3 - 7x - 1 = 0$ has for roots the cosines of the angles $2h\pi/7$; $h = 1, 2, \cdots, 7$. But $\cos (14\pi/7) = 1$, and $\cos (2h\pi/7) = \cos [2(7 - h)\pi/7]$; hence the equation of the seventh degree (compare Wallis) reduces to $8x^3 + 4x^2 - 4x - 1 = 0$, — (5) below. (This method of derivation is indicated by G. Marqfoy, *Nouvelles Annales de Mathématiques*, vol. 9, 1850, pp. 52–53).

[2] H. Hippauf, "Lösung des Problems der Trisection mittelst der Conchoïde auf circularen Basis," *Zeitschrift für mathematischen und naturwiss. Unterricht*, vol. 3, 1872, pp. 215–240; on page 537 of this volume, C. Albricht points out that he solved the same problem by the same

with one of Hamilton's,

$$c^3 + \frac{1}{2}c^2 - \frac{1}{2}c - \frac{1}{8} = 0, \tag{5}$$

where $c = \cos(2h\pi/7)$, $h = 1, 2, 3$, which he reduced to Huygens's equation (2) by the substitution $2c = y - 1/3$.

Freeth showed [1] that regular polygons of 7, 21, 35, $\cdots$ sides could be constructed, by means of the curve $r = a[1 + 2\sin(\theta/2)]$ which he called a nephroid, although in the previous year this term was applied by Proctor to the two-cusped epicycloid.

E. Pascal showed,[2] in 1887, that we could reduce the solution of the problems to construct polygons of 7 or 13 or 97 sides, to the solution of cubic equations which, for the heptagon, was found to be

$$x^3 + x^2 - 2x - 1 = 0, \tag{6}$$

whose roots are double of those of equation (5). Setting $x = y - 1/3$ he reduced this to equation (2), which he solved, as Descartes did,[3] by means of a circle and a parabola.

By trigonometric considerations Dickson was led [4] to equations (4) and (6) in papers of 1894 and 1914, when he proved the impossibility of the construction of the heptagon with ruler and compasses.

Feldblum employed [5] the Gaussian method to reduce the cyclotomic equation to equation (6), and proceeded to give a construction based upon the trisection of an angle. Compare Affolter's method above.

It is well known that any plane geometrical problem, which leads to a cubic equation, can be solved with ruler and compasses alone, if a parabola, or an ellipse, or a hyperbola is first drawn in the plane. Vahlen gave the construction for a regular heptagon by means of an ellipse.[6]

As an application of Lill's method [7] Adler[8] and Mitscherling[9] showed, by means of equation (6), that the problem of the construction of a regular heptagon could be carried through with the aid of two right-angled rulers.

method in his "Die Fusspunktlinien der Kegelschnitte und ihre Anwendungen," Progr. Hermannstadt, 1863–1864.

[1] T. J. Freeth, "The nephroid, heptagon, etc.," *Proceedings of the London Mathematical Society*, vol. 10, pp. 130, 228–229, 1879.

[2] E. Pascal, "Costruzioni geometriche di tre poligoni regolari," *Giornale di matematiche*, Naples, vol. 25, pp. 82–97.

[3] R. Descartes, *Oeuvres de Descartes* publiées par C. Adam and P. Tannery, vol. 6, Paris, 1902, pp. 469f.

[4] L. E. Dickson in this MONTHLY: (*a*) "The inscription of regular polygons," *1894*, 299–300; (*b*) "On the construction of regular polygons of 7 and 9 sides," *1914*, 260–262.

[5] M. Feldblum, *Ueber Elementar-Geometrische Constructionen* (Diss. Göttingen), Warsaw, 1899, pp. 33–40.

[6] R. T. Vahlen, "Ueber kubische Konstructionen," *Archiv der Mathematik und Physik*, (3), vol. 3, pp. 116–117, 1902.

[7] E. Lill, "Résolution graphique des équations numériques de tous les degrés à une seule inconnue et description d'un instrument inventé dans ce but," *Nouvelles Annales de Mathématiques*, vol. 26, 1867, pp. 359–362.

[8] A. Adler, *Theorie der geometrischen Konstruktionen*, Leipsic, 1906, pp. 209–210; 262–263.

[9] A. Mitscherling, *Das Problem der Kreisteilung*, Leipsic, 1913, p. 73.

In the course of discussion of the inscription of a regular heptagon by means of an equilateral hyperbola, C. B. Haldeman was led[1] to $z^3+Rz^2\sqrt{7}-R^3\sqrt{7}=0$, where one root z is the side of the heptagon, and R is the radius of the circle. The equation whose roots are the squares of the roots of this equation is $z^3 - 7R^2z^2 + 14R^4z - 7R^6 = 0$, which is equivalent to one of the equations discussed by Agnesi. Elsewhere in this issue[2] Mr. Haldeman returns to the discussion of the heptagon and is led, as Wallis was in the seventeenth century, to the expression of sin $7A$ in terms of sin A.

It is appropriate next to consider various approximations for the side of a regular heptagon. The first of these, so far as we know, occurs in a work on surveying by Heron of Alexandria who flourished between 50 B.C. and 100 A.D. Introductory to finding the area of the inscribed heptagon he gives a lemma[3] in which he states that the length of the perpendicular from the center of the circle on the side, r, of the regular inscribed hexagon is approximately equal to the side, s_7, of the regular inscribed heptagon; that is, $s_7 = (\sqrt{3}/2)r$, approximately. Of the basis for this statement of approximation, apparently well known, we are ignorant. But with Cantor we may surmise[4] a connection with a work "On a heptagon in a circle" which some Arabian writers attribute to Archimedes[5] (d. 212 B.C.). Heron does not formulate the approximate relation $s_7 = s_3/2$, where s_3 is the length of the side of an equilateral triangle inscribed in the circle, although this readily follows, since $s_3 = \sqrt{3}\,r$—as Heron shows.

In a book of geometrical constructions by Aboûl Wafâ (900–998) we find[6] a construction for the inscribed heptagon in which the side of the heptagon is definitely taken as half the side of the equilateral triangle inscribed in the same circle. It is notable that he adds[7] : "But this is an approximation and not an exact construction." Leonardo da Vinci (1452–1519) maintained the exactness of this construction[8] in his *Codice Atlantico*. The construction is called the Indian method (quaestio Indorum) by Jordanus Nemorarius (d. 1236) in his *De triangulis*[9] where the construction of the regular heptagon is considered on pages 42–44.

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[1] C. B. Haldeman, "Geometrical construction of the roots of a cubic, and inscription of a regular heptagon in a circle" in this MONTHLY, *1919*, 391.

[2] Department of Questions and Discussions.

[3] *Herons von Alexandria Vermessungslehre und Dioptra.* Griechisch und Deutsch von H. Schöne. Leipsic, 1903, pp. 54–55. See also *Heronis Alexandrini Opera*, vol. 4, *Geometrica*, ed. Heiberg, Leipsic, 1912, pp. 384–385.

[4] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 1, third ed., Leipsic, 1907, pp. 377, 307.

[5] J. L. Heiberg, *Quaestiones Archimedeae*, Copenhagen, 1879, p. 29.

[6] F. Woepcke, "Analyse et extrait d'un recueil de constructions géométriques par Aboûl Wafâ," *Journal Asiatique* (5), vol. 5, pp. 218–256, 309–359; reprinted, Paris, 1855, pp. 89.

[7] Many refer in recent times to A. G. Kästner, ("Unrichtige Verzeichnung des Siebenecks" in *Geometrische Abhandlungen*, Erste Sammlung, Göttingen, 1790, pp. 249–250) as the first to point out that the above construction for the heptagon was inexact.

[8] Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 2, second edition, 1900, pp. 298–300 and 465.

[9] *Mitteilungen des Coppernicus-Vereins für Wissenschaft und Kunst zu Thorn*, herausgegeben von M. Curtze, vi. Heft, Thorn, 1887. See also Cantor, vol. 2, *l.c.*, p. 83.

This same method is the basis of constructions or statements in: (*a*) an anonymous work, *Geometria deutsch*, the time of composition of which is also unknown;[1] (*b*) the *Underweysung der messung mit dem zirckel uñ richtscheyt* . . ., 1525, by Albrecht Dürer[2] (1471–1528); (*c*) Vieta's *Pseudomesolabum et alia quaedam adiuncta capitula*,[3] 1596; (*d*) the *Geometrica Practica*, Rome, 1604, of Christoph Clavius (Schlüssel, 1537–1612), pages 407–409, with reference to one Carolus Marianus Cremonensis;[4] (*e*) D. Schwenter, *Geometriae Praticae Novae*, Nürnberg, 1618, p. 171; (*f*) the work of 1619, referred to above, by Kepler (1571–1630); and (*g*) a problem of C. Mydorge[5] (1585–1647).

With reference to Hamilton's exact equation (5) above, Günther explained[6] an interesting relation with the approximation we have been discussing. He substituted for the number in the last term of equation (6), namely for $1/8 = 64/512$, $65/512$, and found that the resulting equation had the root $\cos(2\pi/7) = (5/8)$ or $\sin(\pi/7) = (\sqrt{3}/4)$, which gives $s_7 = \sqrt{3}\, r/2$ as before. The angle subtended by a side of a regular heptagon at the center of its circumscribed circle is about $51° 25' 43''$; the angle subtended by the above approximation to the side is $51° 19' 4''$,—about 6 minutes too small.

The particular case when $n = 7$ of the following construction of Antoine de Ville,[7] revised by Bosse[8] for the side of a polygon of n sides, gives a result about

[1] See Cantor, vol. 2, *l.c.*, p. 451.

[2] Compare H. Staigmüller, *Dürer als Mathematiker*, Stuttgart, 1891, pp. 5, 23. De Morgan wrote in *The Athenaeum*, London, September 12, 1863, in his "Notes on the history of perspective, no. VIII": "There is a very neat way of approximately dividing a circle into seven equal parts which I have traced through writers on perspective up to Albert Dürer, beyond whom I cannot carry it; I do not find it in books upon other kinds of practical geometry, though I am told it has re-appeared in a work of our own time. Half the line which joins the two intersections of the circles in Euclid's first proposition is very nearly the side of the inscribed heptagon: it is too small; but any one who would feel satisfied with 1*l.* as composition for a debt of 1*l.* 0*s.* 0½*d.* ought to be a trifle better satisfied with Albert Dürer's heptagon. An error of less than one inch in 40 feet is good drawing."

[3] Vieta, *Opera Mathematica*, 1646, pp. 283–284; compare Cantor, vol. 2, *l.c.*, p. 583.

[4] Compare Cantor, vol. 2, *l.c.*, p. 581; also Mitscherling, *l.c.*, p. 61.

[5] Compare Cantor, vol. 2, *l.c.*, p. 673.

[6] S. Günther, *Die Geometrischen Näherungskonstruktionen Abrecht Dürers*, Ansbach, 1886, p. 9.

[7] *Les Fortifications du chevalier Antoine de Ville, contenans la maniere de fortifier toute sorte de places* . . . Lyon, 1629 (achevé d'imprimer, 1er août, 1628); there were other editions in 1636, 1640 (some title pages have 1641), 1668 (at Paris), pp. 34–36, and 1672 (at Amsterdam).

[8] A. Bosse, *Traité des pratiqves geometrales et perspectives*, Paris, 1665, p. 62. De Ville joined C to D', the *first* point of division of the diameter AB, and thus determined E' on the circumference; the chord AE'' of an arc the double of AE' was, according to De Ville, approximately equal to the side of the polygon. The De Ville-Bosse construction is often ascribed to C. Renaldini, who gave it in his *De resolutione et compositione mathematica*, Padua, 1688, pp. 367–368. Renaldini considered this construction accurate in all cases, but his error was subsequently pointed out in Schultz, *Dissertatio de circuli divisione*, Königsberg, 1691. Another work, to which credit in connection with De Ville-Bosse's construction is often given, is N. Bion, *Traité de la construction et des instruments de mathématiques*, Paris, 1709; fourth edition, 1752, p. 22. The matter was also considered by Jean Bernoulli and Gabriel Cramer, *Jacobi Bernoulli, Basileensis, Opera*, vol. 2, Geneva, 1744, p. 765; compare A. G. Kästner, "Renaldins allgemeine unrichtige Regel, jedes ordentliche Vieleck im Kreise zu beschreiben," in *Geometrische Abhandlungen*, Erste Sammlung, Göttingen, 1790, pp. 266–281. Among scores of other references which might be given, I simply add: [Note by R. Wolf, on a ms. of 1716 by J. J. Sheuchzer, giving Bosse's construction] *Viertel-iahrsschrift der Naturforschenden Gesellschaft in Zürich*, vol. 29, 1884, pp. 39–40; and E. Catalan, *Théorèmes et Problèmes de Géométrie Elémentaire*, 6e éd., Paris, 1879, pp. 278–279.

6 minutes too large, namely[1] 51° 31′ 5″. De Ville-Bosse's construction: On the diameter AB of a circle, construct an equilateral triangle ABC; divide AB into n equal parts; join D, the second point of division from A, to the point C, and produce CD through D to meet the circle in E; then AE is either exactly or approximately equal to the length of the side of the required regular inscribed polygon of n sides. De Ville remarks (page 29): "Ce problem se demonstre point: et au calcul il ne revient pas tout à fait precisément aux figures qui ont grand nombre de costez . . . il est à estimer pour la facilité et justesse plus grande que de tous les autres qui ont esté escrits pour ce sujet."

A much less inaccurate construction for the side of a regular polygon, discovered by Karl Bernard, Duke of Saxe-Weimar, was published in 1828 (*Correspondance Mathématique et Physique* (Quetelet), volume 4, p. 350). It is as follows: Let BOA at right angles to DOC be diameters of a circle in which a polygon of n sides is to be inscribed; divide BA into n equal parts and produce OA through A to M, and OC through C to N such that $AM = CN = BA/n$. Let P be the point nearer to A in which MN cuts the circle. Join this to the end, Q, of the third point of division from A along AB; then PQ is approximately the length of the side of the polygon for $n \geqq 5$. This construction leads (I. Ghersi, *Matematica Dilettevolle e Curiosa,* Milan, 1913, p. 414), for the circle of unit radius, to the formula

$$PQ = \frac{1}{n}\sqrt{[n^2 - 8n + 48 - (n-6)\sqrt{(n^2 - 4n - 4)}]}.$$

A table of values $n = 5$ to $n = 30$ is given by Ghersi on page 415. For $n = 5$ the angle subtended at the center of the circle by PQ is too small by about 39′ 46″; for $n = 6$ the construction is exact; for $n = 7$, the angle is about 56″ too small; for $n = 8$, 2′ 47″ too small; the error increases to 3′ 48″ for $n = 10$ then decreases to 7″ for $n = 21$; for $n = 22$ to $n = 30$, PQ is too large and the subtended angle-excess varies from about 4″ to about 56″.

In the case of the De Ville-Bosse construction a formula corresponding to that given above for PQ was found by Gabriel Cramer (*l.c.,* 1744) to be $\sqrt{[n^2 + 4n + 16 - \sqrt{(n^4 + 8n^3 - 144n^2 + 512n - 512)}]}/\sqrt{(2n^2 - 4n + 8)}$. For $\sqrt{(n^4 + 8n^3 - 144n^2 + 512n - 512)}$ A. G. Kästner wrote (*Geometrische Abhandlungen,* Erste Sammlung, Göttingen, 1790, p. 271) $(n - 4)\sqrt{(n^2 + 16n - 32)}$.

[1] Housel considered the accuracy of De Ville-Bosse's construction, for $n = 3$ to $n = 17$, in "Division pratique de la circonférence en parties égales," *Nouvelles Annales de Mathématiques,* vol. 12, 1853, pp. 77–79; for $n = 5$ to $n = 17$ the errors range from $-$ 5′ 48″ to + 36′ 37″. Tempier proposed another construction which gave very much better results when n was large (*Nouvelles Annales* . . ., vol. 12, pp. 345–347, and vol. 13, p. 295). For $n > 8$ Tempier proposed the following construction: On the diameter AB of a circle, center O, construct an equilateral triangle ABC; divide the radius OB into n equal parts, the end of the fourth part from O, being D; join CO and CD, producing them to meet the circle in E and F respectively; then $\angle EOF = \delta$ is approximately the angle subtended by a side of the inscribed polygon. Tempier gave the formula

$$\sin\delta = \frac{12n + \sqrt{48n^2 - 512}}{3n^2 + 16}.$$

From this it was found that the formula was exact for $n = 12$, and that for $n = 8$ to $n =$ the error in δ was never greater than 2′ 48″, while for $n = 100$ it was only 53″.

Röber's construction of 1854 is discussed below.

In 1864 Henry Norton suggested [1]

$$\sin\frac{\pi}{7}=\frac{105}{242}=\frac{3\times5\times7}{11\times11\times2},$$

which for construction might be thought of as

$$\left(\frac{3\times5}{2}\right)\Big/\left(\frac{11\times11}{7}\right).$$

On this supposition $\sin(\pi/7) = .4338842975$ which is correct to six places of decimals.

In the same year S. M. Drach gave the following construction also correct to six places of decimals[2]: Let CD be the perpendicular from the vertex C of an equilateral triangle on the base AB; cut off $CM = (1/7 - 1/9)$ $(= 2/63)CD$ and make $\angle DMK = \frac{1}{8} \angle DCA$. Then CK (K is a point in AB) is the approximate length of the side of heptagon inscribed in the circle of radius AB.

E. Collignon applied the discussion of certain series of numbers to the construction of the regular heptagon (plate), in "Examen de certaines séries numériques et application a la géomètrie," *Compte Rendu de l'Association Française pour l'Avancement des Sciences*, 1888, p. 22.

In 1891 A. A. Robb showed [3] that, "by the aid of Peaucellier-cells a machine may be constructed which will solve the problem of the inscription of a regular heptagon in a circle, within the limits of Euclidean geometry."

In 1892 Efremoff published a paper[4] giving $s_7 = (2\sqrt{5} - 1)/4$, which is readily constructed, and hence within a minute the central angle subtended by a side of the heptagon.

In 1894 J. D. Everett described a linkage (*Report of the . . . British Association for the Advancement of Science*, for 1894, pp. 559–561) for the automatic determination of the vertices of regular polygons.

The following construction of a fourteen-year-old youth leads to a chord subtending at the center of a circle an angle $2' \, 31.7''$ in excess of the true value (*Educational Times*, March, 1907, vol. 60, p. 143; also *Mathematical Questions and Solutions from the Educational Times*, new series, vol. 12, 1907, pp. 51–52). Given a circle with center O and radius $OA = r$. Let the tangent at A meet in C the circle with center A and radius equal to r. Let this latter circle be met in D by a circle with center C and radius CA. Let DC cut the given circle with center O in B. Then AB is the approximation to the side of the heptagon inscribed in the given circle.

[1] "On fractional values for the heptagon and circle," *Philosophical Magazine*, (4), vol. 27, 1864, p. 281.

[2] "On Albert Dürer's heptagon-chord.—Second notice," *Philosophical Magazine*, (4), vol. 27, 1864, p. 320.

[3] *Mathematical Questions and Solutions from the "Educational Times,"* vol. 55, London, 1891, p. 61.

[4] D. Efremoff, ["Construction of the sides of the regular heptagon and nonagon to within 0.001 of the radius of the circumscribed circle," *Spaczinski's Messenger*], Odessa, no. 146, pp. 32–33 (in Russian).

The following pair of constructions of K. Hagge leads to remarkably close approximations ("Benaderingsconstructies voor den regelmatigen zevenhoek," *Wiskundig Tijdschrift*, vol. 10, 1913–1914, pp. 165–166). (*a*) With A as center construct circles of radius r, $2r$, $4r$, and $9r$. Let a straight line through A meet the circle of radius $2r$ in B and C, and the circle of $4r$ in D and E. Make $BF = CF = 7r$. Then DF is approximately the side of a regular heptagon inscribed in the circle of radius $9r$. The angle subtended by the side at A is only $9''$ in excess of what it should be. In the second construction the excess is only $0.44''$. (*b*) With center A construct circles of radii r, $3r$, $6r$, $7r$. Let a straight line through A meet the circle of radius $3r$ in B and B', of radius $6r$ in D and C, and of radius $7r$ in F and E. Make $CG = DG = 8r$, $FH = BG$ (H on circle of radius $7r$), $EK = EH$ (K on EF). With center A and radius $AK - r$, describe a circle. Then in this circle the chord of length $4r$ is approximately the side of a regular heptagon.

Ghersi gave other constructions (*l.c.*, 1913, pp. 416–417).

We conclude with Sir William Rowan Hamilton's discussion[1] of F. G. Röber's construction[2] of the heptagon. Sir William took up this question at the request of C. B. Wale, son-in-law of the Archbishop of Dublin, who asked his opinion of the value of a couple of essays of Röber. The following are extracts from Hamilton's letter to Wale (*l.c.*):

"A wish to gratify the archbishop and yourself was the first motive for my attempting to examine to some extent the Essays of Röber which you had the goodness to leave for me a few days ago, and to form some opinion on their value, unimportant as that opinion might be. But the Memoir on the ancient temples of Egypt (Röber, Dresden, 1854) has interested me profoundly. Indeed I have scarcely been able, since I opened it, to attend to anything else; and it led me into some long calculations which I have only just completed to my satisfaction. As I have paid no special attention to Egyptian Antiquities, nor meditated much on such mystical guesses as some have made at their inner meaning, the only point which I could hope to study usefully was the *geometrical discovery* announced in the first memoir, namely, 'the construction of the regular heptagon,' which the elder Röber appears to have *divined*, from the study of the ancient Temple Architecture.[3]

"I entered on the subject, perhaps with prejudice; for like most (if not all) modern geometers, I have been accustomed to hold, and indeed still do hold, that it is *impossible* to construct such a heptagon with the 'right line' and 'circle' *alone*. Yet, to my great surprise, I found no error in Röber's *numbers*; and on repeating the calculations on another plan, with Taylor's seven-figure logarithms, I found myself quite unable to pronounce whether Röber's *arc* erred in excess or in defect from the *exact* seventh part of the circumference; for that it *must err* I felt assured.

"It seemed, therefore, worth while to go much more closely to work; and laying tables entirely aside, to perform the *whole* of the work for myself, by *arithmetic alone*, and especially by extractions of *square roots*. And to be quite sure of a high degree of accuracy in the final result, I made it a rule to work with not fewer than *fifteen decimal places*, besides employing all verifica-

[1] R. P. Graves, *Life of Sir William Rowan Hamilton*, vol. 3, Dublin, 1889, pp. 141–147 (including a letter, dated September 15, 1862, to C. B. Wale); 584–587 (including extracts of letters exchanged by Hamilton and De Morgan in 1862). "On Röber's construction of the heptagon" by W. R. Hamilton, *Philosophical Magazine*, (4), 27, 1864, pp. 124–132.

[2] First given, apparently, in his son's *Beiträge zur Erforschung der geometrischen Grundformen in den alten Tempeln Aegyptens, und deren Beziehung zur alten Naturkenntniss*, Dresden, 1854, pp. 15–16, and repeated in a posthumous work (p. 20) edited by the younger Röber, and published at Leipsic in 1861, entitled *Elementar-Beiträge zur Bestimmung des Naturgesetzes der Gestaltung und des Widerstandes, und Anwendung dieser Beiträge auf Natur und alte Kuntsgestaltung.*

[3] The subject of the Second Essay was "The Pyramids and Parthenon."

tions that I could think of in the progress of calculation, which, thus laboriously conducted, has covered many sheets of paper, and cost many hours on two or three successive days.

"At last, however, it is finished; and I should have no hesitation to commit myself publicly to the result, which is, technically expressed, that the natural *cosine* of the angle assigned by Röber's construction is, to thirteen decimals, 0.62349 00759 241, whereas the true cosine of the seventh part of four right angles deduced to a corresponding accuracy from a known cubic equation I find to be *a little less*, namely, 0.62348 98018 587. Admitting, though I do not believe it, that the two or three last of these of these decimals *may* be wrong after all the precautions taken, I am quite satisfied that the cosine of the *Egyptian Angle*—for really Röber seems to make it likely that the Egyptians did employ it—is *somewhat greater*, and that the cosine of the *true* or *geometrical* angle (for of course we can in geometrical *conception* divide the circumference into *any number* of equal parts) is somewhat less than 0.62349, and consequently that the supposed *Egyptian rule* of the *heptagon* is *not mathematically perfect*, though Röber seems to suppose it to be so. And if you should ever think me worth citing on the subject, I request you to bear in mind that such is *one* of the results of my investigation.

"But now let us turn the tables and inquire *how near* does the supposed ancient rule come to the truth? *How small*, in *practice*, is the *error* which *theory* pronounces to *exist*? And I answer that in *practice* the *error* does not exist *at all*. I do not think that *experiments* of measurement, &c., *could* be so conducted by men, at least in the present age, as to prove to sight that there was any error. For *practical purposes*, then, the elder of the Röbers, or the old Egyptian sage whose secrets he supposed himself to have divined, has *done the impossible*. . . . Yet the *practical success* of the rule is to me absolutely *wonderful*: and it is long since any discovery in science produced in me such a sensation of *surprise*. . . . I may thus recapitulate:

"(1) The (alleged) *Egyptian Rule* for the construction of the Regular Heptagon is, in rigour of theory, erroneous.

"(2) The *same rule* of construction of the Heptagon is, however, for *all practical purposes, perfect*.

"No artist of the present day, I feel sure, would undertake to divide a circle into 7 (seven) equal parts, with a superior, or even an *equal* accuracy, to that which the construction, if fully carried out, would give."

Hamilton's article on the subject describes Röber's diagram as "not very complex, and may even be considered elegant"; and then continues: "but the essential parts of the construction are sufficiently expressed by the following formulæ: in which p denotes a side of a regular pentagon; r, r' the radii of its inscribed and circumscribed circles; r'' the radius of a third circle, concentric with but exterior to both; p' a segment of the side p; and q, s, t, u, v five other derived lines. The result is, that in the right-angled triangle of which the inner diameter $2r$ is the hypotenuse, and u, v supplementary chords, the former chord (u) is *very nearly* equal to a side of a regular heptagon, inscribed in the interior circle; while the latter chord (v) makes with the diameter ($2r$) an angle ϕ, which is *very nearly* equal to the vertical angle of an isosceles triangle, whereof each angle at the base is triple of the angle at the vertex. In symbols, if we write $u = 2r \sin \phi$, $v = 2r \cos \phi$, then ϕ is found to be very nearly $= \pi/7$. It will be seen that the equations can all be easily constructed by right lines and circles alone, having in fact been formed as the expression of such a construction; and that the numerical ratios of the lines, including the numerical values of the sine and cosine of ϕ, can all be arithmetically computed, with a few extractions of square roots." The formulæ are: $(r + r')^2 = 5r^2$, $p^2 = 4(r'^2 - r^2)$, $(p'/p) = (r + \frac{1}{2}r')/(r + r')$, $q^2 = p^2 - p'^2$, $s^2 + ps = (p - q + r)^2$, $r''^2 = r^2 + s^2$, $t^2 = (r'r''/r)^2 - (r'' - r)^2$, $u^2 = 2r(2r - t)$, $v^2 = 2rt$, $u = 2r \sin \phi$, $v = 2r \cos \phi$.

Hamilton showed that

$$\sin \frac{\pi}{7} = 0.43388\ 37391\ 17558\ 1205,$$

that

$$\sin \phi = 0.43388\ 35812\ 03469\ 1138,$$

and that

$$\frac{\pi}{7} - \phi = +\ 0''.03615\ 23230\ 806.$$

"Another way of rendering conceivable the extreme smallness of the practical error of that process, is to imagine a series of seven successive chords inscribed in a circle, according to the construction in question, and to inquire *how near* to the initial point the final point would be. The answer is, that the *last* point would fall *behind* the *first*, but only by about *half a second* (more exactly by 0″.506). If then we suppose, for illustration, that these chords are *seven successive tunnels*, drawn *eastward* from station to station of the *equator of the earth*, the last tunnel would emerge to the *west* of the first station, but only by about *fifty feet.*"

On September 15, 1862, Hamilton wrote to De Morgan as follows:

"Are you aware that it is possible with only *six* extractions of *square roots* (*two* being those required for chord and tangent of 36°) to approach so nearly to the cosine of the *seventh part* of four right angles, or to the positive root of the *cubic*,

$$2x^3 + x^2 - x = \tfrac{1}{4},$$

that the *sevenfold arc* resulting shall err (in defect) by only about *half a second* from the true and whole circumference? so that for *all practical purposes* a *regular heptagon* can be constructed by the right line and circle *alone.*"

De Morgan replied in part:

"I can believe anything of *six* square roots. But I cannot believe that the Egyptians employed them—though they may have hit on a method which requires six square roots to represent it arithmetically."

Hamilton and De Morgan arrived independently at the result

$$\cos\frac{2\pi}{7} = 0.62348\ 98018\ 58733\ 53052\ 50$$

which on September 29 brought forth De Morgan's comment: "I take it that you and I are the only persons who know cos $2\pi/7$ to 22 places."

*

Problem (*d*)—Let a, b, and c be the given integers and n, o and p the corresponding remainders after their division into the required integer. Huygens finds that the integer $bcfn + acgo + abhp$ satisfies all the conditions, if f is an integer such that bcf divided by a gives the remainder unity, and similarly for g and acg/b, and for h and abh/c. As a numerical example Huygens refers to the Julian period, proposed by the noted J. J. Scaliger (1540–1609), and a rule given by P. de Billy; it was translated as, "A problem for finding the year of the Julian period by a new and very easy method," and published in *Philosophical Transactions of the Royal Society*, 1666, p. 324. The rule is as follows: Multiply the year of the solar cycle [n] by 4845 [$= bcf$], that of the moon [o] by 4200 [$= acg$], that of the cycle of indiction [p] by 6916 [$= abh$]. Then divide the sum of the products by 7980 [$= abc$] which is the Julian period. What remains from the division, neglecting the quotient, will be the year sought. Huygens takes $a = 28$ for the whole solar cycle; $b = 19$ for the whole lunar cycle; and $c = 15$ for the cycle of indiction. He takes also $n = 13$ for the number in a solar cycle of the particular year chosen; and similarly $o = 4$ for the lunar cycle and $p = 9$ for the cycle of indiction. Then

$$\begin{aligned}[bcfn + acgo + abhp]/abc &= [62985 + 16800 + 62244]/7980 \\ &= 142029 \div 7980 = 17 + 6369/7980.\end{aligned}$$

Hence 6369 is the year of the Scaligerian period corresponding to the data.

This same problem is discussed in the section on Julian period in the article

"Calendar" of the *Encyclopædia Britannica,* ninth and eleventh editions. It is there shown that the year 1 of our era "had 10 for its number in the solar cycle, 2 in the lunar cycle, and 4 in the cycle of indiction"; hence "the question is therefore to find a number such, that when it is divided by the three numbers 28, 19, and 15 respectively the three quotients shall be 10, 2, and 4." Without recourse to the general discussion of Huygens it is found that the number is 4714.

Compare, F. K. Ginzel, *Handbuch der mathematischen und technischen Chronologie der Zeitrechnungsweise der Völker,* Leipsic, vol. 3, 1914, p. 182.

R. C. Archibald

SOLUTIONS.

499 (Geometry) [1916, 341; 1919, 414; 1920, 187]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

Solution by J. L. Walsh, Harvard University.

This problem was solved by Professor Cairns [*1920,* p. 187], who interpreted the term *surface* as *algebraic surface.* Inasmuch as there exist other interesting surfaces (analytic, regular, etc.), perhaps it is worth while to give a geometric proof of the theorem:

If every plane section of a point set S is a circle, then S is a sphere.

We restrict ourselves to the real domain, and interpret our hypothesis to mean that whenever a plane actually intersects S, the intersection is a circle, which may be a null circle.

If a section of S by a sphere Σ contains three distinct points, then the section also contains the entire circle through these points. For through these three points we can pass a plane which will cut S in a circle and also cut Σ in a circle. These two circles have three points in common and hence are identical.

Let P be any point of S. Transform S into a point set S' by means of an inversion in space with P as center of inversion. The point P is transformed into P', the point at infinity; we consider as is habitual in the geometry of inversion a single point P' to lie at infinity. Every plane section of S' corresponds to a plane section or a spherical section of S. Every straight line of points belonging to S' corresponds to a circle of points belonging to S; there is no straight line all of whose points belong to S, for a straight line lies in a plane and every plane section of S is a circle.

If S' consists merely of the point P' every plane which cuts S cuts it in the null-sphere P, and the theorem is proved. If S contains another point Q besides P every plane through these points will intersect S in a circle which is not a single point, and there will be more than one of these circles. If then besides P' another point Q' belongs to S', there must be lines through Q' belonging to S', more than one of them.

Likewise, if S contains two distinct points besides P these points must lie on a circle through P, the intersection of S with the plane which contains the three points. Therefore if S' contains two distinct points besides P' it must contain the line determined by these two points.

Now we have shown that if S' contains one point Q' distinct from P' it contains more than one line through this point. It contains then the line determined by any two points of two lines through Q', and therefore all the points of at least one plane through Q'.

But a plane belonging entirely to S' corresponds in the inversion to a sphere belonging to S. Thus we have proved that if S contains more than one point it contains all the points of a sphere. S cannot contain a sphere and any point outside of the sphere, for a plane through such a point intersecting the sphere would intersect S in more than a single circle.

The method of proof used in the present note easily yields the following theorem:

If a point set consists of more than two points and is such that every spherical section which contains three points of the set also contains the circle through those three points, then the set is a circle, a plane, a sphere, or every point of space.

NOTE BY THE EDITORS.—Strictly speaking the problem as worded is more general than the theorem proved by J. L. Walsh, as it includes surfaces which are cut by a plane in more than one circle; for example, a combination of several spheres. Neither W. D. Cairns nor J. L. Walsh has solved this more general problem. Indeed there might be some further difficulty in its interpretation. For we interpret circle as including the case of a single point, and any intersection of two figures might be regarded as consisting of point circles. We might exclude such circles, but we would not want to exclude the case of a plane tangent to a sphere.

ELIJAH SWIFT gave, in effect, the same discussion as J. L. Walsh above.

2814 [1920, 134]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

The bisectors of the angles formed by the diagonals of an inscribed quadrilateral are: (1) parallel to the lines joining the midpoints of the arcs subtended by the opposite sides of the quadrilateral on its circumcircle; (2) parallel to the bisectors of the angles formed by any pair of opposite sides of the quadrilateral; (3) equally inclined to pairs of sides of the quadrilateral.

SOLUTION BY J. W. CLAWSON, Ursinus College.

Let $L_1L_2L_3L_4$ be the inscribed quadrangle. L_2L_1, L_3L_4 intersect at N'; L_1L_3, L_2L_4 at N''. D_{12}, D_{34} are the midpoints of arcs L_1L_2, L_3L_4, respectively. Let the bisector of $L_3N''L_4$ cut the circle at X; let $D_{12}D_{34}$ cut L_1L_3 at Y; let the internal and external bisectors of $L_1N'L_4$ cut L_1L_3 in W, V.

Now

$$L_3N''X = \tfrac{1}{2}L_3N''L_4 = \tfrac{1}{2}(L_3L_1L_4 + L_1L_4L_2).$$

Also

$$L_3YD_{34} = L_3D_{12}D_{34} + L_1L_3D_{12} = \tfrac{1}{2}(L_3L_1L_4 + L_1L_3L_2).$$

Again

$$L_3VN' = \frac{\pi}{2} - N'WV = \frac{\pi}{2} - \frac{1}{2}L_1N'L_3 - L_4L_3L_1 = \frac{\pi}{2} - L_4L_3L_1 - \frac{1}{2}(L_2L_1L_3 - L_4L_3L_1)$$
$$= \frac{\pi}{2} - \frac{1}{2}(L_4L_3L_1 + L_2L_1L_3).$$

But

$$L_4L_3L_1 + L_1L_3L_2 + L_2L_1L_3 + L_3L_1L_4 = \pi.$$

Hence,

$$L_3VN' = \tfrac{1}{2}(L_1L_3L_2 + L_3L_1L_4).$$

Therefore, $D_{12}D_{34}$, $N''X$, and VN' are all parallel.

Similarly, the other facts may be established.

Note. (1) was proved by Neuberg, *Mathesis*, volume 6, (1906), page 14. I believe that (2) was first published in an article of mine "The Complete Quadrilateral," *Annals of Mathematics*, volume 20 (1919), page 257. There are several other lines belonging to an inscribed quadrangle which are parallel to the bisectors of the angles between its diagonals. An account of these will be found in the paper referred to.

Also solved by F. V. MORLEY, H. L. OLSON, A. V. RICHARDSON, and ARTHUR PELLETIER.

2843 (1920, 326). Proposed by E. H. MOORE, University of Chicago.

Show that the maximum of the absolute value of $2(a + ib)(x + iy) + i(a + ib)(z + iw) + i(c + id)(x + iy)$, where $i = \sqrt{-1}$, and a, b, c, d, x, y, z, w are real numbers for which $a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2 + w^2 = 1$, is $1 + \sqrt{2}$. Study the locus of the point-pairs, $P = (a, b, c, d)$, $Q = (x, y, z, w)$ of the unit-sphere in real four-space for which this absolute value assumes its maximum value.

PARTIAL SOLUTION BY F. L. WILMER, Omaha, Neb.

As usual let a and b denote the abscissa and ordinate, respectively, of the terminus of the vector $a + ib$ in its standard position, and $|a + ib|$ its absolute value.

Because $a^2 + b^2$ must be equal to the square of the cosine of some angle, say α, therefore $c^2 + d^2 = \sin^2 \alpha$. Similarly for x, y, z, w and an angle, say β.

For given values of the moduli the sum of any number of complex numbers has its maximum absolute value when the arguments are equal or differ by multiples of 2π, and this maximum is the sum of the moduli.

Let the arguments of $a + ib,\quad i(c + id),\quad x + iy,\quad i(z + iw),$
be $\theta_1,\quad \theta_1',\quad \theta_2,\quad \theta_2'$.

The arguments of the three terms of the given expression will be $\theta_1 + \theta_2$, $\theta_1 + \theta_2'$, $\theta_1' + \theta_2$, and these will be equal if $\theta_1' = \theta_1$ and $\theta_2' = \theta_2$ (to multiples of 2π).

The modulus of the given expression will then be

$$\begin{aligned} R &= 2 \cos \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta \\ &= \cos (\alpha + \beta) + \cos (\alpha - \beta) + \sin (\alpha + \beta). \end{aligned}$$

Now in order that R be a maximum we must have $\partial R/\partial \alpha = 0$ and $\partial R/\partial \beta = 0$, simultaneously. That is,

$$- \sin (\alpha + \beta) - \sin (\alpha - \beta) + \cos (\alpha + \beta) = 0,$$

and

$$- \sin (\alpha + \beta) + \sin (\alpha - \beta) + \cos (\alpha + \beta) = 0;$$

whence $\alpha = \beta = \pi/8$, giving[1]

$$R_m = \cos \pi/4 + 1 + \sin \pi/4 = \sqrt{2} + 1.$$

2896. (1921, 227) Proposed by the late L. G. WELD.

A circle is inscribed in a triangle. In each of the three spandrels between this circle and the vertices another circle is described; in each of the three spandrels between the last circles and the vertices another circle; and so on *ad infinitum.* Show that the ratio of the sum of the areas of all the circles to the area of the triangle is

$$\frac{\Sigma}{\Delta} = \frac{\pi \Delta}{4 S^2}\left[\frac{1}{\sin \frac{1}{2}A} + \frac{1}{\sin \frac{1}{2}B} + \frac{1}{\sin \frac{1}{2}C} - 2 + \sin \tfrac{1}{2}A + \sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C\right].$$

This problem is identical with problem 483 Geometry already proposed by Professor Weld (*1916*, 79), and solved by J. A. Caparo (*1916*, 344–346). It was re-proposed through an oversight.

NOTES AND NEWS.

It is to be hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

At the University of Pennsylvania, Mr. J. D. ESHLEMAN, of the University of Chicago, has been appointed instructor of mathematics. Mr. J. M. THOMAS resigned his instructorship to accept a Harrison Fellowship in the Graduate School.

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[1] We might say, $R = \sqrt{2} \cos (\pi/4 - \alpha - \beta) + \cos (\alpha - \beta)$, and as α and β are independent this expression has a maximum equal to $\sqrt{2} + 1$ when $\alpha + \beta = \pi/4$ and $\alpha - \beta = 0$.

The condition that the given expression have its maximum absolute value is that the points (x, y, z, w) and (a, b, c, d) shall be two points of the circle whose equations are

$$z = y \tan \alpha, \qquad w = - x \tan \alpha, \qquad x^2 + y^2 + z^2 + w^2 = 1.$$

—EDITOR.

Dr. E. F. Nichols, inaugurated as president of Massachusetts Institute of Technology in June (*1921*, 336), resigned in November, on account of ill health. The Institute is directed by a committee consisting of Professors H. P. Talbot, E. F. Miller, and E. B. Wilson.

A. V. Miller, associate professor of drawing and descriptive geometry at the University of Wisconsin, has been appointed assistant dean of the college of engineering.

At the University of Wisconsin, Assistant Professor Arnold Dresden has been promoted to an associate professorship and Mr. E. B. Miller has been appointed instructor of mathematics.

At the University of Illinois, Dr. J. M. Stetson, of Yale University, Dr. Beulah Armstrong, of the University of Illinois, and Dr. C. C. Camp, formerly assistant professor at Iowa State College, have been appointed instructors of mathematics.

Dr. Harlow Shapley, whose appointment as observer at the Harvard College Observatory has already been noted in this Monthly (*1921*, 233), was, in November, made director of the Observatory—a post which has been vacant since the death of E. C. Pickering in 1919 (*1919*, 134). The recent acting director, Professor S. I. Bailey (*1921*, 233), expects in a few months to return to Harvard's South American astronomical station at Arequipa, Peru. Dr. Shapley was born at Nashville, Mo., thirty-five years ago. Graduating from the University of Missouri, he was a fellow at Princeton University, 1912–1914, where he received his doctorate in 1913. He was at the Mt. Wilson Observatory, 1914–1921. He perfected methods of measuring star distances photometrically, and applied these methods to the problem of the distances and structures of great star-clusters. "Dr. Shapley is also known as an entomologist, and has done interesting work in investigating the ants of the California mountains. He discovered that the speed at which these creatures move depends on the temperature, and that for some species the time of running through a 'speed-trap,' as shown by the stop-watch, gives the temperature of the surrounding air within one degree. He found that the ants went twelve times as fast at 100 degrees as at 50 degrees." —*Harvard Alumni Bulletin*, November 10, 1921.

Judson Boardman Coit, teacher of mathematics and astronomy in Boston University since 1882, died on July 26, 1921. He was born in Oswego County, New York, June 5, 1849, and graduated from Syracuse University (A.B., 1875; A.M., 1878; Ph.D., 1881). He was professor of mathematics at Dickinson Seminary, Williamsport, Pa., 1875–1879, assistant in the observatory at Ann Arbor, 1879–1880, and teacher of mathematics in a Cleveland high school, 1880–1882. Appointed assistant professor of mathematics and astronomy in Boston University in 1882, he was promoted to a professorship in 1884. In 1915 he was made professor of astronomy. He was acting dean of the graduate school 1911–12. His son, Dr. W. A. Coit, has been professor of mathematics at Acadia College, Wolfville, Nova Scotia, since 1908.

The Ziwet Collection.

By L. C. Karpinski, University of Michigan.

The development of a modern university library requires over long periods of time the devoted service of scholars in many fields. Even in a limited field, such as mathematics, the building of a working library is a serious task of many years.

For a third of a century Professor Alexander Ziwet has given scholarly attention to the library needs of the University of Michigan. Unusual linguistic ability coupled with equally unusual devotion to bibliographical and scholarly affairs combine to make Professor Ziwet's record of service to the library a notable one. At the same time Professor Ziwet was building up a private collection, strong in general mathematics and in the first rank of collections on mechanics. This personal library, comprising a total of over five thousand volumes, Professor Ziwet has recently given to the University of Michigan.

The library is composed as follows: Miscellaneous volumes, about equally mathematics and mechanics, 2092; Miscellaneous pamphlets, 838; Serials, sets of volumes, etc., 1414; Classical volumes, 364; and Classical pamphlets, 127.

Any one who has worked in the University of Michigan Library is aware of the fact that this recent gift by Professor Ziwet is only the culmination of a series of notable gifts extending over many years.

It would take more space than is at my disposal to list noteworthy volumes included. As indicated, the mechanics collection is particularly complete including the best Dutch, Scandinavian, German, French, Spanish, Italian, Russian, and English works which have appeared during the past fifty years. In addition there are numerous first editions of classical works on mechanics such as those by Newton, Lagrange, D'Alembert, Euler, D. Bernoulli, Hermann, Marie, Coulomb, Carnot, and Chasles. A practically complete set of works by Duhem is worth noting.

The collection adds many items on the history of the mathematical and astronomical sciences, to a collection now one of the first in the United States. This collection has been built up through the coöperation of Professors Beman, Ziwet, Hussey, and the writer with the continued active assistance of the librarians at Michigan.

With characteristic generosity Professor Ziwet has made no conditions in regard to his gift. In consequence the duplicates will in all likelihood be used for exchange purposes to further strengthen the mathematical library.

The permanent and increasing value of such collections is beyond dispute. Generations of students to come will have cause to be grateful to the collector, Alexander Ziwet, whose devotion to science is reflected in this munificent and enduring collection.